BROKEN SCALE INVARIANCE AND ANOMALOUS DIMENSIONS*

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ABSTRACT

Mack and Kastrup have proposed that broken scale invariance is a symmetry of strong interactions. There is evidence from the Thirring model and perturbation theory that the dimensions of fields defined by scale transformations will be changed by the interaction from their canonical values. We review these ideas and their consequences for strong interactions.

Symmetry is of fundamental importance to our understanding of elementary particles. Lorentz invariance and isospin invariance are taken for granted. In the last decade we have learned from Gell-Mann the importance of broken symmetries: SU(2) × SU(2), SU(3), and SU(3) × SU(3).[1] It is natural to look for further exact or broken symmetries, especially in strong interactions where SU(2) × SU(2), etc., are important and the dynamics is not understood. The search for further symmetry includes study of possible further space-time symmetries, extending Lorentz invariance. The study of free field equations (Klein-Gordon equation, Maxwell's equations, etc.) has provided two suggestions for such symmetries: scale invariance and conformal invariance.[2] I will discuss only scale invariance here for I have not studied conformal invariance in detail. Scale invariance is an invariance of the free field equations only for zero mass. The zero mass equations such as $\nabla_\mu \nabla^\mu \phi(x) = 0$ contain no parameters with the dimensions of a length; this fact leads to scale invariance of the solution. Since there are no zero mass particles in strong interactions, it is not obvious how scale invariance would be relevant. But with the success of the broken symmetry SU(3) × SU(3) one has learned to derive useful physics from symmetries which are far from exact.

The hypothesis that scale invariance would be a broken symmetry of strong interactions was first clearly stated by G. Mack.[3] Mack was encouraged by H. Kastrup who had been writing about scale invariance in strong interactions for some time.[4] Since Mack's work, a number of people have become interested in the idea. However, to date the hypothesis has not been nearly so successful as broken SU(3) × SU(3). The problem has been to find experimental predictions resulting from broken scale invariance. At present the score is: one prediction, not yet tested; one explanation, which is untestable, and one clarification of

theoretical interest only. The prediction is that the total cross section for $e^+ - e^-$ annihilation into all possible hadron states will behave as $1/q^2$ for large q^2, where q is the four-momentum transfer to the hadrons.[5]

The explanation is an explanation of the $\Delta I = 1/2$ rule in nonleptonic weak decays ($K \rightarrow \pi\pi$, etc.).[6] The clarification is the idea that there be an $SU(3) \times SU(3)$ singlet field $w(x)$ in the Lagrangian which acts as a nucleon mass term and breaks scale invariance,[7] in addition to the $SU(3) \times SU(3)$ breaking terms[8] which give the π and K masses.

There is one extraordinary feature of scale invariance which makes it interesting regardless of the little contact it has with experiment. The extraordinary feature is the "anomalous dimension". When one makes a scale transformation on a field $\phi(x)$, it goes into $\phi(sx)$ times a scale factor s^d; s is the scale factor and d is called the dimension of the field. In quantum mechanics the transformation is accomplished by a unitary transformation $U(s)$:

$$U^+(s) \; \phi(x) \; U(s) = s^d \; \phi(sx) \tag{1}$$

The number d is a quantum number defining a representation of the group of scale transformations, just as the angular momenta j and m are quantum numbers for the rotation group. The unique feature of the dimension is that it can vary with a coupling constant. This is true only of scale invariance; the behavior of a field under Lorentz transformations, isospin, $SU(2) \times SU(2)$, etc. is unchanged by varying coupling constants. The reason d can change while other representations do not is that d is a continuous variable (i.e., any value of d is permitted by the scaling group) while the representations of other symmetries are described by discrete variables like j and m. This distinction does not mean that d must change as a coupling constant changes; the initial discovery of the changes in d was a complete surprise.

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Anomalous dimensions are found in the Thirring model (the Fermi interaction in one space and one time dimension)[9] and in ordinary perturbation theory.[10,11] In these examples they always arise together with an infinite wave function renormalization. The connection of anomalous dimensions to renormalization means that renormalization has a more fundamental significance than one might suspect. The anomalous dimensions are of practical significance: for example, they are crucial for the explanation of the $\Delta I = 1/2$ rule,[6] and may determine the leading corrections to the $e^+ e^-$ annihilation cross section at high q^2 (see below).

In the remainder of this talk, we shall first review some of the ideas of broken scale invariance. Then two examples of anomalous dimensions will be cited. Finally, the implications of anomalous dimensions will be sketched, for example, the nonexistence of a scale-invariant S matrix and an explanation of the $\Delta I = 1/2$ rule.

To illustrate the ideas of broken scale invariance, consider a simple example: a free scalar field $\phi(x)$.[2] To start with consider the zero mass limit which is exactly scale invariant. The free field theory can be defined by a field equation

$$\nabla_\mu \nabla^\mu \phi(x) = 0 \tag{2}$$

and an equal-time commutator

$$[\phi(\underset{\sim}{x},t), \ \overset{\circ}{\phi}(\underset{\sim}{y},t)] = + i\delta^3(\underset{\sim}{x} - \underset{\sim}{y}) \tag{3}$$

One can also specify the Hamiltonian

$$H = \frac{1}{2} \int d^3\underset{\sim}{x} \left\{ \overset{\circ}{\phi}{}^2(\underset{\sim}{x},t) + \underset{\sim}{\nabla}\phi^2(\underset{\sim}{x},t) \right\} \tag{4}$$

The easiest way to see that the theory might be scale invariant is to note that if $\phi(\underset{\sim}{x},t)$ satisfies the field equation, then so does $\phi(s\underset{\sim}{x},\ st)$ where s is a constant scale multiplying $\underset{\sim}{x}$ and t. (Similarly one motivates rotational invariance by noting

that $\phi(R\underset{\sim}{x},t)$ is a solution where R is any rotation matrix.) The field $\phi'(x,t) = \phi(s\underset{\sim}{x},st)$ however does not satisfy the commutation relations: one finds

$$\left[\phi'(\underset{\sim}{x},t), \; \overset{\circ}{\phi}'(\underset{\sim}{y},t)\right] = s\left[\phi(s\underset{\sim}{x},st), \; \overset{\circ}{\phi}(s\underset{\sim}{y},st)\right]$$

$$= s \; i\,\delta^3(s\underset{\sim}{x}-s\underset{\sim}{y}) = s^{-2} \; i\,\delta^3(\underset{\sim}{x}-\underset{\sim}{y}) \tag{5}$$

It is easy to restore the commutation relations: we redefine ϕ' to be

$$\phi'(x,t) = s\phi(sx,st) \tag{6}$$

The field equation and commutation relations uniquely define the quantum field up to a unitary transformation, so there must be a unitary operator U(s) satisfying

$$U^+(s) \; \phi(x,t) \; U(s) = \phi'(x,t) = s\phi(sx,st) \tag{7}$$

which is Eq. (1) with d=1.

I have avoided discussing the Hamiltonian, because it is not invariant to scale transformations. The reason is that H has dimensions so when lengths are scaled, H must be scaled also. In fact, if H' is the Hamiltonian for ϕ', then

$$H' = \frac{1}{2}\int d^3\underset{\sim}{x} \left\{ \overset{\circ}{\phi}'(\underset{\sim}{x},t)^2 + \left[\nabla\phi'(\underset{\sim}{x},t)\right]^2 \right\}$$

$$= \frac{1}{2}\int d^3\underset{\sim}{x} \left\{ s^4\overset{\circ}{\phi}^2(s\underset{\sim}{x},st) + s^4\,\nabla\phi^2(s\underset{\sim}{x},st) \right\} \tag{8}$$

By changing variables from $\underset{\sim}{x}$ to $s\underset{\sim}{x}$ one finds

$$H' = sH \tag{9}$$

This means that

$$U^+(s) \; HU(s) = H' = sH \tag{10}$$

A symmetry whose transformations do not leave H invariant is nothing new. The Lorentz transformations are even more destructive, transforming H into a linear combination of H and the momentum operator $\underset{\sim}{P}$. However, in both cases the

transformation law for H is fixed by the symmetry. In the case of scale transformations one can see that $U^+(s)\,HU(s)$ must be sH by considering the formula

$$i\,\frac{\partial\phi}{\partial t}(x,t) = [\phi(x,t),\, H] \tag{11}$$

Transforming this formula with $U^+(s)\ldots U(s)$ gives

$$s^2\, i\,\frac{\partial\phi}{\partial t}(sx, st) = [s\,\phi(sx, st),\, U^+(s)\,HU(s)] \tag{12}$$

For this to agree with the previous equation requires that $U^+(s)\,HU(s)$ be sH.

The vacuum is invariant to scale transformations:

$$U(s)|\Omega\rangle = |\Omega\rangle \tag{13}$$

The transformations of particle states will not be discussed here.

One can use scale invariance to determine scaling laws for vacuum expectation values. For example consider the propagator

$$D(x) = \langle\Omega|T\,\phi(x)\,\phi(0)|\Omega\rangle \tag{14}$$

For future use let d be unspecified in Eq. (1). Using Eqs. (1) and (13) and the unitarity of U, one can write

$$D(x) = \langle\Omega|U^+(s)\,T\,\phi(x)\,U(s)\,U^+(s)\,\phi(0)\,U(s)|\Omega\rangle$$

$$= s^{2d}\,\langle\Omega|T\,\phi(sx)\,\phi(0)|\Omega\rangle$$

$$= s^{2d}\,D(sx) \tag{15}$$

It follows that D(x) scales as $(x^2)^{-d}$. Another scaling calculation gives the behavior of the propagator D(p) in momentum space:

$$D(p) = \int e^{ip\cdot x}\,D(x)\,d^4x = \int e^{ip\cdot x}\,s^{2d}\,D(sx)\,d^4x \quad; \tag{16}$$

a change of variable to $y=sx$ in the integral gives

$$D(p) = s^{2d-4} \int e^{ip \cdot (y/s)} D(y) \, d^4y = s^{2d-4} D(s^{-1}p) \tag{17}$$

which means $D(p)$ scales as $(p^2)^{d-2}$. With $d=1$ this gives the usual zero mass form of the free propagator, namely $(p^2)^{-1}$. From these formulae it is easy to see why d is called a dimension. The propagator $D(x)$ behaves as $(x^2)^{-d}$ times a dimensionless constant. Hence dimensional analysis gives the dimensions of ϕ as $-d$ in units of length or d in units of mass ($\hbar$ and c are 1 as usual). The constant multiplying $(x^2)^{-d}$ cannot carry dimensions because there are no dimensional constants in the theory; the vacuum state is dimensionless because of the dimensionless normalization condition $\langle \Omega | \Omega \rangle = 1$.

What happens to scale invariance in the finite mass free field theory? We can see from considering the propagator $D(p)$ that for $p \sim m$ the exact propagator $(p^2-m^2)^{-1}$ is quite different from the zero mass propagator; but for p^2 large (i.e., large virtual mass) the propagator reverts to the zero mass form. In x space the equivalent result is that $D(x)$ is almost scale invariant at small distances:

$$D(x) = -\frac{1}{4\pi x^2} + (m^2/8\pi^2) \ln(m^2x^2) + \dots \tag{18}$$

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for small x. The idea that scale invariance becomes exact in the limit of small distances[12] is similar to the hypothesis that equal-time commutators are exact to SU(3) × SU(3).[1] If scale invariance becomes exact only at large virtual masses, as is suggested by the form of the free propagator, it is a severe limitation on the usefulness of scale invariance, since it means the on-mass-shell S matrix is unaffected by scale invariance. We shall consider the S matrix problem later, concluding from more detailed analysis that there is indeed little apparent connection between scale invariance and the S matrix if there are anomalous dimensions. Problems such as e^+-e^- annihilation which involve only large virtual masses and no on-mass-shell variables can still be very much affected by scale invariance.

There is much to be said about broken scale invariance which I must omit. Further questions include defining the infinitesimal generator[2] and the hypothesis of a partially conserved dilation current,[3] coupling of the dilation current to scalar mesons and possible Goldberger-Treiman type relations, Ward identities, etc., i.e., all the apparatus familiar from SU(3) × SU(3).[13] There are also questions of what scale invariance would imply for the S-matrix if the S-matrix were invariant.[14] Here we shall specialize on the idea of anomalous dimensions and its implications.

First we should look at the evidence for the existence of anomalous dimensions. Two examples will be cited here, one from the Thirring model and one from renormalized perturbation theory. Consider the exact propagator of the Thirring model, derived by Johnson:[9]

$$G(z) = i \langle \Omega | T \, \psi(z) \, \overline{\psi}(0) | \Omega \rangle = \exp \left\{ -4\pi i b \left[D_0(z) - D_0(0) \right] \right\} G_0(z) \qquad (19)$$

where $G_0(z)$ is the free propagator (in a space with one space and one time dimension) and $D_0(z)$ is the free propagator for a scalar field:

$$D_0(z) = -(i/4\pi) \ln(-z^2 + i\epsilon) \tag{20}$$

Also

$$b = \lambda^2/4\pi^2 (1 - \lambda^2/4\pi^2)^{-1} \; ; \tag{21}$$

where λ is the Fermi coupling constant.

The formula given above is the unrenormalized formula; subtracting $D_0(0)$ ensures that $G(z)$ is consistent with the canonical commutation rules as $z \to 0$. Unfortunately $D_0(0)$ is infinite; to remove this infinity requires an infinite wave function renormalization, after which $G(z)$ is inconsistent with the canonical commutation rules (see Johnson[9]). The exponential of $D_0(z)$ is a power of z^2; as a result $G(z)$ scales as

$$G(z) \sim (z^2)^{-1/2-b} \tag{22}$$

This means the dimension of ψ is

$$d = \frac{1}{2} + \lambda^2/4\pi^2 (1 - \lambda^2/4\pi^2)^{-1} \tag{23}$$

So d varies with λ, changing from 1/2 (the canonical value in one space dimension) for $\lambda=0$ to ∞ for $\lambda=2\pi$.[15] The singularity in $G(z)$ at $z=0$ is spectacular for λ near 2π![16]

What happens in ordinary perturbation theory? Consider the $\lambda\phi^4$ interaction of a zero mass pseudoscalar field ϕ.[17] This interaction is scale invariant according to canonical field theory[18]: the added term in H is

$$\lambda \int \phi^4(x) \, d^3\underset{\sim}{x}$$

Under the scale transformation U this becomes

$$\lambda \, U^+(s) \int \phi^4(\underset{\sim}{x}, t) \, d^3\underset{\sim}{x} \, U(s) = \lambda \, s^4 \int \phi^4(s\underset{\sim}{x}, st) \, d^3\underset{\sim}{x}$$

$$= s \left\{ \lambda \int \phi^4(\underset{\sim}{x}, st) \, d^3\underset{\sim}{x} \right\} \tag{24}$$

so it transforms like the rest of H. In order λ the field ϕ does not show an anomalous dimension, but the composite field $\phi^4(x)$ does. Consider the matrix element

$$W(p_1, p_2, p_3, p_4) = \int_{x_1} e^{ip_1 \cdot x_1} \dots \int_{x_4} e^{ip_4 \cdot x_4} \langle \Omega | T\phi(x_1) \, \phi(x_2) \, \phi(x_3) \, \phi(x_4) \, \phi^4(x) | \Omega \rangle \tag{25}$$

calculated to order λ. The connected part W_c of W is found to be[17]

$$W_c(p_1, p_2, p_3, p_4) = 24 \, D_0(p_1) \dots D_0(p_4) \left\{ 1 + (3\lambda/4\pi^2) \, \ell n \left[-(p_1 + p_2)^2/\Lambda^2 \right] \right.$$

$$\left. + 5 \text{ permutations of p's in the } \lambda \text{ term} \right\} \tag{26}$$

where Λ is a cutoff; $D_0(p)$ is the free zero mass propagator for the scalar field. The cutoff dependence can be removed by a wavefunction renormalization. If the dimension of ϕ^4 is called d_I and the dimension of ϕ is 1, the scaling law for W_c is

$$W_c(sp_1 \dots sp_4) = s^{4 + d_I - 16} \, W_c(p_1 \dots p_4) \tag{27}$$

In order 1, $d_I = 4$, but in order λ it must change to account for the logarithms in Eq. (26). The easiest way to see this is to note that to order λ, Eq. (26) is equivalent to

$$W_c(p_1, p_2, p_3, p_4) = 24 \, D_0(p_1) \dots D_0(p_4) \left\{ \left[-(p_1 + p_2)^2/\Lambda^2 \right]^{3\lambda/4\pi^2} \times 5 \text{ permutations} \right\} \tag{28}$$

From this formula one has

$$W_c(sp_1 \dots sp_4) = s^{(9\lambda/\pi^2 - 8)} \, W_c(p_1, p_2, p_3, p_4) \tag{29}$$

Comparison of Eq. (29) with Eq. (27) gives

$$d_I = 4 + 9\lambda/\pi^2 \tag{30}$$

This means that $\phi^4(x)$ no longer has dimension 4 in the presence of the perturbation. This in turn means that the $\lambda\phi^4$ term in H no longer scales properly: H does not go into sH under a scale transformation. As a result the theory ceases to be scale invariant: the breakdown occurs in order λ^2. (So one cannot expect Eq. (28) to hold beyond order λ.) This breakdown of scale invariance for zero mass renormalized perturbation theory is true of other standard theories, e.g., quantum electrodynamics or pseudoscalar meson theory.

One sees in the second example especially the connection of renormalization to the anomalous dimension; with a cutoff the logarithm must involve the cutoff in order to have a dimensionless argument.

What are the consequences of anomalous dimensions for strong interactions? First, we observe that the prediction of the asymptotic behavior of the e^+-e^- annihilation cross section is unaffected by anomalous dimensions. The total cross section for annihilation into hadrons is[5]

$$\sigma_{TOT}(q^2) = -16\pi^2 \, \alpha^2 (q^2)^{-2} \int e^{iq\cdot x} \, \rho_\mu^\mu(x) \, d^4x \tag{31}$$

with

$$\rho_{\mu\nu}(x) = \langle \Omega | j_\mu(x) \, j_\nu(0) | \Omega \rangle \tag{32}$$

where q is the momentum transfer to the hadrons, α is the fine structure constant, and $j_\mu(x)$ is the electromagnetic current of the hadrons. The scaling law for $j_\mu(x)$ is fixed by Gell-Mann's current commutators. For example, if $j_{\mu W}^+(x)$ is the charge + 1 component of the weak current, one has

$$\left[j_0(\underset{\sim}{x},t), \, j_{0W}^+(\underset{\sim}{y},t) \right] = j_{0W}^+(\underset{\sim}{y},t) \, \delta^3(\underset{\sim}{x}-\underset{\sim}{y}) \tag{33}$$

Let

$$U^+(s) \, j_\mu(\underset{\sim}{x}, t) \, U(s) = s^d \, j_\mu(s\underset{\sim}{x}, st) \tag{34}$$

$$U^+(s) \, j^+_{\mu W}(\underset{\sim}{x}, t) \, U(s) = s^d \, j^+_{\mu W}(s\underset{\sim}{x}, st) \tag{35}$$

(It is assumed that all components of all $SU(3) \times SU(3)$ currents transform alike.)
Transforming Eq. (33) gives

$$s^{2d} \left[j_0(s\underset{\sim}{x}, st), \, j^+_{0W}(s\underset{\sim}{y}, st) \right] = s^d \, j^+_{0W}(s\underset{\sim}{y}, st) \, \delta^3(\underset{\sim}{x} \underset{\sim}{-} \underset{\sim}{y}) \tag{36}$$

But

$$\left[j_0(s\underset{\sim}{x}, st), \, j^+_{0W}(s\underset{\sim}{y}, st) \right] = j^+_{0W}(s\underset{\sim}{y}, st) \, \delta^3(s\underset{\sim}{x} - s\underset{\sim}{y})$$

$$= s^{-3} \, j^+_{0W}(s\underset{\sim}{y}, st) \, \delta^3(\underset{\sim}{x} \underset{\sim}{-} \underset{\sim}{y}) \tag{37}$$

For the two equations to agree one must have d=3.

Substitution of the scaling law (34) with d=3 into Eqs. (31) and (32) gives the prediction that $\rho_{\mu\nu}(x)$ scales as x^{-6} and

$$\sigma_{TOT}(q^2) \propto 1/q^2 \tag{38}$$

With broken scale invariance the scaling law should hold for small x; this means the scaling law for σ_{TOT} should hold for large q^2. The proportionality constant cannot be predicted.[19]

Second, we examine a negative consequence of anomalous dimensions. Namely, there are no single particle states in the scale invariant zero mass limit of strong interactions, if the fields of strong interactions have anomalous dimensions. The problem is the usual infrared problem; in the zero mass theory every particle is surrounded by a cloud of infrared particles, and one can never separate a particle from its cloud. One consequence of this is that there is no S matrix in the scale invariant limit. Thus there is no reason to expect the S matrix to be approximately

scale invariant at large energies in the finite mass theory.[20] To see that there are no single particle states in the scale invariant limit consider the pion propagator $D_\pi(p)$ as an example. By the analysis described earlier, $D_\pi(p)$ is proportional to $(p^2)^{\Delta-2}$ where Δ is the dimension of the pion field. If Δ is not equal to the free field value 1, it must be larger than 1, since for large p^2, $D_\pi(p)$ cannot be smaller than $(p^2)^{-1}$.[21] But if $\Delta > 1$, $D_\pi(p)$ has no pole for finite or zero p; it has only a branch point at $p^2=0$ which is less singular than a pole. This is a typical symptom of infrared problems.

With anomalous dimensions one possible application of broken scale invariance is ruled out; one cannot look for scale invariance in high energy large angle processes unless one can somehow disentangle infrared effects.[20] Kastrup[4] has worked on this problem but one is a long way from a solution.

The absence of single particle states in the zero mass limit gives another negative result. Namely, one cannot predict the Bjorken scaling laws for deep inelastic electron scattering[22] from the assumption of broken scale invariance.[23] The validity of the Bjorken scaling laws has been shown by Callan and Gross[24] to depend on properties of the equal time commutator of j_μ and $\partial j_\mu/\partial t$. If broken scale invariance holds, this commutator must be scale invariant. However one does not know a priori what local fields will occur in the commutator, or what the dimensions of these fields will be, and this information is crucial. See a forthcoming paper by G. Mack (Center for Theoretical Physics, University of Miami).

A third consequence of anomalous dimensions is that they make possible an explanation of a universal $\Delta I=1/2$ rule.[6] I can only give the essence of the explanation here. For simplicity imagine that the weak interactions are mediated by an intermediate boson of mass m_W. We assume that m_W is large compared to typical strong interaction masses, i.e., $m_W \gg 1$ GeV. The matrix element for

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a typical nonleptonic decay is

$$T = \left(G m_W^2 \cos \theta \sin \theta / \sqrt{2} \right) \int \langle f | T \, j_{\mu W}^+(x) \, j_{\nu S}(0) | i \rangle \, W^{\mu\nu}(x, m_W) \, d^4x \quad (39)$$

where G is the weak coupling constant, θ is the Cabbibo angle, $|i\rangle$ and $|f\rangle$ are the initial and final hadron states (e.g., $|\Lambda\rangle$ and $|p\pi\rangle$), $j_{\nu S}$ is the strangeness changing weak current, and $W^{\mu\nu}(x, m_W)$ is the propagator for the intermediate boson. It is crucial to the analysis to discuss this matrix element in x space, instead of Fourier transforming the matrix element and integrating over the momentum space propagator. It is also crucial to observe that the boson propagator is negligible at distances large compared to m_W^{-1}. Hence only small distances x are important in the integral and one can expect scale invariance to be relevant. However, the states $|i\rangle$ and $|f\rangle$ are low energy states and far from scale invariant; so to use scale invariance one must isolate properties of the product $T j_{\mu W}^+(x) \, j_{\nu S}(0)$ which do not depend on the states $|i\rangle$ and $|f\rangle$. Such a property has been proposed; it is an operator product expansion.[25] The idea of the expansion is that for small enough x the product $T \, j_{\mu W}^+(x) \, j_{\nu S}(0)$ is indistinguishable from a local field at 0. This expressed by writing the product as a linear combination of fields at the origin:

$$T j_{\mu W}^+(x) \, j_{\nu S}(0) = \sum_n C_{n\mu\nu}(x) \, O_n(0) \quad (40)$$

where the fields $O_n(0)$ are a complete set of local fields; the $C_{n\mu\nu}(x)$ are functions of the separation x. This expansion is an operator relation independent of the states $|i\rangle$ and $|f\rangle$. Scale invariance makes predictions for the behavior of the functions $C_{n\mu\nu}(x)$ just as it does for propagators. Namely, $C_{n\mu\nu}(x)$ scales as x^{-6+d_n} where d_n is the dimension of the field O_n. Since x is small the largest term in the expansion corresponds to the field O_n which has the smallest d_n. In some free field models, such as the quark model, the field of smallest dimension that contributes

is the Wick product $:j^+_{\mu W}(0)\, j_{\nu S}(0):$ which has both $\Delta I=1/2$ and $\Delta I=3/2$ parts.
But with anomalous dimensions one can have the field of lowest dimension be
pure $\Delta I=1/2$, with all $\Delta I=3/2$ fields having high dimensions.[26] If this is the
case the $\Delta I=3/2$ part of $T\, j^+_{\mu}(x)\, j_{\nu S}(0)$ will be smaller by a power of x than the
$\Delta I=1/2$ part. Since x is no larger than m^{-1}_W, the result is to suppress <u>all</u> $\Delta I=3/2$
amplitudes by a power of $\left(m^{-1}_W\right)$.[27] Furthermore if m_W is very large (> 10 GeV)
the suppression could be enormous, with observed $\Delta I=3/2$ amplitudes being
electromagnetic. If the observed $\Delta I=3/2$ amplitudes are not electromagnetic
but come from the $\Delta I=3/2$ part of the weak amplitude then probably m_W cannot
be terribly large; however we don't know what power of m_W occurs in the sup-
pression factor so we cannot give a numerical bound for m_W.

The assumptions of this explanation are as follows:

1. Broken scale invariance. This is the big assumption .

2. Anomalous dimensions. The evidence for this is fairly compelling
    from known field theories.

3. $\Delta I=1/2$ dominance. See Ref. 26.

4. Operator product expansion. The evidence for this from known
    field theories is compelling. (I have not yet completed a paper
    describing a thorough but nonrigorous study in perturbation
    theory.)

Can anomalous dimension be measured? If the theory of corrections to scale
invariance proposed in Ref. 6 is correct, then anomalous dimensions can be
measured in e^+-e^- annihilation, at least in principle. The theory of corrections
to scale invariance works as follows.[28] The part of the Lagrangian which breaks
scale invariance is assumed in Ref. 6 to be

$$\mathscr{L}_I = \lambda_0\sigma_0 + \lambda_8\sigma_8 + \lambda\, w \qquad (41)$$

where $\lambda_0 \sigma_0 + \lambda_8 \sigma_8$ is the SU(3) $\times$ SU(3) breaking term of Gell-Mann, Oakes, and Renner, except that the scalar fields σ_0 and σ_8 used here are normalized by their short distance behavior and differ by a dimensional constant from the fields u_0 and u_8 of Gell-Mann et al. The fields σ_0 and σ_8 have an unknown dimension Δ, the field w has unknown dimension Δ_1, but both Δ and Δ_1 must lie between 1 and 4.[6] In the presence of scale breaking one has dimensional constants, namely the constants, λ_0, λ_8, and λ, whose dimensions must be chosen so that $\mathscr{L}_I$ has dimension 4 in mass units. This means λ_0 and λ_8 have dimension $4-\Delta$ and λ has dimension $4-\Delta_1$. The rule governing scale breaking corrections to $\rho_{\mu\nu}(x)$ at small x is that they must be power series in λ_0, λ_8, and λ, and that they must be consistent with SU(3) $\times$ SU(3) symmetry. The result is that there can be terms of order λ_0^2, $\lambda_0\lambda_8$, etc., but not of order λ_0 or λ_8. If w carries no internal symmetry there can be a term of order λ. An alternative is that w is not invariant to an axial baryon number,[29] in which case there is only a term of order λ^2. A second rule is that the scale breaking terms must be dimensionally correct. Since $\rho_{\mu\nu}(x)$ has dimension 6 this means the terms of order λ^2 in $\rho_{\mu\nu}(x)$ behave as $(x^2)^{-3+(4-\Delta_1)}$ which leads to a correction proportional to $(q^2)^{\Delta_1-5}$ in $\sigma_{TOT}(e^+ + e^- \rightarrow$ hadrons). Likewise there will be corrections proportional to $(q^2)^{\Delta-5}$. So corrections to the asymptotic form of the e^+-e^- total cross section scaling as nonintegral powers of q^2 would be experimental evidence for anomalous dimensions.

In the examples of anomalous dimensions cited earlier they were always accompanied by infinite wave function renormalizations. If one is to have a more detailed theory of anomalous dimensions, it will be necessary to solve the renormalization problem for strongly coupled field theories.[30] It can no longer be wished away!

REFERENCES

1. See, e.g., S. L. Adler and R. F. Dashen, Current Algebras (W. A. Benjamin, Inc., New York, 1968).

2. See, e.g., J. Wess, Nuovo Cimento 18, 1086 (1960).

3. G. Mack, Nucl. Phys. B5, 499 (1968).

4. H. A. Kastrup, Phys. Rev. 150, 1183 (1966), and references cited therein; Nucl. Phys. B15, 179 (1970); and Particles, Currents, Symmetries, P. Urban, Ed. (Springer-Verlag, New York, 1968), p. 452.

5. This prediction for e^+-e^- annihilation was made by J. Bjorken (but without explicit reference to broken scale invariance): J. Bjorken, Phys. Rev. 148, 1467 (1966).

6. K. Wilson, Phys. Rev. 179, 1499 (1969). Divergences in electromagnetic corrections to strong interactions, η-decay, and π^o decay are also discussed from the point of view of scale invariance in this reference. See also, R. Gatto, Rivista d. Nuovo Cimento 1, 514 (1969); L. Kadanoff, Phys. Rev. Letters 23, 1430 (1969).

7. The existence of w was first proposed (to my knowledge) by S. Glashow: see Hadrons and Their Interactions, A. Zichichi, Ed. (Academic Press, New York, 1968), p. 102-103. A second (qualitative) argument predicting the existence of w is given in Ref. 6. Attempts to make the argument quantitative are reported in L. N. Chang and P.G.O. Freund, Cal Tech preprint, and S. P. DeAlwis and P. J. O'Donnell, Toronto University preprint. A good discussion appears in the lecture notes of M. Gell-Mann for the Hawaii Summer School (Cal Tech preprint, 1970).

8. S. L. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

9. W. Thirring, Ann. Phys. (N.Y.) $\underline{3}$, 91 (1958); K. Johnson, Nuovo Cimento $\underline{20}$, 773 (1960). The fact that the Thirring model has anomalous dimensions does not seem to have been recognized explicitly until recently: see Ref. 6 and K. Wilson, Report No. SLAC-PUB-734, Stanford Linear Accelerator Center. For other recent work on the Thirring model, see B. Klaiber in Lectures in Theoretical Physics, Vol. X-A; Quantum Theory and Statistical Physics, A. O. Barut and W. E. Brittin, Eds. (Gordon and Breach Science Publishers, Inc., New York, 1968), p. 141; A. Wightman in High Energy Electromagnetic Interactions and Field Theory, M. Levy, Ed. (Gordon and Breach Science Publishers, Inc., New York, 1967), p. 218; John Lowenstein, Normal Products in the Thirring Model, preprint from Departamento de Fisica, Universidade de Sao Paulo, Sao Paulo, Brasil.

10. R. Jackiw, and S. Coleman (private communication from R. Jackiw); K. Wilson, Report No. SLAC-PUB-741, Stanford Linear Accelerator Center.

11. Actually the idea of dimensions changing with the coupling constant in perturbation theory was first suggested in M. Gell-Mann and F. E. Low, Phys. Rev. $\underline{95}$, 1300 (1954), Section 3. The idea of anomalous dimensions is closely analogous to the idea of "critical exponents" in classical statistical mechanics. There is no good reference on this analogy at the moment: for related work, see, A. Suri, Thesis, Cornell University (1969); A. A. Migdal, Zh. Eksp. Teor. Fiz. $\underline{55}$, 1964 (1968) (Translation: JETP $\underline{28}$, 1036 (1969)). For a review of critical exponents in classical statistical mechanics, see M. Fisher, Repts. Prog. Phys. $\underline{30\text{II}}$, 615 (1967).

12. In this paper short distances always mean short in space and time separately. A separation x lying on the light cone ($x^2 = 0$) a finite distance from the origin is not "short".

13. For a brief discussion of these ideas see Ref. 3. See also, L. N. Chang and P. G. O. Freund (Ref. 7) and references cited therein. In broken scale invariance there are "soft σ theorems" analogous to the soft pion theorems of $SU(2) \times SU(2)$, where the σ is a scalar particle of isospin 0. Such theorems give σ production amplitudes for off-mass-shell σ's of zero mass. The usefulness of these is unclear.

14. See Refs. 4, 3, and N. F. Bali, D. D. Coon, and A. Katz, J. Math. Phys. 10, 1939 (1969); D. J. Gross and J. Wess, Report No. TH. 1076, CERN preprint (1969); C. W. Kim, W. W. Repko, and A. Sato, Johns Hopkins preprint (1970); D. G. Boulware, L. S. Brown, and R. D. Peccei, Report No. RLO-577, University of Washington preprint.

15. In the presence of a cutoff the constant $D_0(0)$ is replaced by a finite but cutoff-dependent factor which will be of order $D_0(\Lambda^{-1})$. In this case the unrenormalized propagator is not scale invariant because the propagator depends on the dimensional parameter Λ which spoils scale invariance. One can still perform dimensional analysis, in which Λ is scaled as well as x; the unrenormalized cutoff propagator has the dimensions one predicts from canonical commutation rules. When the cutoff propagator is renormalized one divides it by the factor $\exp(4\pi i b\, D_0(\Lambda^{-1})) = \Lambda^{-2b}$. Since this factor carries dimensions, the renormalized propagator has a different dimension from the unrenormalized propagator. I thank Dr. L. Stodolsky for a discussion on this point.

16. For further discussion see K. Wilson, Ref. 9.

17. A more detailed analysis of this example is found in K. Wilson (Ref. 10). See also, C. G. Callan, S. Coleman, and R. Jackiw, Report No. CTP 113, MIT preprint.

18. For discussions of scale invariance and conformal invariance in canonical field theory, see G. Mack and A. Salam, Ann. Phys. (N. Y.) $\underline{53}$, 174 (1969), and references cited therein. See also C. J. Isham, A. Salam, and J. Strathdee, Trieste preprints (1970).

19. Is it possible for the proportionality constant to be zero? This question is considered in P. de Mottoni and H. Genz, II Institut fur Theoretische Physik der Universität Hamburg, Hamburg, Germany preprint.

20. It is already made clear in Refs. 3 and 4 that there are infrared effects masking any possible scale invariance in high energy scattering processes. By "infrared effects" I mean the fact of multiple meson production, not any particular theory or precise analogy with soft photon production.

21. This is a consequence of the Kallen-Lehmann representation.

22. J. D. Bjorken, Phys. Rev. $\underline{179}$, 1547 (1969).

23. There are two papers discussing deep inelastic electron scattering which assume single particle states do exist in the scaling limit; C. W. Kim, W. W. Repko, and A. Sato, Johns Hopkins preprint (1970); D. G. Boulware, L. S. Brown, and R. D. Peccei, Report No. RLO-577, University of Washington preprint.

24. C. Callan and D. Gross, Phys. Rev. Letters $\underline{22}$, 981 (1969).

25. See Ref. 6; earlier references are cited therein.

26. There is a priori a 50-50 chance that the field of lowest dimension contributing in Eq. (40) will have a $\Delta I = 1/2$ part. With anomalous dimensions it is unlikely that fields with $\Delta I = 3/2$ will be degenerate in dimension with $\Delta I = 1/2$ fields (see, e.g., K. Wilson, Ref. 10), in which case the chances are close to 50-50 that the field of lowest dimension has only $\Delta I = 1/2$.

27. If the matrix element $\langle f | T j_{\mu W}^+(x) \, j_{\nu S}(0) | \Omega \rangle$ is finite as $x \to 0$, then one can estimate the decay amplitude (39) in momentum space keeping only relatively low mass intermediate states. Such estimates have been made; see Ref. 6 for references. These estimates give reasonable agreement with the observed $\Delta I = 1/2$ decay amplitudes and predict a suppression of the $\Delta I = 3/2$ amplitudes. This suggests that the $\Delta I = 1/2$ field dominating the decay has dimension close to 6. If it had dimension much less than 6, the matrix element would be singular for $x \to 0$; as a result large mass intermediate states would dominate the matrix element and the low mass estimates would be much lower than the actual decay amplitude. If $\Delta I = 1/2$ amplitudes are not enhanced, one must suppress the $\Delta I = 3/2$ amplitudes, which is achieved by having the $\Delta I = 3/2$ fields have dimension > 6. This means the corresponding $C_{n\mu\nu}(x)$ go to zero as $x \to 0$; the $\Delta I = 3/2$ part of the x-space matrix element goes to 0 as $x \to 0$, giving rise to sum rules in momentum space: see Ref. 6. A Bjorken limit analysis in momentum space cannot reproduce this result: the Bjorken limit gives only the singular part of the T-product at short distances. This is why the operator product expansion was used in the analysis instead of the Bjorken limit. The explanation of the $\Delta I = 1/2$ rule offered by V. S. Mathur and J. Subba Rao, Phys. Letters 31B, 383 (1970) requires that $\Delta I = 1/2$ decays be enhanced rather than $\Delta I = 3/2$ decays be suppressed.

28. For details on the rules for symmetry breaking, see Ref. 6.

29. In Ref. 6 the problem of the axial baryon number was ignored. If it exists one expects w to break it. See S. Glashow or M. Gell-Mann (Ref. 7).

30. The classic paper on renormalization of strongly coupled fields is M. Gell-Mann and F. E. Low (Ref. 11). Recent papers are M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969); K. Wilson, Report No. SLAC-PUB-733, Stanford Linear Accelerator Center.