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ANOMALOUS DIMENSIONS AND THE BREAKDOWN OF SCALE INVARIANCE IN PERTURBATION THEORY*

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ABSTRACT

Canonical field theory predicts that a zero mass scalar field theory with a $\lambda\phi^4$ interaction is scale invariant. It is shown here that the renormalized perturbation expansion of the $\lambda\phi^4$ theory is not scale invariant in order λ^2. Matrix elements of the divergence of the dilation current $D_\mu(x)$ are computed in order λ^2 using Ward identities; it is found that $\nabla^\mu D_\mu(x)$ is proportional to $\lambda^2\phi^4(x)$. It is also shown that the dimension of the field ϕ^4 differs from the canonical value in order λ and that this result leads one to expect a $\lambda^2\phi^4$ term in $\nabla^\mu D_\mu$. It is also found that matrix elements of the composite field $\phi^4(x)$ in perturbation theory have troublesome singularities at short distances which force one to give careful definitions for equal time commutators and Fourier transforms of T products in the Ward identities involving this field.

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I. INTRODUCTION

In a previous paper a new theory of the short distance behavior of strong interactions was proposed.[1] The theory involved several unfamiliar ideas, in particular the idea of an "operator product expansion" and the idea that the dimensions of quantum fields are changed by interactions between the fields. The present paper is one of a series[2] designed to make these ideas come alive. These papers concern nontrivial problems in perturbation theory or soluble models; they show how operator product expansions or dimensions changing with the coupling constant are involved in the solution of these problems.

The purpose of this paper is to study a puzzle in renormalization theory. The puzzle is as follows. Normally, when the unrenormalized Lagrangian is invariant to a symmetry, the renormalized perturbation expansion for the Lagrangian is also invariant to the symmetry. This is true for internal symmetries such as isotopic spin; it is also true of Lorentz invariance. However, there is an exception, the exception being scale invariance.[3] For example, the unrenormalized Lagrangian for the electrodynamics of zero mass electrons is scale invariant (because the only parameter in the zero mass Lagrangian is the bare coupling constant e_0, which is dimensionless). However the renormalized perturbation expansion for zero mass electrodynamics is not scale invariant. The renormalized zero mass perturbation expansion was defined by Gell-Mann and Low.[4] The photon propagator in the zero mass theory has the approximate form:[5]

$$D(k) = (k^2)^{-1}\left[1 - \left(e_\kappa^2/12\pi^2\right)\ln(k^2/\kappa^2)\right]^{-1} \qquad (I.1)$$

where κ is a reference momentum that is introduced as part of the Gell-Mann-Low renormalization procedure, and e_κ is a renormalized coupling constant defined relative to the reference momentum. The reference momentum is necessary for without it the renormalization procedure would replace ultraviolet divergences by infrared divergences. The form (I.1) is a sum of leading logarithms for each order in e_κ. In contrast if the renormalized perturbation expansion were scale invariant, the leading logarithms would be required to sum to a power of k^2.

A tentative explanation will be proposed here for this puzzle. To simplify matters the $\lambda\phi^4$ interaction of a scalar field ϕ with zero mass will be discussed instead of zero mass electrodynamics. At the heart of the explanation is the result (to be derived in Section III) that when a renormalized Heisenberg composite field is defined starting from the product $\phi^4(x)$, the resulting field changes its dimension in the presence of interaction. However, the dimension of the Lagrangian cannot change, so λ must acquire compensating dimensions. Then λ ceases to be a dimensionless constant, and there is no longer any reason to expect the theory to be scale invariant. This is the essence of the explanation given in Section III of the puzzle. It will also be explained precisely what is meant by a change of dimension for ϕ^4. The idea of the constant λ changing dimensions however will not be discussed in detail; instead it will be argued that the change of dimension of ϕ^4 leads to a term proportional to $\lambda^2\phi^4$ appearing in the divergence of the dilation current, spoiling scale invariance.

In this paper the scaling properties of the $\lambda\phi^4$ theory will be inferred from Ward identities involving vacuum expectation values of the fields $\phi(x)$, $\phi^4(x)$, and the divergence of the dilation current, called S(x). These Ward identities will be used to calculate matrix elements of the divergence S(x), given matrix elements involving only $\phi(x)$ and $\phi^4(x)$. It is possible to calculate matrix elements of S directly without using the Ward identities; doing so would provide a check on the calculations of this paper. A start on such calculations has been made by Callan, Coleman, and Jackiw.[6] Direct calculations of the matrix elements of S are not made in this paper because there are many problems involved with such calculations which do not appear in the calculation of matrix elements of ϕ alone. Some of these problems do appear in the calculation of matrix elements of $\phi^4(x)$ and will be discussed later. But as far as possible this paper relies on uncontroversial Feynman diagram formulae; this is for simplicity and to make clear that the breakdown of scale invariance is an inevitable consequence of these formulae.

In calculating matrix elements of the operator $\phi^4(x)$, and in checking Ward identities involving these matrix elements, problems arise which can be traced to an age-old problem: What does a T product of operators such as $T\,\phi(x)\,\phi^4(y)$ mean when $x=y$? Axiomatic field theorists answer that it is arbitrary in the sense that one is free to add any term proportional to $\delta^4(x-y)$ or derivatives of $\delta^4(x-y)$ to the T product.[7] Other field theorists take it for granted that the T product is uniquely defined, without making clear what that definition is. In order to get consistent results in this paper it will be necessary to specify a definition of the T product which eliminates the arbitrariness. There will be a corresponding, precise definition of the equal time commutators which occur in Ward identities. It will be shown that under normal circumstances the definition of equal time commutators given in this paper agrees with the customary one, but in abnormal cases (one of which occurs later in this paper) the two definitions do not agree. There will also be circumstances where the definition of the T product given here has to be modified to include subtractions; an example of this also occurs later in this paper. The definition of the T product given in this paper may or may not be one that field theorists can agree upon; what is essential is that in all future discussions of Ward identities the definition of the T product be stated, so that one can handle more easily the kind of problem that arises later in this paper.

In Section II of this paper the problem of defining T products is analyzed, with examples showing the problems that can arise. In Section III, which is the heart of this paper, the Ward identities and explicit formulae for vacuum expectation values of ϕ and ϕ^4 are written down. These formulae are used to show that scale invariance holds in order λ and breaks down in order λ^2, to compute the dimension of ϕ^4 in order λ, and to infer that S(x) in order λ^2 is proportional to ϕ^4. In Section IV the operator product expansion for $\phi(x)\ \phi^4(y)$ is discussed; also the dimensions of the composite field $\phi_i(x)\ \phi_j(x)$ in an isospin one ϕ^4 theory are computed and shown to be different for the isospin 0 and isospin 2 components.

II. DEFINITIONS OF T PRODUCTS

The problem of defining T products will be discussed primarily in terms of an example, the example being the T product of two currents.[8] Consider in particular the propagator

$$D_{\mu\nu}(p) = \int_x e^{ip\cdot x} D_{\mu\nu}(x) \tag{II.1}$$

$$D_{\mu\nu}(x) = \langle \Omega | T\, j_\mu(x)\, j_\nu(0) | \Omega \rangle \tag{II.2}$$

where j_μ is a conserved current in an unspecified field theory and $\int_x$ means $\int d^4x$. The problem to be discussed here is this: How is the integral in Eq. (II.1) to be calculated, assuming the function $D_{\mu\nu}(x)$ is known? This is a question which does not arise much in practice since one is more likely to have an explicit formula for $D_{\mu\nu}(p)$ (via Feynman graphs, or whatever) than for $D_{\mu\nu}(x)$. However, Ward identities are derived in x space and then Fourier transformed to momentum space; if one is deriving a Ward identity for $D_{\mu\nu}(p)$, then $D_{\mu\nu}(p)$ is <u>defined</u> by Eq. (II.1) and it becomes a legitimate question to ask whether ambiguities arise in computing the integral, and how to avoid them if they do occur.

The reason the integral in Eq. (II.1) can cause difficulties is that $D_{\mu\nu}(x)$ is singular at x=0; the singularity at x=0 is such that the integral may be conditionally convergent or divergent at x=0. If the integral is conditionally convergent, it can be defined by specifying an <u>order of integration</u> for the four integrations (over the components of x), but the result may depend on which order is chosen. If the integral is divergent then it can only be defined by subtracting the divergent terms.

An example of conditional convergence is provided by a free vector meson propagator. In this case it will be shown below that the integral in Eq. (II.1) gives different answers depending on whether the $\underset{\sim}{x}$ integral or the x_0 integral is performed first. It will also be shown that the usual noncovariant form of $D_{\mu\nu}(p)$ is obtained by doing the $\underset{\sim}{x}$ integral first. These results will be shown by using one of the standard derivations of the noncovariant propagator and being careful when the order of integration is changed. The standard derivation will first be stated without being careful; the careful derivation will be given afterwards.

The non-time ordered matrix element

$$\rho_{\mu\nu}(x) = \langle \Omega | j_\mu(x)\, j_\nu(0) | \Omega \rangle \qquad \text{(II.3)}$$

(where j_μ is now the vector meson field) is

$$\rho_{\mu\nu}(x) = \int_p e^{-ip\cdot x}\, \rho_{\mu\nu}(p) \qquad \text{(II.4)}$$

$$\rho_{\mu\nu}(p) = 2\pi\, \theta(p_0)\, \delta(p^2-m^2) \left(-g_{\mu\nu} + p_\mu p_\nu / m^2\right) \qquad \text{(II.5)}$$

where m is the vector meson mass, $\int_p$ means $(2\pi)^{-4} \int d^4p$, and $\theta(x_0)$ is the usual θ function.[9] The T product $D_{\mu\nu}(x) = \rho_{\mu\nu}(x)$ when $x_0 > 0$; for $x_0 < 0$ $D_{\mu\nu}(x) = \rho_{\nu\mu}(-x)$. The propagator $D_{\mu\nu}(p)$ is

$$D_{\mu\nu}(p) = \int_0^\infty dx_0 \int d^3\underset{\sim}{x}\, e^{+ip\cdot x} \int_q e^{-iq\cdot x}\, \rho_{\mu\nu}(q)$$

$$+ \int_{-\infty}^0 dx_0 \int d^3\underset{\sim}{x}\, e^{ip\cdot x} \int_q e^{+iq\cdot x}\, \rho_{\nu\mu}(q) \qquad \text{(II.6)}$$

Exchanging the order of integration so that the $\underset{\sim}{x}$ integral is done first one gets a δ-function (either $\delta^3(\underset{\sim}{p}-\underset{\sim}{q})$ or $\delta^3(\underset{\sim}{p}+\underset{\sim}{q})$). Doing the $\underset{\sim}{q}$ integration next eliminates the δ-function; then one does the x_0 integral, leaving

$$D_{\mu\nu}(p) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dq_0 \left\{ \frac{1}{q_0 - p_0 - i\epsilon}\, \rho_{\mu\nu}(q_0, \underset{\sim}{p}) + \frac{1}{q_0 + p_0}\, \rho_{\nu\mu}(q_0, -\underset{\sim}{p}) \right\} \qquad \text{(II.7)}$$

Using the explicit form for $\rho_{\mu\nu}(p)$ gives

$$D_{\mu\nu}(p) = \left\{-g_{\mu\nu} + p_\mu p_\nu/m^2\right\} i(p^2-m^2+i\epsilon)^{-1} - i\,\delta_{\mu 0}\,\delta_{\nu 0}/m^2 \quad (\text{II. 8})$$

where $\delta_{\mu 0}$ is the Kronecker δ; the $\delta_{\mu 0}\delta_{\nu 0}$ term is the noncovariant piece.

The integrals in Eq. (II. 6) can be done more carefully using convergence factors to make all integrals absolutely convergent. The orders of integration can then be exchanged legitimately. With convergence factors one has

$$D_{\mu\nu}(p) = \int_\eta^\infty dx_0 \int_\theta^\infty r^2 dr \int d\Omega \int_q \left\{e^{i(p-q)\cdot x} + e^{-i(p+q)\cdot x}\right\} \rho_{\mu\nu}(q)$$

$$\times \exp\left\{-\alpha x_0 - \alpha r - \epsilon|q_0| - \epsilon|\underset{\sim}{q}|\right\} \quad (\text{II. 9})$$

where the $\underset{\sim}{x}$ integral has been written in terms of polar coordinates ($d\Omega$ is the solid angle differential). The constants α and ϵ (which must be positive) make the integrals absolutely convergent. By putting lower limits η and θ on the x_0 and $\underset{\sim}{x}$ integrations one can study different orders of integration for the x integral. Thus to find the result of performing the $\underset{\sim}{x}$ integral before the x_0 integral in Eq. (II. 1), one takes the following limit in Eq. (II. 9): $\epsilon \to 0$ first (to get the q integration correct before taking any other limits), $\alpha \to 0$ second, $\theta \to 0$ third, and $\eta \to 0$ last. To do the $\underset{\sim}{x}$ integration last one takes the limits in the order $\epsilon \to 0$ first, then $\eta \to 0$, then $\alpha \to 0$, then $\theta \to 0$.

The integral with convergence factors present can be computed explicitly when α, ϵ, η, and θ are small, neglecting small terms. The result is

$$D_{\mu\nu}(p) = D_{S\mu\nu}(p) + (\pi m^2)^{-1} \left\{-\frac{1}{3} g_{\mu\nu} + \frac{4}{3}\delta_{\mu 0}\delta_{\nu 0}\right\} \left\{\ln\left[\frac{(\eta+\theta)}{(\eta-\theta-i\epsilon)}\right] + \frac{2\eta\theta}{\theta^2-\eta^2+i\epsilon}\right\}$$

(II. 10)

where $D_{S\mu\nu}(p)$ is the standard form for $D_{\mu\nu}(p)$ given by Eq. (II.8). If η and θ are both small but of the same order, the second term is of order 1; terms of order η, θ, etc., have been dropped. If $\theta \to 0$ keeping η fixed the second term vanishes, leaving the standard form; but if $\eta \to 0$ keeping θ fixed one gets

$$D_{\mu\nu}(p) = D_{S\mu\nu}(p) + (im^{-2}) \left(-\frac{1}{3} g_{\mu\nu} + \frac{4}{3} \delta_{\mu 0} \delta_{\nu 0}\right) \qquad \text{(II.11)}$$

Hence the order of integration matters in Eq. (II.1); to get the standard form for $D_{\mu\nu}(p)$ one must write

$$D_{\mu\nu}(p) = \mathrm{Lim}\,(\eta \to 0) \left\{ \int_{\eta}^{\infty} dx_0 + \int_{-\infty}^{-\eta} dx_0 \right\} \int d^3\underset{\sim}{x}\, e^{ip\cdot x} D_{\mu\nu}(x) \qquad \text{(II.12)}$$

For any finite η the point x=0 is excluded from the integral. However, except for this point the function $D_{\mu\nu}(x)$ is covariant (the Fourier transform of the noncovariant piece of $D_{\mu\nu}(p)$ is proportional to $\delta^4(x)$ and vanishes if $x \neq 0$). So the noncovariance in $D_{\mu\nu}(p)$ is entirely due to the noncovariant definition of the integral in Eq. (II.12). This result can be shown directly. If $D_{\mu\nu}(p)$ is computed in a Lorentz frame moving in the z direction with velocity v, using Eq. (II.12) in the moving frame, and then Lorentz transformed back to the fixed frame, one gets (by transforming Eq. (II.8)) a function $D_{\mu\nu}(p, v)$:

$$D_{\mu\nu}(p,v) = i(p^2-m^2+i\epsilon)^{-1} \left(-g_{\mu\nu}+p_\mu p_\nu/m^2\right) - i\frac{\left(\delta_{\mu 0} + v\,\delta_{\mu 3}\right)\left(\delta_{\nu 0} + v\,\delta_{\nu 3}\right)}{(1-v^2)m^2} \qquad \text{(II.13)}$$

The function $D_{\mu\nu}(p, v)$ must also result if one transforms the integral of Eq. (II.12) from the moving frame to the fixed frame. Since $D_{\mu\nu}(x)$ is covariant, the only change is in the boundary of integration; one gets

$$D_{\mu\nu}(p,v) = \mathrm{Lim}\,(\eta \to 0) \int_x \theta\left[|x_0 - vx_3| - \eta(1-v^2)^{1/2}\right] e^{ip\cdot x} D_{\mu\nu}(x) \qquad \text{(II.14)}$$

i.e., the region $|x_0 - vx_3| < \eta(1-v^2)^{1/2}$ is excluded from the range of integration. Since the scale of η does not matter one can also specify the excluded region as $|x_0 - vx_3| < \eta$. The difference between $D_{\mu\nu}(p,v)$ and $D_{\mu\nu}(p)$ must come from the difference in the excluded regions. That is

$$D_{\mu\nu}(p,v) - D_{\mu\nu}(p) = \lim_{\eta \to 0} \left\{ \int_{R_1} - \int_{R_2} \right\} d^4x \, e^{ip\cdot x} D_{\mu\nu}(x) \tag{II.15}$$

where R_1 is the region $|x_0| < \eta$, $|x_0 - vx_3| > \eta$ and R_2 is the region $|x_0| > \eta$, $|x_0 - vx_3| < \eta$.

The regions R_1 and R_2 both collapse in the limit $\eta \to 0$, so for the limit $\eta \to 0$ to be nonzero $D_{\mu\nu}(x)$ has to be singular within these regions. Both regions are spacelike relative to the origin except for a region of linear size η. The function $D_{\mu\nu}(x)$ is singular only on the light cone and at x=0; these singularities lie in the region of linear size η, and must be strong enough to overcome the small volume of integration. It is worth showing explicitly how the singularity of $D_{\mu\nu}(x)$ at x=0 results in a nonzero limit, for in doing so one can deduce a general rule for when the integral of a T product may be noncovariant.

The explicit form of $D_{\mu\nu}(x)$ is known; it is[10]

$$D_{\mu\nu}(x) = \left[-g_{\mu\nu} - \frac{1}{m^2} \nabla_\mu \nabla_\nu \right] D_0(x) \tag{II.16}$$

where $D_0(x)$ is the free propagator in x space for a scalar particle:

$$D_0(x) = \int_p e^{-ip\cdot x} \, i(p^2 - m^2 + i\epsilon)^{-1} \tag{II.17}$$

For small x

$$D_0(x) = -(4\pi^2)^{-1} (x^2 - i\epsilon)^{-1} \tag{II.18}$$

The most singular term in $D_{\mu\nu}(x)$ for small x is

$$D_{\mu\nu}(x) \simeq (2\pi^2 m^2)^{-1} \left\{ -g_{\mu\nu} x^2 + 4x_\mu x_\nu \right\} (x^2 - i\epsilon)^{-3} \tag{II.19}$$

Without affecting the limit (II.15) the regions R_1 and R_2 can be redefined to lie within the region $|x_0| < t_0$, $|\underset{\sim}{x}| < r_0$ where t_0 and r_0 are small but held fixed as $\eta \to 0$. Within this region both $D_{\mu\nu}(x)$ and $e^{ip\cdot x}$ can be approximated by small x expansions; as will be shown later only the leading terms from these expansions contribute to the limit (II.15). Only the leading terms will be discussed explicitly. Also for simplicity only the 00 component of $D_{\mu\nu}(0)$ will be discussed. Approximating $D_{00}(0)$ by Eq. (II.19) gives

$$\Delta = D_{00}(0,v) - D_{00}(0) = \lim_{\eta \to 0} \left\{ \int_{R_1} - \int_{R_2} \right\} d^4x \, (2\pi^2 m^2)^{-1} (-x^2 + 4x_0^2)$$

$$\times \left(x_0^2 - \underset{\sim}{x}^2 - i\epsilon \right)^{-3} \tag{II.20}$$

The regions R_1 and R_2 are now

R_1: $|x_0| < \eta$, $|x_0 - vx_3| > \eta$, $|x_0| < t_0$, and $|\underset{\sim}{x}| < r_0$

R_2: $|x_0| > \eta$, $|x_0 - vx_3| < \eta$, $|x_0| < t_0$, and $|\underset{\sim}{x}| < r_0$

The $\underset{\sim}{x}$ integrations can be done explicitly; it is easily seen that terms depending on r_0 will not contribute in the limit $\eta \to 0$. With such terms dropped, the integrals have the explicit form

$$\Delta = -(2\pi m^2)^{-1} \int_0^\eta dx_0 \left\{ \frac{1}{|r_a| + x_0} + \frac{1}{|r_a| - x_0 + i\epsilon} + \frac{1}{r_b + x_0} + \frac{1}{r_b - x_0 + i\epsilon} \right\}$$

$$+ (2\pi m^2)^{-1} \int_\eta^{t_0} dx_0 \left\{ \frac{1}{r_a + x_0} + \frac{1}{r_a - x_0 + i\epsilon} - \frac{1}{r_b + x_0} - \frac{1}{r_b - x_0 + i\epsilon} \right\} \tag{II.21}$$

where

$$r_a = (x_0 - \eta)/v \tag{II.22}$$

$$r_b = (x_0 + \eta)/v \tag{II.23}$$

(the symmetry for $x_0 \to -x_0$ of Eq. (II.20) was used to eliminate integrals with $x_0 < 0$). The x_0 integrals can also be done explicitly; the result is independent of t_0 when η is small and gives

$$\Delta = (-i/m^2)\, v^2(1-v^2)^{-1} \tag{II.24}$$

which agrees with Eq. (II.13).

The reason one can generalize the above calculation easily is that its qualitative features can all be determined by scaling arguments. The terms in Δ which stay finite for $\eta \to 0$ are unaffected by t_0 and r_c, and in the leading approximation $D_{\mu\nu}(x)$ depends only on x not on m^2 except as an overall factor. Hence η becomes the only dimensional parameter in the integrals. So to get qualitatively the dependence of the integrals on η one can replace x_0 and $\underset{\sim}{x}$ by the dimensionless variables $y_0 = x_0/\eta$, $\underset{\sim}{y} = \underset{\sim}{x}/\eta$, and collect factors of η. When y_0 and $\underset{\sim}{y}$ are of order 1 the limits defining R_1 and R_2 do not depend on η. So in Eq. (II.20) the substitution gives

$$\Delta = \left\{\int_{R_1} - \int_{R_2}\right\} \eta^4 d^4y\, (2\pi^2 m^2)^{-1} \eta^2\left(-y^2 + 4y_0^2\right) \eta^{-6}\left(y_0^2 - \underset{\sim}{y}^2 - i\epsilon\right)^{-3}$$

$$= \left\{\int_{R_1} - \int_{R_2}\right\} d^4y\, (2\pi^2 m^2)^{-1} \left(-y^2 + 4y_0^2\right)\left(y_0^2 - \underset{\sim}{y}^2 - i\epsilon\right)^{-3} \tag{II.25}$$

which is independent of η; the regions R_1 and R_2 are[11]

$$R_1:\ |y_0| < 1,\quad |y_0 - vy_3| > 1$$

$$R_2:\ |y_0| > 1,\quad |y_0 - vy_3| < 1$$

Thus from a scaling argument one sees that Δ will be a constant for $\eta \to 0$ (however only an explicit calculation can show that the constant does not vanish). One might worry about the effects of the light cone singularity ($y_0 = |\underset{\sim}{y}|$ but $y_0 \neq 0$) on the scaling analysis, but one can see by tracing through the detailed calculation that the $i\epsilon$ in $x^2 - i\epsilon$ makes the light cone singularity integrable and does not destroy the scaling arguments (provided one does not choose t_0 and r_0 so that $t_0^2 - r_0^2 = 0$!)

The importance of the scaling argument is that if one had extra powers of $\underset{\sim}{x}$ or x_0 in the numerator of Eq. (II.20), the scaling argument shows that Δ would vanish. This can be verified by explicit calculation. This means that Δ does not change if one puts $e^{ip\cdot x}$ in the integral, since the terms $p\cdot x$, $(p\cdot x)^2$, etc., in the expansion of $e^{ip\cdot x}$ do not contribute in the limit $\eta \to 0$. Likewise less singular terms in $D_{\mu\nu}(x)$ do not contribute to the limit. Hence the explicit calculation gives the more general result

$$D_{00}(p, v) - D_{00}(p) = (-i/m^2)\, v^2/(1-v^2) \tag{II.26}$$

in agreement with Eq. (II.13).

Even more generally one deduces the following general rule. Let $TO_1(x)\, O_2(0)$ be a T product of two arbitrary local operators $O_1(x)$ and $O_2(0)$. It does not matter whether these operators are scalars, spinors, tensors, or whatever. Let

$$M(p) = \int_x e^{ip\cdot x} \langle A | TO_1(x)\, O_2(0) | B \rangle \tag{II.27}$$

be the Fourier transform of an arbitrary matrix element of the T product. If the matrix element itself scales as x^{-4+d} as $x \to 0$, with $d > 0$, then $M(p)$ is covariant and independent of the order of integration. The hypothesis of operator product expansions[1] predicts that no matter what matrix element is considered leading short distance behavior of the matrix element will be a function of x only except for an overall factor (as was the case for $D_{\mu\nu}(x)$), so that the scaling analysis applies.

The conventional integral for $D_{\mu\nu}(p)$ can be divergent. The current of a free Dirac field gives a simple example of this. The divergence is simply the well-known divergence in the lowest order vacuum polarization diagram for electrodynamics. However, we are not calculating vacuum polarization here, so the divergences cannot be removed by a renormalization. The calculation here is of the Fourier transform of the propagator of the current; to remove these divergences, the Fourier transform integral must be subtracted. As usual with subtractions, there is some arbitrariness in the exact form of the subtracted integral. The calculation will be described briefly. The current $j_\mu(x)$ is

$$j_\mu(x) = :\overline{\psi}(x)\,\gamma_\mu\,\psi(x): \tag{II.28}$$

where ψ is a free Dirac field and $:\ldots:$ denotes Wick ordering. The propagator $D_{\mu\nu}(x)$ is now

$$D_{\mu\nu}(x) = -\mathrm{Tr}\,\gamma_\mu\,S_0(x)\,\gamma_\nu\,S_0(-x) \tag{II.29}$$

where

$$\begin{aligned} S_0(x) &= i\int_p e^{-ip\cdot x}\left(\gamma^\mu p_\mu + m\right)(p^2-m^2+i\epsilon)^{-1} \\ &= \left(i\gamma^\mu \nabla_\mu + m\right) D_0(x) \end{aligned} \tag{II.30}$$

When x is small the most singular term in $S_0(x)$ is

$$S_0(x) \approx i(2\pi^2)^{-1}\gamma^\mu x_\mu (x^2-i\epsilon)^{-2} \tag{II.31}$$

As a result

$$D_{\mu\nu}(x) \simeq \pi^{-4} \left(g_{\mu\nu} x^2 - 2x_\mu x_\nu\right) (x^2 - i\epsilon)^{-4} \tag{II.32}$$

for small x. The integral $\int D_{\mu\nu}(x)\, e^{ip\cdot x}\, d^3\underset{\sim}{x}$ diverges as $x_0 \to 0$; from a scaling argument the divergence should be proportional to x_0^{-3}. The divergence can come only from $|\underset{\sim}{x}| \sim x_0$ in the integral so it is legitimate to use the approximation (II.32) in doing the calculation of the divergence. The integral can now be done explicitly and gives

$$\int d^3\underset{\sim}{x}\, e^{ip\cdot x} D_{\mu\nu}(x) \simeq (i/6\pi^2)\, |x_0|^{-3} \left\{-g_{\mu\nu} + \delta_{\mu 0}\delta_{\nu 0}\right\} \tag{II.33}$$

There can also be terms of order $|x_0|^{-2}$, $|x_0|^{-1}$, etc. So computing the integral of Eq. (II.12) gives a divergent result. The way to avoid this divergence is to subtract the integral so that the scaling argument predicts convergence. The simplest subtraction is to subtract a Taylor's series expansion of $e^{ip\cdot x}$: one defines[12]

$$D_{\mu\nu}(p) = \int_p \left\{e^{ip\cdot x} - 1 - ip\cdot x + \frac{1}{2}(p\cdot x)^2\right\} D_{\mu\nu}(x) \tag{II.34}$$

The leading singularity of the integrand now scales as x^{-3} instead of x^{-6}. As a result the scaling arguments show that $D_{\mu\nu}(p)$ is finite and covariant. The terms subtracted are a quadratic polynomial in p. In effect one has subtracted infinite constants multiplying p^2, p, and 1 from the old form of $D_{\mu\nu}(p)$. As usual, one is always free to add finite constants times p^2, p, or 1 to $D_{\mu\nu}(p)$; to keep $D_{\mu\nu}(p)$ covariant the added terms must also be covariant.

Even for cases like the free vector meson propagator where the unsubtracted integral is finite, one is free to use a subtracted integral to define $D_{\mu\nu}(p)$. One can make as many subtractions as one likes, but one subtraction is sufficient to define a covariant form for $D_{\mu\nu}(p)$.

Axiomatic field theorists have shouted since prehistoric times that the Fourier transforms of T products are ambiguous. There is an excellent discussion of the role of these ambiguities in renormalization theory in Bogoliubov and Shirkov.[5] Nevertheless the popular view is that a Fourier transform such as $D_{\mu\nu}(p)$ is a unique and even physical quantity at least relative to a given Lorentz frame. The axiomatic view must in the end replace the popular view, since the ambiguity in $D_{\mu\nu}(p)$ in examples like the Dirac current of a free fermion field is beyond question. Unfortunately, much experience has been acquired with the unsubtracted form of the definition of $D_{\mu\nu}(p)$ and more general transforms like M(p) in Eq. (II.27). One must now distinguish two problems. The first is, given that the standard definition of the Fourier transform exists, to show in practical situations that no physics is changed by using a subtracted formula instead. This may not be trivial to demonstrate but is not a very rewarding subject to pursue. The second question is what happens to the physics when subtractions are necessary. There is already one example known where the necessity for a subtraction changes a current algebra prediction, namely the Adler-Bell-Jackiw-Schwinger anomaly which changes the current algebra prediction of the π^0 lifetime.[13] One must be prepared to find other applications where subtractions have nontrivial effects. It is certainly worth looking for such effects, especially when the use of conventional Ward identities gives unsatisfactory results, as in η decay.[14]

It may help in understanding the problem of the ambiguity in $D_{\mu\nu}(p)$ if one can understand why it was possible for nonaxiomaticists to conclude that $D_{\mu\nu}(p)$ is unique. The reason lies, I believe, in a conscious or unconscious assumption that nonaxiomaticists make about the nature of field theory. The assumption is this: any local operator, such as a current, becomes an observable when averaged over a region of space, the time being held fixed. By an "observable", I mean an operator which can be multiplied by itself or by other fields, without producing singularities. The best way to show that this assumption is made is to look at the popularly accepted form for an equal time commutator. The equal time commutator of two local fields $O_1(x)$ and $O_2(y)$ is expected to be a sum of δ-functions and derivatives of δ-functions in the spatial variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$. These δ-functions can be eliminated by averaging $O_1(x)$ say over a region of space; if $\rho(\underset{\sim}{x})$ is an averaging function then $\left[\int \rho(\underset{\sim}{x})\, O_1(x_0, \underset{\sim}{x})\, d^3\underset{\sim}{x},\ O_2(x_0, \underset{\sim}{y})\right]$ is completely free of singularities. Even more, one assumes that the unequal time commutator $\left[\int \rho(\underset{\sim}{x})\, O_1(x_0, \underset{\sim}{x})\, d^3\underset{\sim}{x},\ O_2(y_0, \underset{\sim}{y})\right]$ is continuous and differentiable in y_0 for $y_0 = x_0$. This assumption is implicit in the equal time commutator formula

$$\left[O_1(x_0, \underset{\sim}{x}),\ \dot{O}_2(x_0, \underset{\sim}{y})\right] = i\left[O_1(x_0, \underset{\sim}{x}),\ \left[H,\ O_2(x_0, \underset{\sim}{y})\right]\right] \qquad \text{(II.35)}$$

where H is the Hamiltonian and the double commutator is again expected to be a sum of δ-functions. If the unequal time commutator were not differentiable in y_0 at $y_0 = x_0$ then the equal time commutator with $\dot{O}_2$ would diverge.

Given the assumption that integration with $\underset{\sim}{x}$ makes operator products be smooth in time, it is easy to derive the usual form of the Ward identity for $D_{\mu\nu}(p)$ from the definition (II.12). One writes

$$p^\mu D_{\mu\nu}(p) = \mathrm{Lim}\,(\eta \to 0) \int_x \left(-i\nabla^\mu e^{ip\cdot x}\right) D_{\mu\nu}(x)\ \theta\left[|x_0| - \eta\right] \qquad \text{(II.36)}$$

Integrating by parts, one gets

$$p^\mu D_{\mu\nu}(p) = \lim\,(\eta \to 0) \int_x e^{ip\cdot x}\left[i\nabla^\mu D_{\mu\nu}(x)\right]\ \theta\left[|x_0| - \eta\right]$$

$$+\, i \int_x e^{ip\cdot x} D_{0\nu}(x)\left[\delta(x_0 - \eta) - \delta(x_0 + \eta)\right] \qquad \text{(II.37)}$$

Since j_μ is assumed to be conserved, $\nabla^\mu j_\mu(x)$ is zero, and since x_0 is never zero in the integral, $\nabla^\mu \langle \Omega | T\, j_\mu(x)\, j_\nu(0) | \Omega \rangle = \langle \Omega | T\, \nabla^\mu j_\mu(x)\, j_\nu(0) | \Omega \rangle = 0$. So the first term vanishes and one is left with the surface terms. These terms may be written as follows. Let

$$Q(\underset{\sim}{p}, x_0) = \int d^3\underset{\sim}{x}\, e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, j_0(x_0, \underset{\sim}{x}) \tag{II.38}$$

Then

$$p^\mu D_{\mu\nu}(p) = \mathrm{Lim}\,(\eta \to 0)\; i \,\langle \Omega | \left\{ e^{ip_0\eta}\, Q(\underset{\sim}{p}, \eta)\, j_\nu(0) - e^{-ip_0\eta}\, j_\nu(0)\, Q(\underset{\sim}{p}, -\eta) \right\} | \Omega \rangle \tag{II.39}$$

According to the assumption stated above, the products $Q(\underset{\sim}{p}, \eta)\, j_\nu(0)$ and $j_\nu(0)\, Q(\underset{\sim}{p}, -\eta)$ should be free of any singularity for $\eta \to 0$, in which case the limit gives

$$p^\mu D_{\mu\nu}(p) = i \,\langle \Omega | \left[Q(\underset{\sim}{p}, 0),\; j_\nu(0) \right] | \Omega \rangle \tag{II.40}$$

which is the usual Ward identity relating $p^\mu D_{\mu\nu}(p)$ to an equal time commutator. If the assumption that $Q(p, \eta)$ is an observable breaks down,[15] the limit (II.39) may not behave like a commutator, since the expression for finite η is not a commutator. An example of this occurs in Section III.

The assumption that integrating an operator over space only gives an observable is a basic tenet of canonical field theory, since one builds the Hamiltonian of a canonical theory out of space-averaged operators, and the Hamiltonian has to be an observable. The assumption has been rejected by axiomatic field theory from the beginning since the currents and other local products in free field theories violate this assumption (as is shown by the example of a divergent propagator discussed earlier). In axiomatic field theory one assumes only that operators averaged over space and time give observables; this hypothesis was formally stated by Wightman but the idea dates back to the discussion of measurability of fields by Bohr and Rosenfeld.[16] Unfortunately the assumption that space-time averages give observables is not very helpful in dealing with the specific problems posed by the singularities of T products.

Some general conclusions of this section are as follows:

1. The precise definition for the Fourier transform of a T product in common usage is exemplified by Eq. (II.12).

2. T products in x-space are covariant; any noncovariance in their Fourier transforms are entirely due to the noncovariant η-limit chosen to define the Fourier integral.

3. The definition (II.12) is capable of giving divergent results in which case a subtracted definition, as in Eq. (II.34), will have to be used instead.

4. If the integral of a T product is defined as in Eq. (II.12), then the equal time commutators appearing in Ward identities must be defined as a limit as in Eq. (II.39).

III. SCALE INVARIANCE AND PERTURBATION THEORY

To begin this section the commutators of the generator of scale transformations will be derived. Ward identities for the dilation current will then be written for matrix elements involving the fields ϕ and ϕ^4 of the $\lambda\phi^4$ theory. It will be assumed to start with that all integrals of T products are conventionally defined and all Ward identities have their customary form. The exceptions will be discussed later.

If the field theory is scale invariant,[17] then there exists a set of unitary transformations U(s) with the property

$$U^{+}(s)\ \phi(x)\ U(s) = s^{d}\ \phi(sx) \tag{III.1}$$

The constant d is called the dimension of ϕ. The unitary transformations U(s) can be written in terms of an infinitesimal generator D:

$$U(s) = e^{-i(\ln s)D} \tag{III.2}$$

The logarithm of s appears in the exponent so that U(s) will satisfy the composition law

$$U(s)\ U(s_1) = U(ss_1) \tag{III.3}$$

Let s be $1+\epsilon$ with ϵ small. Then from Eq. (III.1) one derives

$$i[D,\ \phi(x)] = \left(d + x^{\mu}\nabla_{\mu}\right)\ \phi(x) \tag{III.4}$$

For each composite field in the theory there will be a corresponding commutator. In particular

$$i\left[D,\ \phi^4(x)\right] = \left(d_I + x^{\mu}\nabla_{\mu}\right)\ \phi^4(x) \tag{III.5}$$

where d_I is the dimension of $\phi^4(x)$. The generator D is expected to be the integral of a local "dilation current" $D_{\mu}(x)$:

$$D = \int D_0(x)\ d^3\underset{\sim}{x} \tag{III.6}$$

The current D_μ must be conserved if scale invariance holds, in which case D is time independent.

Now consider the Ward identities. To allow for the breakdown of scale invariance, let D_μ have a divergence S:

$$\nabla^\mu D_\mu(x) = S(x) \tag{III.7}$$

and consider the matrix element

$$M(x_1 \ldots x_n) = \int_y \langle\Omega|T\,\phi(x_1)\ldots\phi(x_n)\,S(y)|\Omega\rangle \tag{III.8}$$

where $|\Omega\rangle$ is the vacuum state. Substituting $\nabla^\mu D_\mu$ for S and integrating by parts, the conventional calculation gives[18]

$$\begin{aligned} M(x_1 \ldots x_n) = \int_y \nabla_\mu \langle\Omega|T\,\phi(x_1)\ldots\phi(x_n)\,D^\mu(y)|\Omega\rangle \\ + i\langle\Omega|T\left\{(d+x_1\cdot\nabla_1)\,\phi(x_1)\right\}\phi(x_2)\ldots\phi(x_n)|\Omega\rangle \\ +\ldots+ i\langle\Omega|T\,\phi(x_1)\,\phi(x_2)\ldots\left\{(d+x_n\cdot\nabla_n)\,\phi(x_n)\right\}|\Omega\rangle \end{aligned} \tag{III.9}$$

The integral of the gradient vanishes and one is left with the commutators. It is convenient to bring the derivatives ∇_1, etc., outside the T product, which results in further equal time commutator terms. However, these further commutators cancel in pairs.[19] Consider the case n=2, for example. Then the result of moving the gradients is

$$\begin{aligned} M(x_1, x_2) = i\,(2d+x_1\cdot\nabla_1+x_2\cdot\nabla_2)\langle\Omega|T\,\phi(x_1)\,\phi(x_2)|\Omega\rangle \\ -i\,x_{10}\,\delta(x_{10}-x_{20})\langle\Omega|\left[\phi(x_1),\ \phi(x_2)\right]|\Omega\rangle \\ -i\,x_{20}\,\delta(x_{10}-x_{20})\langle\Omega|\left[\phi(x_2),\ \phi(x_1)\right]|\Omega\rangle \end{aligned} \tag{III.10}$$

The two commutator terms cancel. This is true for all n; so

$$M(x_1 \ldots x_n) = i\,(nd + x_1 \cdot \nabla_1 + \ldots + x_n \cdot \nabla_n)\, K(x_1 \ldots x_n) \qquad \text{(III.11)}$$

where

$$K(x_1 \ldots x_n) = \langle \Omega | T\, \phi(x_1) \ldots \phi(x_n) | \Omega \rangle \qquad \text{(III.12)}$$

The Ward identity (III.11) is the starting point of the analysis of this section. If scale invariance is exact, M must vanish. So we shall try to make the functions $M(x_1 \ldots x_n)$ vanish in perturbation theory. The dimension d will be treated as a fudge factor chosen to make M vanish if possible. This will be possible in order λ but not in order λ^2. Having found that the functions M cannot vanish in order λ^2, they will be calculated explicitly and used to infer the form of $\nabla^\mu D_\mu$.

Next some explicit perturbation formulae will be written out for vacuum expectation values involving $\phi(x)$ and $\phi^4(x)$. Only connected graphs will be considered (disconnected graphs will be discussed later). Let $K_c(x_1 \ldots x_n)$ be the connected part of $\langle \Omega | T\, \phi(x_1) \ldots \phi(x_n) | \Omega \rangle$ and let $W_c(x_1 \ldots x_n, y)$ be the connected part of the matrix element $\langle \Omega | T\, \phi(x_1) \ldots \phi(x_n) :\phi^4(y): | \Omega \rangle$. By $:\phi^4(y):$ is meant a Heisenberg field which reduces to the Wick product $:\phi^4(x):$ in the free field limit. In the interaction representation one defines (before renormalization)

$$W(x_1 \ldots x_n, y) = \langle \Omega | T\, \phi_I(x_1)\, \phi_I(x_2) \ldots \phi_I(x_n)\ :\phi_I^4(y):\ \exp\left\{ -i\lambda \int_z :\phi_I^4(z): \right\} | \Omega \rangle \qquad \text{(III.13)}$$

where $\phi_I(x)$ is the scalar field in the interaction representation. W_c is the connected part of W. The matrix elements K_c will be quoted to order λ^2, the matrix elements W_c to order λ only. The vacuum expectation value $W_c(y)$ will not be computed since it can be renormalized to zero by subtracting a c-number from the Heisenberg field $:\phi^4:$. Matrix elements involving products of two or more Heisenberg fields $:\phi^4:$ will not be discussed; hopefully the analysis of the W_c functions is sufficient to determine the properties of $:\phi^4:$. The nonzero, unrenormalized graphs for K_c and W_c (to order λ^2 and λ respectively) are:

$$K_c(x_1,\ldots,x_n) = \int_{p_1}\int_{p_2}\cdots\int_{p_{n-1}} e^{-p_1(x_1-x_n)}\cdots e^{-ip_{n-1}(x_{n-1}-x_n)} K_c(p_1,\ldots,p_{n-1}) \tag{III.14}$$

$$K_c(p) = D(p) = D_0(p) + 96\, i\lambda^2\, D_0^2(p)\, \Sigma(p^2,\Lambda^2) \tag{III.15}$$

where D(p) is the interacting meson propagator, $D_0(p)$ the free meson propagator with zero mass, and $\Sigma(p^2,\Lambda^2)$ is the Feynman graph shown in Fig. 1a computed with a cutoff Λ. Formulae are:

$$D_0(p) = i(p^2 + i\epsilon)^{-1} \tag{III.16}$$

$$\Sigma(p^2,\Lambda^2) = \int_q \rho(q^2,\Lambda^2)\, D_0(q-p) \tag{III.17}$$

$$\rho(q^2,\Lambda^2) = i\int_k D_0(k)\, D_0(q-k)\, D_0(k,\Lambda)\, D_0(q-k,\Lambda) \tag{III.18}$$

$$D_0(k,\Lambda) = \Lambda^2(\Lambda^2-k^2-i\epsilon)^{-1} \tag{III.19}$$

$\rho(q^2,\Lambda^2)$ is the Feynman graph shown in Fig. 1b, also with a cutoff. Calculation of ρ and Σ in the limit of large cutoff gives (see the Appendix)

$$\rho(q^2,\Lambda^2) = -(16\pi^2)^{-1}\, \ln\left[(-q^2-i\epsilon)/\Lambda^2\right] \tag{III.20}$$

$$\Sigma(q^2,\Lambda^2) = -(512\pi^4)^{-1}\, q^2\, \ln\left[(-q^2-i\epsilon)/\Lambda^2\right] + c\Lambda^2 + c_1 q^2 \tag{III.21}$$

where c and c_1 are numerical constants; terms of order q^2/Λ^2 or smaller for large Λ have been dropped. These formulae are relatively simple because the mass of ϕ is zero. Further formulae:

$$K_c(p_1, p_2, p_3) = -24\, i\lambda\, D_0(p_1)\, D_0(p_2)\, D_0(p_3)\, D_0(-p_1-p_2-p_3)$$
$$\times \left\{1 - 12\,\lambda\, \rho\left[(p_1+p_2)^2, \Lambda^2\right] - 12\lambda\, \rho\left[(p_1+p_3)^2, \Lambda^2\right] - 12\,\lambda\, \rho\left[(p_2+p_3)^2, \Lambda^2\right]\right\} \tag{III.22}$$

It is a nuisance to write out terms which differ only by a permutation of the momenta so in the following formulae only the number of such terms will be given:

$$K_c(p_1 \ldots p_5) = -576\,\lambda^2\, D_0(p_1) \ldots D_0(p_5)\, D_0(-p_1- \ldots -p_5) \left\{D_0(p_1+p_2+p_3) + 9 \text{ permutations}\right\} \tag{III.23}$$

$$W_c(x_1 \ldots x_n, y) = \int_{p_1} \ldots \int_{p_n} e^{-ip_1\cdot(x_1-y)} \ldots e^{-ip_n\cdot(x_n-y)}\, W_c(p_1 \ldots p_n) \tag{III.24}$$

$$W_c(p_1, p_2) = -96\,\lambda\, D_0(p_1)\, D_0(p_2) \left\{\Sigma\left(p_1^2, \Lambda^2\right) + \Sigma\left(p_2^2, \Lambda^2\right)\right\} \tag{III.25}$$

$$W_c(p_1, p_2, p_3, p_4) = 24\, D_0(p_1)\, D_0(p_2)\, D_0(p_3)\, D_0(p_4) \left\{1 - 12\,\lambda\,\rho\left[(p_1+p_2)^2, \Lambda^2\right] + 5 \text{ permutations of the } \lambda \text{ term}\right\} \tag{III.26}$$

$$W_c(p_1, \ldots, p_6) = -576\, i\,\lambda\, D_0(p_1) \ldots D_0(p_6) \left\{D_0(p_1+p_2+p_3) + 19 \text{ permutations}\right\} \tag{III.27}$$

The renormalized formulae for K_c and W_c are obtained by modifying Σ and ρ and redefining the coupling constant but otherwise using the formulae given above. The renormalized Σ is obtained by dropping the constants c and c_1 and replacing Λ^2 by an arbitrarily chosen but fixed "reference momentum" κ^2. Likewise the renormalized ρ is obtained by replacing Λ^2 by κ^2. The renormalized functions Σ_R and ρ_R are

$$\Sigma_R(q^2) = -\left(512\,\pi^4\right)^{-1} q^2\, \ln\left[(-q^2 - i\epsilon)/\kappa^2\right] \tag{III.28}$$

and

$$\rho_R(q^2) = -(16\pi^2)^{-1}\ \ln\left[(-q^2-i\epsilon)/\kappa^2\right] \tag{III.29}$$

The rationalization of these modifications is as follows.

The function Σ occurs in two different formulae; the modifications have a different significance in the two cases. This is also true of the function ρ. When Σ is a correction to the propagator, the modifications amount to a mass and wave function renormalization. In particular, replacing c by zero ensures that the renormalized mass is zero through order λ^2; replacing c_1 by 0 and Λ^2 by κ^2 are both wave function renormalizations. It is necessary to introduce the arbitrary parameter κ (which has the dimensions of a mass) into the theory because there is no naturally occurring parameter with the dimensions of a mass to replace the cutoff inside the logarithm. The value of κ is unimportant since changing κ only changes the normalization of the field ϕ, which is arbitrary. Similarly, when ρ is a correction to $K_c(p, p_1, p_2)$ the modification of ρ is a coupling constant renormalization; when ρ is replaced by ρ_R one must also replace λ by a renormalized coupling constant λ_κ. The renormalized coupling constant depends on κ in the sense that if κ is changed to κ' one must change λ_κ to $\lambda_{\kappa'}$, with

$$\lambda_{\kappa'} = \lambda_\kappa + \left(9\lambda_\kappa^2/4\pi^2\right)\ \ln(\kappa'^2/\kappa^2) + \text{order } (\lambda_\kappa^3) \tag{III.30}$$

in order that $K_c(p_1, p_2, p_3)$ be independent of the choice of κ.[20]

When Σ is a first order contribution to $W_c(p_1, p_2)$ the modifications have a different interpretation. If the unrenormalized formula for $W_c(p_1, p_2)$ is Fourier transformed to x-space, one obtains (see the appendix):

$$W_c(x_1,x_2,y) = (3/16\pi^6)\,\lambda \left\{ D_0(x_2-y)\left[(x_1-y)^2 - i\epsilon\right]^{-2} + D_0(x_1-y)\left[(x_2-y)^2 - i\epsilon\right]^{-2} \right\}$$

$$- 192\,\lambda(c\Lambda^2)\, D_0(x_1-y)\, D_0(x_2-y)$$

$$- 96\,\lambda\, ic_1 \left\{ D_0(x_2-y)\,\delta^4(x_1-y) + D_0(x_1-y)\,\delta^4(x_2-y) \right\} \qquad \text{(III.31)}$$

where $D_0(x)$ is the Fourier transform of $D_0(p)$, and the first term is correct only for x_1-y and x_2-y nonzero. The term proportional to c can be rewritten $-96\lambda c\Lambda^2 \langle\Omega|T\,\phi_I(x_1)\,\phi_I(x_2)\,:\phi_I^2(y):|\Omega\rangle$: Replacing c by 0 is equivalent to subtracting $-96c\lambda\Lambda^2\,:\phi^2(x):$ from the unrenormalized operator $:\phi^4(x):$. This subtraction is one of two needed to define a finite renormalized form of the Heisenberg field $:\phi^4(x):$. The other subtraction needed to define the renormalized form of $:\phi^4(x):$ is a subtraction proportional to $\lambda_\kappa\ :\phi^4:$. This subtraction is generated when one replaces Λ by κ in the function ρ, ρ being considered as a correction to the function $W_c(p_1,p_2,p_3,p_4)$. Replacing c_1 by 0 in $W_c(p_1,p_2)$ is simply a redefinition of the Fourier transform of $W_c(x_1,x_2,y)$:

When $W_c(x_1,x_2,y)$ is Fourier transformed, the c_1 term in $W_c(x_1,x_2,y)$ will not contribute because by definition the points $x_1=y$ and $x_2=y$ are excluded from the region of integration (see Section II). However, the unsubtracted Fourier transform of $W_c(x_1,x_2,y)$ diverges because of the singularities $\left[(x_1-y)^2 - i\epsilon\right]^{-2}$ and $\left[(x_2-y)^2 - i\epsilon\right]^{-2}$ in the first term of Eq. (III.31).[21] This means the Fourier transform must be subtracted. The unsubtracted Fourier transform would be

$$W_c(p_1,p_2) = \int_{x_1}\int_{x_2} e^{ip_1\cdot x_1} e^{ip_2\cdot x_2}\, W_c(x_1,x_2,0) \qquad \text{(III.32)}$$

The singular term for $x_1 \longrightarrow 0$ in the integrand has the form

$$e^{ip_2 \cdot x_2} (3/16\pi^6) \lambda D_0(x_2) \left(x_1^2 - i\epsilon\right)^{-2} .$$

The singular term in x_1 is present for any x_2 so one cannot approximate the x_2 dependence of the singular term. One cannot subtract this term unchanged because it does not go to zero fast enough when $x_1 \longrightarrow \infty$. To avoid an infrared divergence one subtracts

$$e^{i\kappa \cdot x_1} e^{ip_2 \cdot x_2} (3/16\pi^6) \lambda D_0(x_2) \left(x_1^2 - i\epsilon\right)^{-2}$$

where κ_μ is any four-vector with magnitude $\kappa_\mu \kappa^\mu = -\kappa^2$. Putting in the factor $e^{i\kappa \cdot x_1}$ does not change the dependence of the subtraction on p_1 and p_2, so it is a legitimate modification. The renormalized, subtracted formula for $W_c(p_1, p_2)$ is

$$\begin{aligned} W_c(p_1,p_2) = \int_{x_1} \int_{x_2} \Big\{ & e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_2} W_c(x_1, x_2, 0) \\ & - (3/16\pi^6) \lambda_\kappa e^{i\kappa \cdot x_1} e^{ip_2 \cdot x_2} D_0(x_2) \left(x_1^2 - i\epsilon\right)^{-2} \\ & - (3/16\pi^6) \lambda_\kappa e^{i\kappa \cdot x_2} e^{ip_1 \cdot x_1} D_0(x_1) \left(x_2^2 - i\epsilon\right)^{-2} \Big\} \end{aligned} \qquad \text{(III.33)}$$

with λ replaced by λ_κ in $W_c(x_1, x_2, 0)$ (and the c and c_1 terms dropped). This formula reproduces the renormalized form of $W_c(p_1, p_2)$ (given by Eq. (III.25) with λ_κ replacing λ and Σ_R replacing Σ).

The subtractions in Eq. (III.33) depend on p_1 and p_2 in the form $\left\{D_0(p_2) + D_0(p_1)\right\}$; hence one is always free to change the formula for $W_c(p_1, p_2)$ by adding a finite constant times $\left\{D_0(p_1) + D_0(p_2)\right\}$. Changing Σ_R back towards Σ by replacing κ by Λ and adding the c_1 term is exactly a change in $W_c(p_1, p_2)$ of this type. Hence c_1 is a subtraction constant which one is free to set equal to zero.

Now study the matrix elements of the divergence of the dilation current, using the Ward identity (III.11). First note that

$$(nd + x_1 \cdot \nabla_1 + \ldots + x_n \cdot \nabla_n)\, e^{-ip_1 \cdot (x_1 - x_n)} \ldots e^{-ip_{n-1} \cdot (x_{n-1} - x_n)}$$
$$= \left(nd + p_1 \cdot \nabla_{p_1} + \ldots + p_{n-1} \cdot \nabla_{p_{n-1}}\right) e^{-ip_1 \cdot (x_1 - x_n)} \ldots e^{-ip_{n-1} \cdot (x_{n-1} - x_n)} \qquad \text{(III.34)}$$

Using Eqs. (III.11) and (III.14) and an integration by parts one gets

$$M(x_1 \ldots x_n) = \int_{p_1} \ldots \int_{p_{n-1}} e^{-ip_1 \cdot (x_1 - x_n)} \ldots e^{-ip_{n-1} \cdot (x_{n-1} - x_n)} M(p_1 \ldots p_{n-1}) \qquad \text{(III.35)}$$

with

$$M(p_1 \ldots p_{n-1}) = i\left(nd - 4(n-1) - p_1 \cdot \nabla_{p_1} - \ldots - p_{n-1} \cdot \nabla_{p_{n-1}}\right) K(p_1 \ldots p_{n-1}) \qquad \text{(III.36)}$$

The connected part of M is related to the connected part of K by the same equation. One can also define

$$V(x_1 \ldots x_n, y) = \int_z \langle \Omega | T\, \phi(x_1) \ldots \phi(x_n) : \phi^4(y) : S(z) | \Omega \rangle \qquad \text{(III.37)}$$

and obtain

$$V(x_1 \ldots x_n, y) = \int_{p_1} \ldots \int_{p_n} e^{-ip_1(x_1 - y)} \ldots e^{-ip_n \cdot (x_n - y)} V(p_1 \ldots p_n) \qquad \text{(III.38)}$$

with

$$V(p_1 \ldots p_n) = i\left(nd + d_I - 4n - p_1 \cdot \nabla_{p_1} - \ldots - p_n \cdot \nabla_{p_n}\right) W(p_1 \ldots p_n) \qquad \text{(III.39)}$$

It is straightforward to obtain explicit formulae for the connected parts of M to second order in λ_κ and the connected parts of V to first order in λ_κ. The dimensions d and d_I will be left as unknowns for the moment. For example

$$M_c(p) = i\,(2d - 4 - p\cdot\nabla_p)\, D(p) = i\,(2d - 4 - p\cdot\nabla_p) \times i\,(p^2 + i\epsilon)^{-1}$$

$$\times \left\{1 + \left(3\lambda_\kappa^2/16\,\pi^4\right) \ln\left[(-p^2 - i\epsilon)/\kappa^2\right]\right\} \qquad \text{(III.40)}$$

Separating the term where ∇_p acts on $(p^2+i\,\epsilon)^{-1}$ from the term where ∇_p acts on the logarithm, this becomes

$$M_c(p) = i(2d-2)\, D(p) - i\left(6\lambda_\kappa^2/16\pi^4\right) D_0(p) \qquad \text{(III.41)}$$

But to order λ_κ^2, one can replace $D_0(p)$ by $D(p)$ in the second term. The resulting formula for $M_c(p)$ and analogous formulae for other M_c and V_c functions are:

$$M_c(p) = i\left[2d - 2 - 3\,\lambda_\kappa^2/(8\pi^4)\right] D(p) \qquad \text{(III.42)}$$

$$M_c(p_1, p_2, p_3) = i\,(4d - 4 - 9\,\lambda_\kappa/2)\; K_c(p_1, p_2, p_3) \qquad \text{(III.43)}$$

$$M_c(p_1 \ldots p_5) = i\,(6d - 6)\; K_c(p_1 \ldots p_5) \qquad \text{(III.44)}$$

$$V_c(p_1, p_2) = i\,(2d + d_I - 6)\; W_c(p_1, p_2) + 3\,\lambda_\kappa\,(8\pi^4)^{-1}\left[D_0(p_1) + D_0(p_2)\right] \qquad \text{(III.45)}$$

$$V_c(p_1 \ldots p_4) = i\,(4d + d_I - 8 - 9\,\lambda_\kappa/\pi^2)\; W_c(p_1 \ldots p_4) \qquad \text{(III.46)}$$

$$V_c(p_1 \ldots p_6) = i\,(6d + d_I - 10)\; W_c(p_1 \ldots p_6) \qquad \text{(III.47)}$$

Equation (III.45) for $V_c(p_1, p_2)$ is incorrect because its derivation assumes that $W_c(p_1, p_2)$ is unsubtracted. The correct formula will be derived later.

The first application of Eqs. (III.42) - (III.47) is to show that scale invariance breaks down in order λ_κ^2. To determine the validity of scale invariance the equations for M_c will be discussed order by order (the equations for V_c will be discussed later). In the free field limit the only nonzero M_c is $M_c(p)$ and it too is zero if d=1. This agrees with the known result that the free field theory is scale invariant and ϕ has dimension 1. To first order in λ_κ, $M_c(p)$ and $M_c(p_1,p_2,p_3)$ do not trivially vanish, but by setting d=1 both are zero. So we infer that scale invariance holds to order λ_κ and d is 1 to this order. In order λ_κ^2 the situation is as follows. $M_c(p_1 \ldots p_5)$ vanishes because $K_c(p_1 \ldots p_5)$ is already of order λ_κ^2 and 6d-6 is zero to order 1. The function $M_c(p_1,p_2,p_3)$ cannot vanish: $K_c(p_1,p_2,p_3)$ is of order λ_κ and d is already determined to be 1 through order λ_κ so

$$M_c(p_1,p_2,p_3) = -i\,(9\lambda_\kappa/2)\,K_c(p_1,p_2,p_3) \tag{III.48}$$

The function $M_c(p)$ vanishes to order λ_κ^2 if d is

$$d = 1 + 3\lambda_\kappa^2/(16\,\pi^4) \tag{III.49}$$

The nonvanishing of $M_c(p_1,p_2,p_3)$ in order λ_κ^2 means S(x) is nonzero in order λ_κ^2, so scale invariance breaks down in this order. It does not help to change d in order to make $M_c(p_1,p_2,p_3)$ vanish in order λ_κ^2; this would require a change in d of order λ_κ which would make $M_c(p)$ nonzero in order λ_κ, which would be even worse. It will be assumed in what follows that d is given by Eq. (III.49).[22]

It appears that scale invariance is exact through order λ_κ. If so the quantities V_c must vanish to order λ_κ. Consider first $V_c(p_1,p_2,p_3,p_4)$. Since $W_c(p_1,p_2,p_3,p_4)$ is of order 1, V_c vanishes only if

$$4d + d_I = 8 + 9\lambda_\kappa/\pi^2 \tag{III.50}$$

Since d is already known, this gives

$$d_I = 4 + 9\,\lambda_\kappa/\pi^2 \tag{III.51}$$

So the dimension of $:\phi^4(x):$ changes in order λ_κ. To order λ_κ, $V_c(p_1 \ldots p_6)$ vanishes (note that $W_c(p_1 \ldots p_6)$ is itself of order λ_κ).

Before examining $V_c(p_1, p_2)$, the correct Ward identity for $V_c(p_1, p_2)$ must be obtained. To do so requires careful attention to the definition of Fourier transforms.[23] For $V_c(p_1, p_2)$ we shall use the standard definition ($V_c(x_1, x_2, y)$ will turn out to be zero so the standard definition exists). So

$$V_c(p_1, p_2) = \mathrm{Lim}\,(\eta \to 0) \int_{|x_{10}|>\eta} d^4x_1 \int_{|x_{20}|>\eta} d^4x_2\, e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_2} V_c(x_1, x_2, 0) \tag{III.52}$$

The region $|x_{10}-x_{20}| < \eta$ is also excluded from the integral. By analogy with Eq. (III.11)

$$V_c(x_1, x_2, 0) = i\,(2d + d_I + x_1 \cdot \nabla_1 + x_2 \cdot \nabla_2)\, W_c(x_1, x_2, 0) \tag{III.53}$$

When this is substituted in Eq. (III.52) one can integrate by parts giving

$$V_c(p_1, p_2) = \mathrm{Lim}\,(\eta \to 0)\; i\left(2d + d_I - 8 - p_1 \cdot \nabla_{p_1} - p_2 \cdot \nabla_{p_2}\right) \int_{x_1} \int_{x_2} e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_2} W_c(x_1, x_2, 0)$$

$$+ \mathrm{Lim}\,(\eta \to 0)\; E(\eta, p_1, p_2) \tag{III.54}$$

where the integral over x_1 and x_2 still excludes $|x_{10}| < \eta$, $|x_{20}| < \eta$, and $|x_{10}-x_{20}| < \eta$. The term $E(\eta, p_1 p_2)$ is the sum of surface terms. It turns out that the surface terms at $|x_{10}-x_{20}| = \eta$ are negligible but the surface terms at $x_{10} = \pm\,\eta$ or $x_{20} = \pm\,\eta$ have to be computed giving

$$E(\eta, p_1, p_2) = i \int_{x_1} \int_{x_2} \Big\{ -x_{10}\,\delta(x_{10}-\eta) + x_{10}\,\delta(x_{10}+\eta) - x_{20}\,\delta(x_{20}-\eta)$$

$$+ x_{20}\,\delta(x_{20}+\eta) \Big\}\, e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_2} W_c(x_1, x_2, 0) \tag{III.55}$$

with the regions $|x_{10}| < \eta$, etc., excluded still. Because of the δ-functions, the factors x_{10} and x_{20} are of order η, so only the singular part of $W_c(x_1, x_2, 0)$ is is important in the integral; for example the integrals with $x_{10} = \pm\eta$ come predominately from small x_1. Hence E is approximately

$$E(\eta, p_1, p_2) = -i\eta \int_{x_1}\int_{x_2} \left\{\delta(x_{10}-\eta) + \delta(x_{10}+\eta)\right\} e^{ip_2\cdot x_2} (3\lambda_\kappa/16\pi^6)\, D_0(x_2)(x_1^2-i\epsilon)^{-2}$$

$$-i\eta \int_{x_1}\int_{x_2} \left\{\delta(x_{20}-\eta) + \delta(x_{20}+\eta)\right\} e^{ip_1\cdot x_1} (3\lambda_\kappa/16\pi^6)\, D_0(x_1)(x_2^2-i\epsilon)^{-2} \tag{III.56}$$

These integrals can be performed explicitly giving

$$E(\eta, p_1, p_2) = -(3\lambda_\kappa/8\pi^4)\left\{D_0(p_1) + D_0(p_2)\right\} \tag{III.57}$$

To complete the construction of the Ward identity one must replace the unsubtracted Fourier transform of W_c in Eq. (III.54) by its subtracted form. The result is

$$V_c(p_1, p_2) = i\left(2d + d_I - 8 - p_1\cdot\nabla_{p_1} - p_2\cdot\nabla_{p_2}\right) W_c(p_1, p_2)$$

$$-(3\lambda_\kappa/8\pi^4)\left\{D_0(p_1) + D_0(p_2)\right\} + \mathrm{Lim}\,(\eta \to 0)\ F(\eta, p_1 p_2) \tag{III.58}$$

with

$$F(\eta, p_1, p_2) = i\left(2d + d_I - 8 - p_1\cdot\nabla_{p_1} - p_2\cdot\nabla_{p_2}\right)\int_{x_1}\int_{x_2} (3\lambda_\kappa/16\pi^6) \tag{III.59}$$

$$\times\left\{e^{i\kappa\cdot x_1} e^{ip_2\cdot x_2} D_0(x_2)\,(x_1^2-i\epsilon)^{-2} + e^{i\kappa\cdot x_2} e^{ip_1\cdot x_1} D_0(x_1)(x_2^2-i\epsilon)^{-2}\right\}$$

with $|x_{10}| < \eta$, etc., omitted from the integral. The integrals give η-dependent constants multiplying the functions $D_0(p_2)$ and $D_0(p_1)$. Using the values $d=1$, $d_I=4$ to lowest order, one finds that $F(\eta, p_1, p_2)=0$. Using these values for d and d_I in

Eq. (III.58), one has (correct through order λ_κ):

$$V_c(p_1,p_2) = i\left(-2 - p_1\cdot\nabla_{p_1} - p_2\cdot\nabla_{p_2}\right) W_c(p_1,p_2)$$
$$- (3\lambda_\kappa/8\pi^4)\left\{D_0(p_1) + D_0(p_2)\right\} \qquad \text{(III.60)}$$

This Ward identity has an extra term which does not appear in the conventional form [Eq. (III.39)]. It is not caused by the subtractions in $W_c(p_1,p_2)$. It came from the surface terms $E(\eta,p_1,p_2)$ arising when $x_1\cdot\nabla W_c(x_1,x_2,0)$ and $x_2\cdot\nabla W_c(x_1,x_2,0)$ were integrated by parts in the integral of Eq. (III.52). According to the conventional analysis given earlier (cf. Eq. (III.10)) these surface terms should have cancelled. They would have vanished had the assumption underlying the conventional analysis been correct. Namely if $\int d^3\underset{\sim}{x}_1\, W_c(x_1,x_2,0)$ were a smooth function of x_{10} at $x_{10}=0$ (and likewise for $\int d^3\underset{\sim}{x}_2\, W_c(x_1,x_2,0)$ at $x_{20}=0$) then the integral (III.55) for $E(\eta,p_1,p_2)$ would have been of order η. In practice the integral $\int d^3\underset{\sim}{x}_1\, W_c(x_1,x_2,0)$ is of order $|x_{10}|^{-1}$ for $x_{10}\to 0$ and cancels the explicit factor x_{10} in Eq. (III.55); hence $E(\eta,p_1,p_2)$ has a finite, nonzero limit for $\eta\to 0$.

Using the explicit renormalized formula for $W_c(p_1,p_2)$ to order λ_κ, one finds that Eq. (III.60) gives $V_c(p_1,p_2) = 0$. So all the functions V_c vanish to order λ_κ, as expected, and the field $:\phi^4(x):$ has a dimension d_I given by Eq. (III.51).

Since $M_c(p_1,p_2,p_3)$ does not vanish, the operator $S(x)$ (the divergence of $D_\mu(x)$) is nonzero. Can it be identified? It has been shown that all connected matrix elements of $S(x)$ vanish in order λ_κ^2 except for $M_c(p_1,p_2,p_3)$, and $M_c(p_1,p_2,p_3)$ is proportional to $K_c(p_1,p_2,p_3)$, or to be precise M_c in order λ_κ^2 is proportional to K_c in order λ_κ. Transforming to x space, and using the perturbation formula which defines K_c in order λ_κ, Eq. (III.48) becomes

$$M_c(x_1, x_2, x_3, x_4) = -(9\lambda_\kappa^2/2) \int_z \langle\Omega| T\, \phi_I(x_1)\, \phi_I(x_2)\, \phi_I(x_3)\, \phi_I(x_4) :\phi_I^4(z):|\Omega\rangle \quad \text{(III.61)}$$

A comparison of this formula with Eq. (III.8) suggests that

$$S(x) = -(9\lambda_\kappa^2/2) :\phi^4(x): \quad \text{(III.62)}$$

This hypothesis gives back Eq. (III.61) and also makes all other connected matrix elements M_c vanish to order λ_κ^2.

Can one understand how a term proportional to $:\phi^4(x):$ appears in the divergence of D? It will be shown that this is to be expected, given that the operator $:\phi^4(x):$ changes its dimension in order λ_κ. To simplify matters consider not S(x) but the integral

$$\int d^3\underset{\sim}{x}\, S(x) = dD/dx_0 \quad \text{(III.63)}$$

The operator D must contain an explicit time dependence proportional to $x_0 H$, where H is the Hamiltonian;[17] this is necessary to give the $x_0 \nabla_0 \phi(x)$ term in the commutator of D with ϕ. So let

$$D = x_0 H + D_A \quad \text{(III.64)}$$

The formula for dD/dx_0 is

$$\frac{dD}{dx_0} = \frac{\partial D}{\partial x_0} - i[D, H] = H - i\left[D_A, H\right] \quad \text{(III.65)}$$

The Hamiltonian contains the interaction term

$$H_I = \lambda_\kappa \int d^3\underset{\sim}{x} :\phi^4(x): \quad \text{(III.66)}$$

The contribution of H_I to dD/dx_0 is $H_I - i\left[D_A, H_I\right]$. The commutator of D_A with $:\phi^4(x):$ is

$$\left[D_A,\ :\phi^4(x):\right] = -i\,(d_I + \underset{\sim}{x}\cdot\underset{\sim}{\nabla})\ :\phi^4(x): \tag{III.67}$$

Integrating over $\underset{\sim}{x}$, and using an integration by parts on the gradient term

$$\left[D_A, H_I\right] = -i\,(d_I - 3)\ H_I \tag{III.68}$$

So the contribution of the interaction to dD/dx_0 is $-(d_I-4)\ H_I$, which is $-\lambda_\kappa\,(d_I-4)\int d^3\underset{\sim}{x}\,:\phi^4(x):$. Using Eq. (III.51), this is $(-9\lambda_\kappa^2/\pi^2)\int d^3\underset{\sim}{x}\,:\phi^4(x):$. According to Eq. (III.62) the total dD/dx_0 is half of this so there must also be a contribution to dD/dx_0 from the unperturbed part of the Hamiltonian. This analysis shows that a term of order $\lambda_\kappa^2\ :\phi^4(x):$ is to be expected in $\nabla^\mu D_\mu(x)$, given that $:\phi^4(x):$ changes its dimension in order λ_κ.

To conclude this section the various assumptions and undiscussed problems will be listed. The above discussion concerned only connected graphs but it can be shown that the conclusions are unchanged by the disconnected graphs (such as the products of two propagators in the four-point function). The matrix elements of two or more $:\phi^4(x):$ fields were not computed (thus avoiding the problems associated with the product $T:\phi^4(x)::\phi^4(y):$ when $x=y$). In deriving Ward identities the surface terms at time $\pm\infty$ were assumed to vanish; this should be checked by explicit calculation of the matrix elements of $D_\mu(x)$, since one is dealing with a zero mass theory. In second order in λ_κ, for which D_μ is not conserved, it was assumed that the equal time commutator of $D(x_0)$ with ϕ could still be computed from the matrix element $M_c(P)$ as if D were conserved; this will have to be checked by explicit calculation.[22] However, even if this assumption is incorrect it will not change the calculation of $M_c(p_1, p_2, p_3)$ to order λ_κ^2, since this calculation involves the commutator of D with ϕ only to order λ_κ. So whatever the commutator of D with ϕ is in order λ_κ^2, there will still be a λ_κ^2 $:\phi^4(x):$ term in S(x); there may be other terms also. The presence of the λ_κ^2 $:\phi^4(x):$ term in S(x) makes it likely that the equal time commutator of $D(x_0)$ with $\phi(x)$ will diverge in order λ_κ^3. This is because the integral (III.8) which defines $M(x_1, x_2)$ diverges in order λ_κ^3 if S(x) is λ_κ^2 $:\phi^4(x):$; this in turn is a consequence of the nonintegrable singularity of $W(x_1, x_2, y)$ for $y \to x_1$ or x_2 in order λ_κ.

Given that the interaction $:\phi^4(x):$ changes its dimension in order λ_κ why does not the free part of the Hamiltonian also change its dimension in order λ_κ? If this were to happen then scale invariance would break down in order λ_κ instead of λ_κ^2. This is another question that will not be discussed here.

The analysis of this section has been carried through for the zero mass $\lambda\phi^4$ theory. One may ask, why not work with the finite mass theory instead? The reason for not using the nonzero mass theory is that when the mass is nonzero the divergence S(x) contains a term proportional to $:\phi^2(x):$, which is nonzero in the free field limit. This means the matrix elements $M(x_1 \ldots x_n)$ will be nonzero in the free field limit. To show that S(x) contains a term proportional to λ_κ^2 $:\phi^4(x):$ in addition one must calculate matrix elements of $:\phi^2(x):$ to order λ_κ^2; one must also argue that terms proportional to $:\phi^4(x):$ are not permitted to occur as part of the renormalization of $:\phi^2(x):$. The argument cannot be rigorous, for if one is flexible enough about how one renormalizes there is no argument that forbids the use of finite $:\phi^4(x):$ counterterms in renormalizing $:\phi^2(x):$. Furthermore the zero mass case is discussed because it is only for the zero mass case that the canonical Lagrangian formulation of the $\lambda\phi^4$ theory predicts scale invariance, so it is only for the zero mass case that there is a contradiction between the prediction and perturbation theory calculations.

In the renormalization of $W_c(p_1, p_2)$ the constant c_1 was interpreted as a subtraction constant. It is possible to give the constant c_1 a different interpretation. If one defines the renormalized form of $:\phi^4(x):$ to include a subtraction proportional to $c_1 :\phi \nabla_\mu \nabla^\mu \phi:$, this will also eliminate the c_1 term from $W_c(p_1, p_2)$. This is because the matrix element

$$\int_{x_1} \int_{x_2} e^{ip_1 \cdot x_1} e^{ip_2 \cdot x_2} \langle \Omega | T \, \phi_I(x_1) \, \phi_I(x_2) \, :\phi_I(0) \, \nabla_\mu \nabla^\mu \, \phi_I(0): | \Omega \rangle \qquad \text{(III. 69)}$$

computed by Feynman rules, is $\left\{-p_1^2 - p_2^2\right\} D_0(p_1) \, D_0(p_2)$. This is proportional to $\left\{D_0(p_1) + D_0(p_2)\right\}$, which is exactly the form of the c_1 term in Eq. (III. 25) (using Eq. (III. 21) for Σ). This procedure for eliminating the c_1 term is more conventional than to interpret c_1 as a subtraction in a Fourier integral. Unfortunately the procedure is nonsensical. The field $:\phi_I \nabla_\mu \nabla^\mu \phi_I:$ vanishes because $\phi_I(x)$ satisfies the free field equation $\nabla_\mu \nabla^\mu \phi_I(x) = 0$. This means that $:\phi \nabla_\mu \nabla^\mu \phi:$ also vanishes in lowest order so subtracting it from $:\phi^4(x):$ does not change $:\phi^4(x):$ in order λ_κ. Furthermore the integral in Eq. (III. 69) should vanish since integrand vanishes. However the Feynman rules give a nonzero result for this integral. There is nothing wrong with this; the term given by the Feynman rules is a term which in x space involves δ-functions of x_1 or x_2, which one is always allowed to add to a T product, even if one of the operators in the T product vanishes. While there is nothing wrong with adding δ-functions to the T product, it is not a sensible thing to do. In any case c_1 is a subtraction constant in a Fourier integral. It does not matter whether it is recognized as such or snuck in by the device of subtracting $:\phi \nabla_\mu \nabla^\mu \phi:$ from $:\phi^4(x):$ and using the Feynman rules to introduce a subtraction in the definition of integrals of T products involving $:\phi \nabla_\mu \nabla^\mu \phi:$.

IV. MISCELLANY

In the previous section, it was necessary to know the behavior of the matrix element $\langle\Omega|T\,\phi(x_1)\;\phi(x_2)\;:\phi^4(y):|\Omega\rangle$ for $x_1 \to y$ or $x_2 \to y$. This behavior was determined by explicit calculation. This is a problem which can be understood in general in terms of operator product expansions.[24] In this section the operator product expansion for $T\,\phi(x)\;:\phi^4(y):$ will be discussed through order λ_κ using the matrix element $W(x_1, x, y)$ of $:\phi^4(y):$. At the end of this section the dimension of the field $:\phi^2(x):$ will be calculated through order λ_κ for the case of an isospin 1 field ϕ; it will be shown that the isospin 2 component of $:\phi^2:$ has a different dimension (in order λ_κ) than the isospin 0 component of $:\phi^2(x):$. A similar isospin splitting was postulated in a previous paper[1] to explain the $\Delta I=1/2$ rule in weak interactions.

In the free field theory the operator product expansion for the product $T\,\phi(x)\;:\phi^4(y):$ is derived from the Wick expansion of this product:

$$\begin{aligned} T\,\phi(x)\;:\phi^4(y): &= 4\,D_0(x-y)\;:\phi^3(y): + :\phi(x)\,\phi^4(y): \\ &= 4\,D_0(x-y)\;:\phi^3(y): + :\phi^5(y): + (x^\mu - y^\mu)\;:\phi^4(y)\,\nabla_\mu\,\phi(y): + \ldots \end{aligned} \tag{IV.1}$$

In the final form of this formula, functions of (x-y) multiply local operators at the point y; any such formula is called an *operator product expansion*. The expansion is an expansion in terms of x-y and makes sense when x-y is small. In perturbation theory one looks for a generalization of Eq. (IV.1) in the form

$$T\,\phi(x)\;:\phi^4(y): = \sum_n C_n(x-y)\;O_n(y) \tag{IV.2}$$

where the $C_n(x-y)$ are functions of x-y and $O_n(y)$ are local fields at y. The functions $C_n(x-y)$ may be singular as $x \to y$. The operators $O_n(y)$ are Heisenberg operators whose matrix elements will be functions of λ_κ; the functions $C_n(x-y)$ can also change with λ_κ. One can separate the two dependencies because only $C_n(x-y)$ can depend on x and because the same functions $C_n(x-y)$ must occur no matter which matrix element of $T\,\phi(x)\,:\phi^4(y):$ one studies. To first order in λ_κ perturbation theory is scale invariant, which restricts the behavior of the functions $C_n(x-y)$. As shown in a previous paper,[1] $C_n(x)$ must scale as

$$C_n(sx) = s^{\left[d_n - d - d_I\right]} C_n(x) \tag{IV.3}$$

where d_n is the dimension of the operator $O_n(x)$. If

$$d_n = d_{n0} + \lambda_\kappa\, d_{n1} \tag{IV.4}$$

and

$$C_n(x) = C_{n0}(x) + \lambda_\kappa\, C_{n1}(x) \tag{IV.5}$$

then the expansion of Eq. (IV.2) to order λ_κ gives

$$C_{n0}(sx) = s^{\left[d_{n0} - 5\right]} C_{n0}(x) \tag{IV.6}$$

$$C_{n1}(sx) = s^{\left[d_{n0} - 5\right]} \left\{ (d_{n1} - 9\lambda_\kappa/\pi)\, C_{n0}(x) \ln s + C_{n1}(x) \right\} \tag{IV.7}$$

(The dimensions d and d_I are taken from Eqs. (III.49) and (III.51).)

To learn something about the functions $C_n(x-y)$ and the operators $O_n(y)$ in order λ_κ, we study the matrix element $W(x_1, x, y)$ for x near y. The function $W(x_1, x, y)$ has no disconnected diagrams (given that the vacuum expectation value $\langle \Omega | :\phi^4(y): | \Omega \rangle$ is renormalized to zero) so $W(x_1, x, y) = W_c(x_1, x, y)$ which is given by the renormalized form of Eq. (III.31):

$$W(x_1, x, y) = (3/16\pi^6)\, \lambda_\kappa \left\{ D_0(x_1 - y) \left[(x-y)^2 - i\epsilon\right]^{-2} + D_0(x-y) \left[(x_1-y)^2 - i\epsilon\right]^{-2} \right\} \tag{IV.8}$$

In terms of z=x-y this is[25]

$$W(x_1, x, y) = (3/16\pi^6)\, \lambda_\kappa \left\{ (z^2 - i\epsilon)^{-2} D_0(x_1 - y) - (4\pi^2)^{-1} (z^2 - i\epsilon)^{-1} \left[(x_1 - y)^2 - i\epsilon \right]^{-2} \right\} \tag{IV.9}$$

There are only two terms when $W(x_1, x, y)$ is expanded in z. Comparing with the operator product expansion, one should have

$$W(x_1, x, y) = \sum_n C_n(z) \langle \Omega | T\, \phi(x_1)\, O_n(y) | \Omega \rangle \tag{IV.10}$$

From the scaling law (IV.6) the term proportional to $(z^2 - i\epsilon)^{-2}$ must involve an operator O_n of dimension $d_{n0}=1$ while the term proportional to $(z^2 - i\epsilon)^{-1}$ must involve an operator O_n of dimension 3. There is only one operator of dimension 1, namely ϕ itself. The coefficient $(z^2 - i\epsilon)^{-1}$ is a Lorentz scalar so it must involve a scalar field O_n. O_n must be odd in ϕ since $\phi :\phi^4:$ is odd. The only possibilities are $\nabla_\mu \nabla^\mu \phi(x)$ and $:\phi^3(x):$. These are not linearly independent because they are related by the field equation of the ϕ^4 theory; it is convenient to regard $\nabla_\mu \nabla^\mu \phi$ as the dependent field, so the only field left is $:\phi^3:$. So the expansion for $W(x_1, x, y)$ should be[26]

$$W(x_1, x, y) = C_1(z) \langle \Omega | T\, \phi(x_1)\, \phi(y) | \Omega \rangle + C_2(z) \langle \Omega | T\, \phi(x_1)\, :\phi^3(y): | \Omega \rangle \tag{IV.11}$$

The first matrix element is in lowest order the free propagator; comparing with Eq. (IV.9) gives

$$C_1(z) = (3/16\pi^6)\, \lambda_\kappa (z^2 - i\epsilon)^{-2} \tag{IV.12}$$

The matrix element $\langle \Omega | T\, \phi(x_1)\, :\phi^3(y): | \Omega \rangle$ vanishes in order 1 and has not been computed here to order λ_κ; the function $C_2(z)$ is known in order 1 from Eq. (IV.1) to be $4D_0(z)$. Comparison of Eqs. (IV.8) and (IV.11) gives

$$\langle\Omega|T\,\phi(x_1)\,:\phi^3(y):|\Omega\rangle = (3/64\pi^6)\,\lambda_\kappa\left[(x_1-y)^2-i\epsilon\right]^{-2} \tag{IV.13}$$

The most singular term in the operator product expansion of $T\,\phi(x)\,:\phi^4(y):$ is the term $C_1(x-y)\;\phi(y)$ because $\phi(y)$ is the field of lowest dimension in the expansion. It is this term that has caused all the troubles with subtractions and breakdown of conventional Ward identities in Section III. To order λ_κ this term does not affect the other connected functions $W_c(x_1, x_2, x_3, x, y)$, etc., because $C_1(z)$ is of order λ_κ and the connected parts of $\langle\Omega|T\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(y)\,|\Omega\rangle$, etc., vanish in order 1.

The analysis of the other connected matrix elements $W_c(x_1, x_2, x_3, x, y)$, etc., for small x-y is complicated and will not be given.

In a previous paper[1] it was postulated that there would be specific local fields of isospin 1/2 and 3/2 involved in nonleptonic weak interactions, and that these fields have different dimensions, the isospin 1/2 field being of lower dimension than the isospin 3/2 field. If this is true it was shown that the $\Delta I=1/2$ rule is universal, with all $\Delta I=3/2$ decays being suppressed by a power of (m/m_W) where m is a strong interaction mass ($\sim$ 1 BeV) and M_W is the weak boson mass or the equivalent. The assumption is not true of the free quark model. In the free quark model the relevant local fields are the isospin 1/2 and 3/2 parts of the Wick product $:j_{\mu\alpha}(x)\,j^+_{\nu\beta}(x):$ with $j_{\mu\alpha}(x)$ being the chiral SU(3) currents of the model; both $\Delta I=1/2$ and $\Delta I=3/2$ components of the Wick product have dimension 6. So it is worthwhile to consider how perturbation theory changes the dimensions of such a Wick product. To simplify the calculation a simple Wick product $:\phi_i(x)\;\phi_j(x):$ is discussed, where $\phi_i(x)$ (i=1, 2, or 3) are the components of an isospin 1 scalar field. The interaction Lagrangian density will be $-\lambda\left[\sum_i\phi_i^2(x)\right]^2$.

Consider the matrix element

$$N_{ijk\ell}(x,y,z) = \langle \Omega | T \ \phi_i(x) \ \phi_j(y) \ :\phi_k(z) \ \phi_\ell(z): | \Omega \rangle \tag{IV.14}$$

To order λ this matrix element (before renormalization) is given by

$$N_{ijk\ell}(x,y,z) = \int_p \int_q e^{-ip\cdot(x-z)} \ e^{-iq\cdot(y-z)} \ N_{ijk\ell}(p,q) \tag{IV.15}$$

$$\begin{aligned} N_{ijk\ell}(p,q) = D_0(p) \ D_0(q) \Big\{ & \delta_{ik} \ \delta_{j\ell} + \delta_{i\ell} \ \delta_{jk} \\ & + (\lambda/2\pi^2) \ (\delta_{ij} \ \delta_{k\ell} + \delta_{ik} \ \delta_{j\ell} + \delta_{i\ell} \ \delta_{jk}) \ \rho \left[(p+q)^2, \Lambda^2 \right] \Big\} \end{aligned} \tag{IV.16}$$

where ρ is defined by Eq. (III.18). The field $:\phi_i(x) \ \phi_j(x):$ has isospin 0 and isospin 2 components. The isospin 0 component is $\sum_i :\phi_i^2:$; the isospin two components can be written as the traceless tensor $:\phi_i\phi_j: - 1/3 \ \delta_{ij} \sum_k :\phi_k^2:$. There is a corresponding decomposition of $N_{ijk\ell}(p,q)$:

$$N_{ijk\ell}(p,q) = \delta_{ij} \delta_{k\ell} \ N_0(p,q) + (\delta_{ik}\delta_{j\ell} + \delta_{i\ell} \ \delta_{jk} - 2/3 \ \delta_{ij} \ \delta_{k\ell}) \ N_2(p,q) \tag{IV.17}$$

where N_0 is the isospin 0 component of $N_{ijk\ell}$ and N_2 the isospin 2 component. Using Eq. (IV.16) and using the renormalized form of ρ (Eq. (III.29)), one gets

$$N_0(p,q) = (2/3) \ D_0(p) \ D_0(q) \left\{ 1 + (5 \ \lambda_\kappa/4\pi^2) \ \ln \left[(-(p+q)^2 - i\epsilon)/\kappa^2 \right] \right\} \tag{IV.18}$$

$$N_2(p,q) = D_0(p) \ D_0(q) \left\{ 1 + (\lambda_\kappa/2\pi^2) \ \ln \left[(-(p+q)^2 - i\epsilon)/\kappa^2 \right] \right\} \tag{IV.19}$$

The renormalization is a wave function renormalization (with different renormalization constants for the isospin 0 and isospin 2 components of $:\phi_i\phi_j:$). Let d_0 and d_2 be the dimensions of the isospin 0 and isospin 2 components, respectively, of $:\phi_i\phi_j:$. The Ward identities which scale invariance imposes in N_0 and N_2

are

$$i(2d + d_0 - 8 - p\cdot\nabla_p - q\cdot\nabla_q)\; N_0(p,q) = 0 \tag{IV.20}$$

$$i(2d + d_2 - 8 - p\cdot\nabla_p - q\cdot\nabla_q)\; N_2(p,q) = 0 \tag{IV.21}$$

As in the case of the neutral field theory of Section III, d is 1 through order λ_κ. Explicit calculation using Eqs. (IV.18) and (IV.19) gives

$$d_0 = 2 + 2.5\,(\lambda_\kappa/\pi^2) \tag{IV.22}$$

$$d_2 = 2 + \lambda_\kappa/\pi^2 \tag{IV.23}$$

so in order λ_κ the dimensions d_0 and d_2 indeed differ.

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APPENDIX

In this appendix the calculation of $\rho(p^2, \Lambda^2)$ and $\Sigma(p^2, \Lambda^2)$ (Eqs. (III.17) and (III.18)) will be described briefly. Then the calculation of the Fourier transform of $W_c(p_1, p_2)$ (Eqs. (III.25) and (III.21)) will be discussed. The calculation of $\rho(p^2, \Lambda^2)$ is a standard Feynman diagram calculation. The answer for finite Λ can be obtained exactly in closed form, the result being

$$\rho(q^2, \Lambda^2) = (1/16\pi^2)\left\{(2 - 2\Lambda^2/q^2)\,\ln(1 - q^2/\Lambda^2) - \ln(-q^2/\Lambda^2) + (1 - 4\Lambda^2/q^2)^{1/2}\,\ln\left[\frac{(1 - 4\Lambda^2/q^2)^{1/2} - 1}{(1 - 4\Lambda^2/q^2)^{1/2} + 1}\right]\right\} \tag{A.1}$$

with q^2 being replaced by $q^2 + i\epsilon$ if necessary. For $q^2 \ll \Lambda^2$ this reduces to

$$\rho(q^2, \Lambda^2) \simeq -(1/16\pi^2)\,\ln(-q^2/\Lambda^2) \tag{A.2}$$

giving Eq. (III.20). For $q^2 \gg \Lambda^2$ ρ is proportional to $\Lambda^4 (q^2)^{-2}\,\ln(q^2/\Lambda^2)$. The formula for $\Sigma(p^2, \Lambda^2)$ is

$$\Sigma(p^2, \Lambda^2) = i\int_q \rho(q^2, \Lambda^2)\left[(q-p)^2 + i\epsilon\right]^{-1} \tag{A.3}$$

The function ρ drops off rapidly enough at large q^2 so that the integral for Σ converges (for finite Λ). The function Σ will be calculated first for spacelike p, and then determined for timelike p through analytic continuation. For spacelike p one can choose a Lorentz frame in which p_0 is 0. In this frame the integral over q_0 can be rotated from the real axis to the imaginary axis (counterclockwise). The result can be written in terms of Euclidean four vectors

$$\Sigma(-p^2, \Lambda^2) = \int_q \rho(-q^2, \Lambda^2)\left[(q-p)^2\right]^{-1} \tag{A.4}$$

where q is the four vector (q_1, q_2, q_3, q_4) (and similarly for p) and q^2 is $q_1^2 + q_2^2 + q_3^2 + q_4^2$ (and similarly for $q \cdot p$ and p^2). The integral over q can be performed in hyperspherical coordinates:

$$q_1 = q \cos \theta \tag{A.5}$$

$$q_2 = q \sin \theta \cos \phi \tag{A.6}$$

$$q_3 = q \sin \theta \sin \phi \cos \psi \tag{A.7}$$

$$q_4 = q \sin \theta \sin \phi \sin \psi \tag{A.8}$$

$$\int_q = (2\pi)^{-4} \int_0^\infty q^3\, dq \int_0^\pi \sin^2 \theta\, d\theta \int_0^\pi \sin \phi\, d\phi \int_0^{2\pi} d\psi \tag{A.9}$$

Performing the angular integrations gives

$$\Sigma(-p^2, \Lambda^2) = (8\pi^2 p^2)^{-1} \int_0^p q^3 \rho(-q^2, \Lambda^2)\, dq + (1/8\pi^2) \int_p^\infty q\, \rho(-q^2, \Lambda^2)\, dq \tag{A.10}$$

When p^2 is small compared to Λ^2, the integrals can be computed using the approximate form for ρ (Eq. (A.2)) except in a constant term (the second integral with p replaced by 0). The result is Eq. (III.21) with

$$c = (8\pi^2 \Lambda^2)^{-1} \int_0^\infty q\, \rho(-q^2, \Lambda^2)\, dq \tag{A.11}$$

and $c_1 = 3(1024\pi^4)^{-1}$. The constant c is independent of Λ because ρ depends only on the ratio (q^2/Λ^2).

In Fourier transforming $W_c(p_1, p_2)$ the only integral which is not already known is an integral of the form

$$u(x) = \int_p e^{-ip\cdot x}\, \ln\left[(-p^2 - i\epsilon)/\Lambda^2\right] \tag{A.12}$$

For x=0 this is highly divergent, but for $x \neq 0$ the exponent serves as a convergence factor. If one wishes to be careful one can insert an explicit convergence factor, say $\exp\left\{-|p_0|\eta - |p_1|\eta - |p_2|\eta - |p_3|\eta\right\}$ with $\eta > 0$, p_0 to p_3 being the components of p. Then one writes

$$\ln\left[(-p^2 - i\epsilon)/\Lambda^2\right] \doteq \int_0^\infty \omega^{-1}\left\{e^{-i\omega\Lambda^2} - e^{i\omega(p^2+i\epsilon)}\right\} d\omega \qquad (A.13)$$

After substituting this formula in Eq. (A.12) the p integration can be done explicitly, leaving

$$u(x) = (i/16\pi^2)\int_0^\infty \omega^{-3}\exp\left\{-ix^2/4\omega\right\} d\omega \qquad (A.14)$$

(if the convergence factor is inserted in Eq. (A.12) the result is to cutoff the integral (A.14) for $\omega < \eta^2$). One can change variables to $\nu = \omega^{-1}$ and then compute the integral giving

$$u(x) = (1/i\pi^2)\,(x^2 - i\epsilon)^{-2} \qquad (A.15)$$

The $i\epsilon$ is present because x^2 needs an imaginary part $-i\epsilon$ to ensure that the integral (A.14) converges.

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2. The other paper in the series is K. Wilson, Report No. SLAC-PUB-734, Stanford Linear Accelerator Center (1970).

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4. M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

5. N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, Inc., New York, 1959), Chapter VIII.

6. C. G. Callan, S. Coleman, and R. Jackiw, Report No. CTP 113, MIT preprint (December, 1969).

7. An excellent discussion of the ambiguity in T products is given in Ref. 5, pp. 144-145 and 168-191.

8. The "noncovariance" of the propagator of a free vector meson field is discussed in Ref. 5, pp. 141-142. For more general currents the problem is discussed by K. Johnson, Nucl. Phys. 25, 431 (1961). For more recent discussions of the "noncovariance" of T products, see R. F. Dashen and S. Y. Lee, Phys. Rev. 187, 2017 (1969), and references cited therein.

9. The metric of this paper is (1, -1, -1, -1).

10. Eqs. (II.16) and (II.18) can be derived from formulae in Appendix I of Ref. 5 (the equations at the top of p. 652 are incorrect by a minus sign and there are factors of i relating the propagators of this paper to those of Ref. 5).

11. The original limits $|x_0| < t_0$, $|\underset{\sim}{x}| < r_0$ become $|y_0| < t_0/\eta$, $|\underset{\sim}{y}| < r_0/\eta$. In the integrals of Eq. (II.25) these upper limits can be replaced by ∞ without

changing Δ, when η is small. In scaling analyses of more general problems (discussed after Eq. (II.26)) replacing t_0/η, r_0/η by ∞ may lead to divergent integrals. Then one must make a more sophisticated analysis, using the scaling argument only for values of $y \sim 1$ and computing explicitly the integral for y large, up to of order η^{-1}. However the large y region will only give terms of order η since this region is away from the singularity of $D_{\mu\nu}$; hence the scaling analysis will still determine whether or not Δ can be nonzero for $\eta \to 0$.

12. The subtraction $ip \cdot x$ might seem unnecessary since the integral of $xD_{\mu\nu}(x)$ should vanish by Lorentz invariance. Unfortunately one often has to use a noncovariant definition of the integral as in Eq. (II.12) in which case the integral of $xD_{\mu\nu}(x)$ might not vanish.

13. J. Schwinger, Phys. Rev. 82, 664 (1951);
J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969);
S. L. Adler, Phys. Rev. 177, 2426 (1969).
For a discussion explicitly in terms of divergences and subtractions in Fourier transforms of T products see K. Wilson, Phys. Rev. 181, 1909 (1969). For further references see R. W. Brown, C-C. Shih, and B. L. Young, Phys. Rev. 186, 1491 (1969).

14. See Ref. 1 for a possible resolution of the η-decay problem and further references. The explanation of η decay offered in Ref. 1 fails if all nine pseudoscalar fields are divergences of currents, as in the quark model. The reason is as follows: According to Ref. 1, the η-decay amplitude when the π_0 four momentum is zero is given by a matrix element $\langle \eta | [f\sigma_3(0), Q_{A3}] | \pi^+\pi^- \rangle$ where f is a coupling constant, σ_3 is the third component of the isovector σ field, and Q_{A3} is the third component of the

axial charge. Since the π_0 has zero four momentum, the full four momentum of the η is carried by the π^+ and π^-. Hence the commutator must not equal a divergence, for any divergence has zero matrix element between states of the same four momentum. But in conventional $SU(3) \times SU(3)$ the commutator is one of the pseudoscalar fields. One can arrange that the commutator is not a divergence by assuming that there are only eight axial currents instead of nine (this was done in Ref. 1) or by assuming that the field w introduced in Ref. 1 does not commute with the ninth axial charge. See S. Glashow in Hadrons and Their Interactions, A. Zichichi, Ed. (Academic Press, New York, 1968) and M. Gell-Mann's Hawaii Summer School lecture notes (Cal Tech preprint, 1970). This difficulty in explaining η decay was pointed out by G. Preparata (see R. Brandt and G. Preparata, to be published).

15. The operator $Q(0, x_0)$ is independent of x_0 because j_μ is conserved; therefore it automatically satisfies the smoothness assumption. But $Q(\underset{\sim}{p}, x_0)$ need not be smooth in x_0 for nonzero $\underset{\sim}{p}$. The problem of defining equal time commutators within the framework of axiomatic field theory is discussed in R. Schroer and P. Stichel, Commun. Math. Phys. 3, 258 (1966) and A. H. Volkel, University of Pittsburgh preprint N-Y-O-3829-36 (1969) and Free University of Berlin preprint.

16. See A. Wightman and L. Garding, Arkiv for Fysik 28, 129 (1965) especially pp. 131-133 and 153-154, and references cited therein.

17. For more detailed discussions of scale invariance, see: for free field theories, J. Wess, Nuovo Cimento 18, 1086 (1960); for interacting field theories (including Ward identities) G. Mack, Nucl. Phys. B5, 499 (1968). See also Refs. 1, 3, and 6, and D. Gross and J. Wess, CERN preprint. Some recent preprints are S. P. DeAlwis and P. J. O'Donnell, Toronto

University preprint (1970), L. N. Chang and P.G.O. Freund, California Institute of Technology preprint (1970), P. de Mottoni and H. Genz, II Institut fur Theoretische Physik der Universitat Hamburg preprint (1970), K. Wilson, Stanford Linear Accelerator Center Report No. SLAC-PUB-737, and M. Gell-Mann (Ref. 14).

18. Surface terms at time $y_0 = \pm\infty$ are neglected. In a zero mass theory this can be a mistake; it is assumed here that the neglect is legitimate.

19. D. Gross and J. Wess (Ref. 17).

20. For further discussion of the dependence of the coupling constant on the parameter κ, see Ref. 4.

21. These singularities cause a logarithmic divergence; this can be shown using the methods of Section II.

22. It was suggested by S. Coleman (private communication via R. Jackiw) that ϕ has a dimension in second order despite the breakdown of scale invariance. See the end of this Section for further discussion.

23. There are many aspects of the derivation of the Ward identity for $V_c(p_1, p_2)$ that should be examined carefully. In practice only one problem seems to cause difficulties, namely the singularity in the product $T\,\phi(x)\,:\phi^4(y):$ for $x \to y$ and only this problem will be discussed.

24. For background, see Refs. 1, 2, and references cited therein. Ideas completely analogous to operator product expansions and scale invariance have been developed independently for classical statistical mechanics by Kadanoff. See L. Kadanoff, Phys. Rev. Letters 23, 1430 (1969) and references cited therein.

25. The zero mass propagator $D_0(z)$ behaves as $(z^2)^{-1}$ for all z.

26. It seems a bit strange that other local fields such as $\nabla_\mu \nabla^\mu\, :\phi^3(y):$ do not occur in this expansion; presumably they will be involved in higher orders in λ_κ.

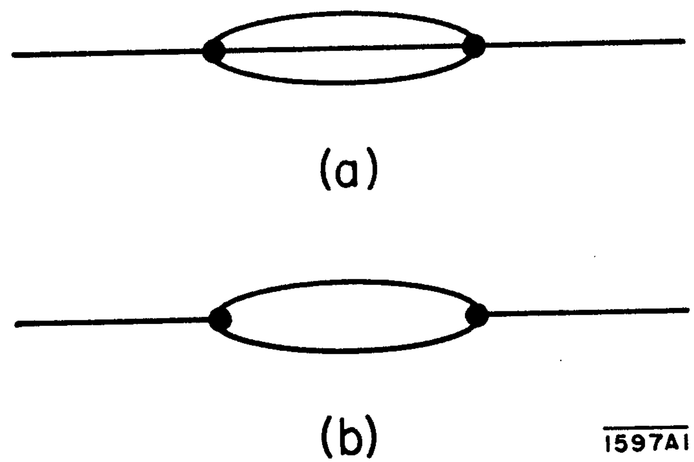

Fig. 1

(a) Feynman graph for self energy function Σ.

(b) Feynman graph for ρ.