SLAC-PUB-707
January 1970
(TH) and (EXP)

THE ORDER α^2 ELECTRODYNAMIC CORRECTIONS TO THE LAMB SHIFT†

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ABSTRACT

The fourth-order radiative correction to the slope at $q^2 = 0$ of the Dirac form factor of the free electron vertex is calculated using computer techniques. The result,

$$m^2 \partial F_1^{(4)}(0)/\partial q^2 = (\alpha/\pi)^2 [0.48 \pm 0.07] ,$$

disagrees with previous calculations, and implies a new theoretical value for the order $\alpha^2(Z\alpha)^4 mc^2$ contribution to the Lamb shift. The new values for the $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$ separation in H and D are increased by (0.35 ± 0.07) MHz and are in good agreement with the results of recent experiments.

(Submitted to Physical Review Letters)

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† Work supported by the U. S. Atomic Energy Commission.

It is rather ironic that the only tests of quantum electrodynamics which still show a serious discrepancy between theory and experiment are the $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$ and $2P_{\frac{3}{2}} - 2S_{\frac{1}{2}}$ separations in atomic hydrogen and deuterium — precisely the levels measured by Lamb[1] and co-workers which gave the theory its start. The disagreement (> 200 ppm) has become more acute with recent measurements and refinements by Robiscoe and Cosens[2] of the Lamb interval in H and D, and three measurements this year[3] of the $2P_{\frac{3}{2}} - 2S_{\frac{1}{2}}$ interval in H. The results are tabulated in Table I.

The only experimentally relevant contribution to the theoretical value of the Lamb shift not checked by independent methods is the fourth-order self-energy correction to the energy levels of the bound electron[4]. The leading contribution, of order $\alpha^2 (Z\alpha)^4 mc^2$ to the level shift formula, may be obtained directly[5] from the Dirac form factor of the free electron in fourth order:[6]

$$\Delta E^{(4)}(n,j,\ell) = \delta_{\ell o} \frac{4(Z\alpha)^4 mc^2}{n^3} \; m^2 \left. \frac{\partial F_1^{(4)}}{\partial q^2} \right|_{q^2=0} .$$

This contribution comes from the same set of fourth-order Feynman diagrams (see Table II) which give the well-known $0.32847\ldots(\alpha/\pi)^2$ contribution to the electron magnetic moment[7].

In this paper we report the results of a new computation of the slope at $q^2 = 0$ of the Dirac form factor in fourth order. In the calculation, all traces, projections, and reductions to Feynman parametric form are performed automatically by REDUCE, an algebraic computation program written by A.C. Hearn[8]. The integrals over the Feynman parameters (up to five dimensions) are performed numerically to a typical precision of 0.1% using a program based on work by G. Sheppey[9]. The integration method is basically a computation of the Riemann sum, but on successive iterations the integration grid is modified by the program to minimize the variance of the integrand within each hypercube.

Our results for each graph are shown in Table II along with those of the previous analytic calculation of Soto[4]. Except for a discrepancy in overall sign, our results for the individual contributions are consistent with the asymptotic infrared behavior of the individual amplitudes given in Refs. (4) and (5), as well as the expectation that the separate sums of ladder plus crossed-ladder contributions [Table II — (a) and (e)] and corner plus self-energy contributions [Table II — (b) and (d)] are finite as the photon mass $\lambda \rightarrow 0$. The results shown in Table II are obtained from several types of least-square fits to the results of numerical integration of the individual amplitudes for $10^{-6} < \lambda^2 < 10^{-2}$. In addition to the discrepancy in overall sign, our numerical results for the corner and cross graph contributions do not agree with Soto's results for the finite (non-infrared divergent) contributions. In the case of the corner graph, this discrepancy is quite large.

The sign of the Lamb shift contribution of the vacuum polarization insertion graph (Table II-c) can be obtained simply without computation. It is easy to check that the second order vertex yields a positive contribution to $F_1'(0)$ independent of photon mass. The spectral integral representation of the photon propagator corresponds to a summation over spin one propagators with mass $\lambda^2 > 4m^2$ and a positive weight function. Thus to all orders, the vacuum polarization insertion in the second order vertex yields a positive contribution to $F_1'(0)$ and hence a positive correction to the Lamb shift[6]. This is opposite to the result of Refs. (4) and (5).

Our results imply an increase of (0.35 ± 0.07) MHz in the theoretical value for the $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$ level separation in H and D. A tabulation of the various contributions to the theoretical result for H, including the revised fourth order contribution of $F_1'(0)$, is given in Table III. The comparison of the revised theory with experiment is given in the last column of Tables Ia and Ib. The majority of the experimental results are within one standard deviation of theory, leaving only one high precision measurement, the first to use the non-atomic beam "bottle" method, in serious disagreement. The one standard deviation error limits used in the comparison of theory and experiment were computed by combining the standard deviation experimental error assigned by Taylor et al.[2] with one-third of the limit of error (L.E.) of the theoretical result.

We have performed several checks on our result including a computation of the contribution of the corner and crossed graphs to the fourth order magnetic moment. The results agreed with Petermann's computation[7]. The reducible corner and ladder graphs were renormalized by a direct subtraction in the integrand of the five dimensional integrals. As a check we have also used "intermediate renormalization"[10] in which the subtraction term is computed at m = 0. A complete discussion of these checks and a description of the calculation will be described elsewhere.

Since our results are in disagreement with the previous calculation and contain error intervals from numerical integration, it is important that this contribution to the Lamb shift be recomputed using independent methods.

Acknowledgement

We wish to thank our colleagues at SLAC for helpful discussions and the SLAC-IBM-91 for its assistance.

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TABLE CAPTIONS

Table I The Lamb shift in Hydrogenic Atoms (in MHz). The experimental results for H and D are from Refs. (1-3). Robiscoe's and Cosen's values include a correction for the non-Maxwellian velocity distribution of the atoms in the beam. (R. Robiscoe and T.W. Shyn, private communication. See B. N. Taylor et al., Ref. 2, note added in proof.). The corresponding correction to the H result of Triebwasser et al.[1] is shown in the second line of the table, although there is some question whether this correction is appropriate. No estimates have been made for the corresponding correction to the deuterium measurements of Ref. (1). The values for $\mathscr{L}_{\text{exp}}$ listed in parenthesis are computed from experimental measurements of the large interval "$\Delta E-\mathscr{L}$" = $\Delta E(2P_{\frac{3}{2}} - 2S_{\frac{1}{2}})$ and the theoretical fine structure (see Table III). The "old" theoretical values are from B. N. Taylor et al. (Ref. 2). The revised theory corresponds to the corrected result for the fourth-order contributions discussed in this paper. We use the conventions of B. N. Taylor et al., Ref. (2), and take the limit of error (L.E.) to be three standard deviations.

Table II Comparison of the results of this calculation and that of Ref. (4) for the Feynman graph contributions to $a_4 = m^2 d\,F_1/dq^2 (q^2=0) \big/ (\alpha^2/\pi^2)$. The corner and self-energy graph results include the contribution of mirror graphs. The infrared behavior is expressed in terms of a photon-mass parametrization for $\lambda^2 << m^2$. The infrared convergent ladder plus cross contributions (0.68 ± 0.04) as well as the corner plus self-energy contribution (- 0.23 ± 0.03) could be obtained without knowledge of the infrared divergent behavior of the individual graphs, which were, however, found to be consistent with the negative of the asymptotic behavior ($\lambda^2 << m^2$) given in Refs. (4) and (5). The individual non-infrared remainders given in the last column were determined from fits with the logarithmic terms constrained to those values plus "background" terms multiplying $\sqrt{\lambda^2}$ and $\lambda^2 \log \lambda^2$.

Table III Revised tabulation of the theoretical contributions to the Lamb interval $\mathscr{L} = \Delta E(2S_{\frac{1}{2}} - 2P_{\frac{1}{2}})$ in H. References to the various entries may be found in G. W. Erickson and D. R. Yennie (Ref. 6) and B. N. Taylor et al, (Ref. 2). The only revision from the compilation of Ref. (2) is the new result for the order $\alpha^2 (Z\alpha)^4 m$ contribution to the energy shift from the slope of the Dirac form factor in fourth order as given in this paper. The result of Ref. (4) is 0.102 MHz. Note that fourth order contributions also arise from the anomalous magnetic moment and vacuum polarization corrections.

TABLE I(a)

THE LAMB SHIFT IN HYDROGENIC ATOMS (in MHz)

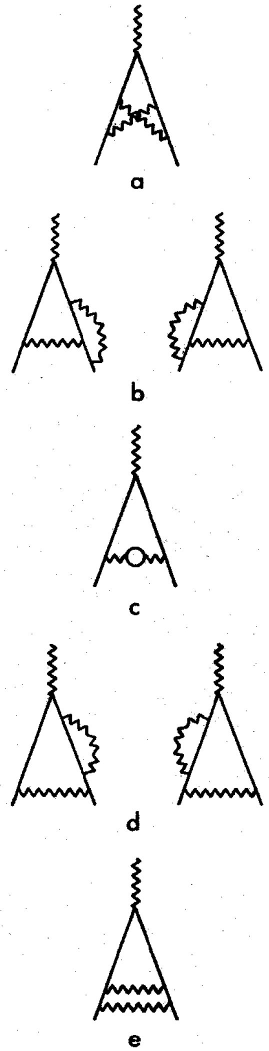
Reference	$\mathscr{L}_{exp}$ (± 1σ)	(Old) Theory (± L.E.)	(Old) Exp–Th (± 1σ)	Revised Theory (± L.E.)	Revised Exp–Th (± 1σ)
H (n = 2)		1057.56 ± 0.09		1057.91 ± 0.16	
Triebwasser, Dayhoff, Lamb (1953)	1057.77 ± 0.06		0.21 ± 0.07		- 0.14 ± 0.08
[- Revision, 1969 by Robiscoe, Shyn]	1057.86 ± 0.06		0.30 ± 0.07		- 0.05 ± 0.08
Robiscoe (Revised, 1969)	1057.90 ± 0.06		0.34 ± 0.07		- 0.01 ± 0.08
Kaufman, Lea, Leventhal, Lamb (1968) [$(\Delta E-\mathscr{L})_{exp}$ = 9911.38 ± 0.03]	(1057.65 ± 0.05)		0.09 ± 0.06		- 0.26 ± 0.07
Shyn, Williams, Robiscoe, Rebane (1969) [$(\Delta E-\mathscr{L})_{exp}$ = 9911.25 ± 0.06]	(1057.78 ± 0.07)		0.22 ± 0.08		- 0.13 ± 0.09
Cosens and Vorburger (1969) [$(\Delta E-\mathscr{L})_{exp}$ = 9911.17 ± 0.04]	(1057.86 ± 0.06)		0.30 ± 0.07		- 0.05 ± 0.08
D (n = 2)		1058.82 ± 0.15		1059.17 ± 0.22	
Triebwasser, Dayhoff, Lamb (1953)	1059.00 ± 0.06		0.18 ± 0.15		- 0.17 ± 0.09
Cosens (Revised, 1969)	1059.28 ± 0.06		0.46 ± 0.08		+ 0.16 ± 0.09

TABLE I(b)

LAMB SHIFT IN OTHER HYDROGENIC ATOMS

Reference	$\mathscr{L}_{exp}$ (± 1σ)	Old Theory (± L.E.)	Old Exp-Th (± 1σ)	Revised Theory (± L.E.)	Revised Exp-Th (± 1σ)
He^+ (n = 2)		14038.9 ± 4.1		14044.5 ± 5.2	
Lipworth, Novick (1957)[a]	14040.2 ± 1.8		1.3 ± 2.2		- 4.3 ± 2.5
Narasimham (1968)[b, c]	14045.4 ± 1.2		6.5 ± 1.8		1.0 ± 2.1
He^+ (n = 3)		4182.7 ± 1.2		4184.4 ± 1.5	
Mader, Levanthal (1969)[d, c]	4182.4 ± 1.0		-0.3 ± 1.1		- 2.0 ± 1.1
Mader, Levanthal (1969)[c]	(4184.0 ± 0.6)		1.3 ± 0.7		- 0.4 ± 0.8
[ΔE-$\mathscr{L}$= 47843.8 ± 0.5]					
He^+ (n = 4)		1768.3 ± 0.5		1769.0 ± 0.6	
Hatfield, Hughes (1968)[e]	1776.0 ± 7.5		-2.3 ± 7.5		- 3.0 ± 7.5
Jacobs, Lea, Lamb (1969)[f, c]	1768.0 ± 5.0		-0.3 ± 5.0		- 2.0 ± 5.0
Jacobs, Lea, Lamb (1969)[c]	(1769.4 ± 1.2)		+1.1 ± 1.3		0.4 ± 1.3
[ΔE-$\mathscr{L}$= 20179.7 ± 1.2]					
Li^{++} (n = 2)		62743.0 ± 45.0		62771.0 ± 50.0	
Fan, Garcia-Munoz, Sellin (1967)[g]	63031.0 ± 327.0		288.0 ± 333.0		260.0 ± 333.0

TABLE II



	Soto	This Calculation
a_{cross}	$\frac{13}{36}\log\lambda^{-2} - 2.314$	$-\frac{13}{36}\log\lambda^{-2} + 2.37 \pm 0.02$
a_{corner}	$-\frac{1}{12}\log^2\lambda^{-2} + \frac{1}{72}\log\lambda^{-2} + 2.432$	$\frac{1}{12}\log^2\lambda^{-2} - \frac{1}{72}\log\lambda^{-2} - 1.91 \pm 0.02$
$a_{vac\text{-}pol}$	-0.0316	$+0.0316 \pm 0.0005$
$a_{self\text{-}energy}$	$\frac{1}{12}\log^2\lambda^{-2} - \frac{1}{72}\log^{-2} - 1.688$	$-\frac{1}{12}\log^2\lambda^{-2} + \frac{1}{72}\log\lambda^{-2} + 1.68 \pm 0.01$
a_{ladder}	$-\frac{13}{36}\log\lambda^{-2} + 1.710$	$+\frac{13}{36}\log\lambda^{-2} - 1.69 \pm 0.02$
a_4	0.108	0.48 ± 0.07

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TABLE III

VARIOUS CONTRIBUTIONS TO THE LAMB SHIFT IN H (n = 2)

DESCRIPTION	ORDER	MAGNITUDE (MHz)
2^{nd}ORDER — SELF-ENERGY	$\alpha(Z\alpha)^4 m\{\log Z\alpha,\ 1\}$	1079.32 ± 0.02
2^{nd}ORDER — VAC. POL.	$\alpha(Z\alpha)^4 m$	- 27.13
2^{nd}ORDER — REMAINDER	$\alpha(Z\alpha)^5 m$	7.14
	$\alpha(Z\alpha)^6 m\{\log^2 Z\alpha,\ \log Z\alpha,\ 1\}$	- 0.38 ± 0.04
4^{th}ORDER — SELF-ENERGY	$\alpha^2(Z\alpha)^4 m$ $F_1'(0)$	0.45 ± 0.07
	$\alpha^2(Z\alpha)^4 m$ $F_2(0)$	- 0.10
	$\alpha^2(Z\alpha)^5 m$	± 0.02
4^{th}ORDER — VAC. POL.	$\alpha^2(Z\alpha)^4 m$	- 0.24
REDUCED MASS CORRECTIONS	$\alpha(Z\alpha)^4 \frac{m}{M} m\{\log Z\alpha,\ 1\}$	- 1.64
RECOIL	$(Z\alpha)^5 \frac{m}{M} m\{\log Z\alpha,\ 1\}$	0.36 ± 0.01
PROTON SIZE	$(Z\alpha)^4 (mR_N)^2 m$	0.13

$$\mathscr{L} = \Delta E(2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}) = 1057.91 \pm 0.16 \quad \text{(L. E.)}$$

$$\alpha^{-1} = 137.03608(26) \qquad \Delta E(2P_{\frac{3}{2}} - 2S_{\frac{1}{2}}) = 9911.12 \pm 0.22 \quad \text{(L. E.)}$$

$$\Delta E(2P_{\frac{3}{2}} - 2P_{\frac{1}{2}}) = 10969.03 \pm 0.12 \quad \text{(L. E.)}$$