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PION DEUTERON ELASTIC SCATTERING AT INTERMEDIATE ENERGIES*

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ABSTRACT

Pion deuteron elastic scattering is calculated via the impulse approximation (including double scattering) at intermediate energies. Good agreement is found with experiment at 87, 142, and 180 MeV/c^2 incident pion energy. Thus the impulse approximation seems to give reliable results even in the region of a large resonance.

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I. INTRODUCTION

The elastic scattering of pions by deuterons is for several reasons a good test of our ability to understand composite systems of strongly interacting particles using a bound state model and a generalized impulse approximation. First, the deuteron is well described as a bound state of a neutron and proton with a wave function whose properties are known. Second, the weak binding of the deuteron and large distance between its components encourages the belief that the scattering amplitude can be well approximated by a simple expression involving amplitudes for pions scattering on free nucleons. Such an expression will be some version of the impulse approximation,[1] and fortunately the requisite accurate information on the pion-nucleon scattering amplitude is available. Third, a theoretical calculation can immediately compared with nature as there are a number of differential cross section measurements, including a recent experiment with an incident pion lab energy of 180 MeV.[2]

If the comparison shows that pion deuteron scattering can be calculated well, then our knowledge of the deuteron wave function is corroborated and we can be more confident of other calculations of scattering from deuterons. In particular, we can be more sanguine in situations where information about scattering on neutrons is gotten by calculation from experiments on scattering by deuterons.

The calculation presented here is based on the impulse approximation. In the simple version of the impulse approximation, the pion-deuteron scattering amplitudes is expressed as a superposition of scattering amplitudes for pions on a set of free neutrons and protons which have the same momentum distribution that they would have inside the deuteron. This leads to formulas similar to

$$A_D = \int d^3p \, A_P(\underset{\sim}{p}) \, \phi_F^*(\underset{\sim}{p}) \, \phi_I(\underset{\sim}{p}) + \text{term for neutron,}$$

where A_D, A_P, and A_N are scattering amplitudes for pions on deuterons, protons, and neutrons, respectively, and ϕ_F and ϕ_I are deuteron wave functions. The proton and neutron scattering amplitudes are customarily brought out of the integral on the grounds that they are slowly varying, and evaluated at some average value of p. However, in a bound system, in addition to the possibility that the pion will undergo just one scattering (which leads to the simple impulse approximation), there may be significant effects due to multiple scattering and the binding potential. These corrections are considered here following a generalization of the impulse approximation by Chew and Goldberger.[3]

The preceeding considerations are described in detail in Section II, and in Section III the calculation is compared to the experimental data.

II. THE IMPULSE APPROXIMATION AND FIRST ORDER CORRECTIONS

Our starting point is the generalized impulse approximation of Chew and Goldberger.[3] We consider a scatterer which is made up of several constituants bound by a potential U, and an incident particle which interacts with the $k^{\underline{th}}$ particle of the scatterer through a potential V_k. The total Hamiltonian is

$$H = K + U + V, \tag{1}$$

where K is the kinetic energy operator and

$$V = \sum_k V_k \tag{2}$$

For convenience, we define operators $\omega_k^{(\pm)}$ such that $\omega_k^{(+)}$, operating to the right on an eigenstate of K with energy E, is given by

$$\omega_k^{(+)} = 1 + (E-K-V_k + i\epsilon)^{-1} V_k, \tag{3a}$$

and $\omega_k^{(-)}$ is a similar operator acting to the left,

$$\omega_k^{(-)} = 1 + V_k(E-K-V_k + i\epsilon)^{-1}. \tag{3b}$$

The scattering operators for two-body scattering are given by

$$t_k^{(+)} = V_k \omega_k^{(+)} \tag{4a}$$

$$t_k^{(-)} = \omega_k^{(-)} V_k, \tag{4b}$$

and the scattering amplitudes are given by matrix elements of the $t_k^{(\pm)}$ between initial and final states consisting of free particles. (We might note that if the initial and final states have the same energies, then there is no difference between $t_k^{(+)}$ and $t_k^{(-)}$).

The impulse approximation relates the scattering operator for scattering on the entire bound system to the two-body operators. The total T-matrix is

$$T^{(+)} = \sum_k t_k^{(+)} + \sum_{k'\neq k}\sum t_{k'}^{(-)}\left(\omega_k^{(+)} - 1\right) + \sum_{k'}\sum_k \left(\omega_{k'}^{(-)} - 1\right)\left[U, \omega_k^{(+)}\right], \tag{5}$$

where this operator is to be evaluated between eigenstates of the unperturbed Hamiltonian for the entire composite system, which is

$$H_0 = K + U. \tag{6}$$

The first term here is just single scattering and gives the simple impulse approximation; the second term gives the multiple scattering corrections; and the last term shows the effects of the binding potential on the individual two-body scatterings.

The necessary matrix elements of the two-body scattering operators $t_k^{(\pm)}$ are known from available analyses of pion-nucleon scattering data, and as a practical matter we note that

$$\omega_k^{(+)} - 1 = (E - K + i\epsilon)^{-1}\, t_k^{(+)} \tag{7}$$

which can be evaluated easily. Thus the single and double scattering terms in the expression for $T^{(+)}$ are written in terms of known quantities. We will argue that the binding corrections are small.

The results of analyses of pion-nucleon scattering are a set of phase shifts[4] which give the scattering amplitude in the center-of-mass system. The necessary relations between these phase shifts and the matrix elements of t_k are given here for the definite example of the reaction $\pi^+ + p \rightarrow \pi^+ + p$. Let us say we are scattering from an initial pion proton state χ_a to a final state χ_b, with the spin projection of the initial proton i, and of the final proton j. The kinematics are shown in Fig. 1. A matrix element of t_k contains a δ-function of three momentum which will be taken out so that it does not appear explicitly in subsequent formulas.

$$\langle \chi_b | t | \chi_a \rangle = (2\pi)^3 \delta^3(\underset{\sim}{p}_1 + \underset{\sim}{q}_1 - \underset{\sim}{p}_2 - \underset{\sim}{q}_2)\, F_{ji}(p_2, q_2; p_1 q_1) \,. \tag{8}$$

In the center-of-mass, there is a conventionally defined scattering amplitude matrix, f, related to F by

$$(\nu_1 \nu_2 E_1 E_2)^{1/2} F_{ji} = -2\pi W f_{ji} \,, \tag{9}$$

where W is the total c. m. energy and the quantity on the left is a Lorentz scalar. If $|i\rangle$ and $|j\rangle$ represent Pauli spinors for the proton spin states, we have[5]

$$f_{ji} = \langle j | f | i \rangle \tag{10}$$

with

$$f = f_1 + \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{q}}_1 \underset{\sim}{\sigma} \cdot \hat{q}_2 f_2 \tag{11a}$$

$$f_1 = \sum_{\ell=0}^{\infty} a_{\ell+}(W)\, P'_{\ell+1}(\cos\theta) + \sum_{\ell=2}^{\infty} a_{\ell-}(W)\, P'_{\ell-1}(\cos\theta) \tag{11b}$$

$$f_2 = \sum_{\ell=1}^{\infty} \Big(a_{\ell-}(W) - a_{\ell+}(W)\Big) P'_{\ell}(\cos\theta) \,, \tag{11c}$$

and finally

$$a_{\ell\pm} = 1/q\; e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm} \,. \tag{12}$$

$\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2$ are the pion momenta in the c. m., q their magnitude and θ the angle between them.

We must also establish the notation to be used for the deuteron wave function. The deuteron has total angular momentum 1, spin 1, and the orbital state is a combination of S-wave and D-wave. Its wave function in co-ordinate space is

$$\phi^{(m)}(\underset{\sim}{x}) = \frac{N}{r}\left\{u(r)\, Y_{00}(\theta, \phi)\, \chi^{m} + w(r) \sum_{n} Y_{2,\, m-n}(\theta, \phi)\, \chi^{n} C_{m-n,\, n}\right\} \tag{13}$$

where χ^{m} is a combination of Pauli spinors in a total spin one state, with a magnetic quantum number m; $C_{m-n,\,n}$ is a Clebsch-Gordon coefficient,

$$C_{m-n,\,n} = (2, m-n;\; 2, n|2, 1;\; 1, m) \tag{14}$$

in Edmond's[6] notation; the normalization constant N is determined by the condition

$$\int dr \left(u^2(r) + w^2(r)\right) = 1\,. \tag{15}$$

The forms of the S- and D-state radial wave functions, u(r) and w(r) are taken from the work of Moravcsik.[7]

A. Single Scattering Terms

Keeping only the single scattering terms in the formula for $T^{(+)}$ gives what is often called the "simple impulse approximation." We have

$$T^{(+)} = \Sigma t_k^{(+)} \tag{16}$$

and this is to be evaluated between initial and final pion-deuteron states. The pion may scatter from either the proton or neutron inside the deuteron, and these processes are diagrammed in Fig. 2.

The kinematics for scattering from a deuteron in the laboratory system are shown in Fig. 3, and the initial and final deuterons have spin projections m and m' respectively. The matrix element of $T^{(+)}$ will contain a δ-function of 3 momentum, which will again be written explicitly so that it will not appear in subsequent formulas,

$$\langle \phi_b | T^{(+)} | \phi_a \rangle = (2\pi)^3\, \delta^{(3)}(\underset{\sim}{P} + \underset{\sim}{q}' - \underset{\sim}{q})\, A_{m'm}(\underset{\sim}{q}, \underset{\sim}{q}')\,. \tag{17}$$

Now it can be seen that

$$A_{m'm}(\mathbf{q},\mathbf{q}') = \int \frac{d^3p}{(2\pi)^3} \, \phi^{*m'}{}_{i'j'}\,(1/2\,\mathbf{P}+\mathbf{p})\,\phi^{m}{}_{ij}(\mathbf{p})$$

$$\times \left[\delta_{j'j}F_{i'i}(\mathbf{P}+\mathbf{p},\ \mathbf{q}';\ \mathbf{p},\mathbf{q}) + \delta_{i'i}G_{j'j}(\mathbf{P}+\mathbf{p},\ \mathbf{q}';\ \mathbf{p},\ \mathbf{q})\right] . \tag{18}$$

G is defined in the same way as F but for the process $\pi^+ + n \longrightarrow \pi^+ + n$.

If F and G are slowly varying as functions of energy, we can remove them from the integral, evaluating them at some average value of $\mathbf{p} = \mathbf{p}_0$. We can find some optimal value of $\mathbf{p}_0$ (following Pendelton[8]) by expanding F and G in a Taylor series,

$$F(\mathbf{P}+\mathbf{p},\mathbf{q}';\mathbf{p},\mathbf{q}) = F(\mathbf{P}+\mathbf{p}_0,\mathbf{q}';\mathbf{p}_0,\mathbf{q}) + \nabla_p F(\mathbf{P}+\mathbf{p}_0,\mathbf{q}';\mathbf{p}_0,\mathbf{q})\,(\mathbf{p}-\mathbf{p}_0) + \ldots \tag{19}$$

If we are not right at a resonance energy, we can assume the term linear in $\mathbf{p}$ will not have a zero coefficient, so that we can choose $\mathbf{p}_0$ by requiring that the linear term not contribute to the integral. Thus,

$$\mathbf{p}_0 = -\frac{1}{4}\mathbf{P}, \tag{20}$$

and

$$A_{m'm}(\mathbf{q}',\mathbf{q}) = \left[\delta_{j'j}F_{i'i}\left(\frac{3}{4}\mathbf{P},\ \mathbf{q}';\ -\frac{1}{4}\mathbf{P},\mathbf{q}\right) + \delta_{i'i}G_{j'j}\left(\frac{3}{4}\mathbf{P},\mathbf{q}';\ -\frac{1}{4}\mathbf{P},\mathbf{q}\right)\right]$$

$$\times \int \frac{d^3p}{(2\pi)^3}\,\phi^{*m'}{}_{i'j'}\left(\frac{1}{2}\mathbf{P}+\mathbf{p}\right)\phi^{m}{}_{ij}(\mathbf{p}) . \tag{21}$$

The integral that remains is a "form factor" that measures the probability that the deuteron will stay together when given a momentum transfer $\mathbf{P}$. Writing the wave function as a combination of S- and D-waves leads to four integrals. The first integral is what the form factor would be if there were no D-state part (and can be integrated analytically for a Hulthén type wave function),

$$E = \frac{1}{4\pi} \int \frac{d^3P}{(2\pi)^3} u^* \left(\frac{1}{2}\underset{\sim}{P}+\underset{\sim}{p}\right) u(\underset{\sim}{p})$$

$$= \frac{1}{4\pi} \int d^3x\, e^{i1/2\underset{\sim}{P}\cdot\underset{\sim}{x}} \left|\frac{u(r)}{r}\right|^2$$

$$= \int dr\, j_0\left(\frac{1}{2}Pr\right) u^2(r)\,. \tag{22a}$$

The other integrals are

$$D = \int dr\, j_2\left(\frac{1}{2}Pr\right) u(r)\, w(r) \tag{22b}$$

$$C = \int dr\, j_0\left(\frac{1}{2}Pr\right) w^2(r) \tag{22c}$$

$$B = \int dr\, j_2\left(\frac{1}{2}Pr\right) w^2(r)\,. \tag{22d}$$

Finally, let us write down the differential cross section in the lab for a calculation where only single scattering is included.

$$\frac{d\sigma}{d\Omega} = R\cdot\frac{1}{3}\sum_{m,m'} \left|A_{m'm}\right|^2 = R\left\{\left|F_{++}+G_{++}\right|^2\mathscr{F}_1^2 + \frac{2}{3}\left|F_{-+}+G_{-+}\right|^2\mathscr{F}_2^2\right\} \tag{23}$$

where

$$\mathscr{F}_1 = \left[(E+C)^2 + 4\left(D - \frac{1}{2\sqrt{2}}B\right)^2\right]^{1/2} \tag{24a}$$

$$\mathscr{F}_2 = E + \frac{1}{\sqrt{2}}D - \frac{1}{2}C + \frac{1}{2}B\,, \tag{24b}$$

and

$$R = (2\pi)^{-2}\nu\nu'\,\frac{E'}{W}\,\frac{q'}{q}\left(1 - \frac{\nu' q}{Wq'}\cos\theta\right)^{-1}. \tag{25}$$

B. Double Scattering Terms

The double scattering contributions to the T-matrix are given by

$$\sum_{k'\neq k}\sum_{k} t_{k'}^{(-)}\,(E - K + i\epsilon)^{-1}\, t_k^{(+)} \tag{26}$$

Three processes may contribute, and these are diagrammed in Fig. 4. For simplicity in this paper, formulas will be written down only for the reaction in Fig. 4a, which consists of a π^+ scattering elastically first off the proton and then off the neutron.

Some approximations will be made. Terms in the double scattering of order $\frac{m_\pi}{m_N}$ will be neglected, and only the δ-function part of the propagator will be kept, so that

$$(E - K + i\epsilon)^{-1} \longrightarrow -i\pi\, \delta(E - K). \tag{27}$$

The deuteron will be taken to be in an S-state only, and again the energy dependence of the pion nucleon scattering amplitude will be neglected. This time, however, we must integrate over an angular variable for the intermediate pion, so that it is necessary to make the expansion

$$F_{\alpha i}\left(\tfrac{1}{2}\underset{\sim}{P} - \underset{\sim}{p}', \underset{\sim}{q}''; -\underset{\sim}{p}, \underset{\sim}{q}\right) = \sum f_{\alpha i}^{\ell_1 m_1} Y_{\ell_1 m_1}(\Omega_1) \tag{28a}$$

$$G_{\beta j}\left(\tfrac{1}{2}\underset{\sim}{P} + \underset{\sim}{p}', \underset{\sim}{q}'; \underset{\sim}{p}, \underset{\sim}{q}''\right) = \sum g_{\beta j}^{\ell_2 m_2} Y_{\ell_2 m_2}(\Omega_2)\,; \tag{28b}$$

where Ω_1 and Ω_2 are the angles between $(\underset{\sim}{q}'', \underset{\sim}{u})$ and $(\underset{\sim}{q}', \underset{\sim}{q}'')$ respectively.

With these approximations, the contribution to πd elastic scattering can be calculated straightforwardly. Let the energy and momentum of the intermediate pion be ν'' and q''. We have

$$A^{(2)}_{m'm} = -i\pi \int \frac{d^3q''}{(2\pi)^3}\, \frac{d^3p}{(2\pi)^3}\; \chi^{m'}{}_{\alpha\beta}\, F_{\alpha i}\left(\tfrac{1}{2}\underset{\sim}{P} - \underset{\sim}{p}', \underset{\sim}{q}''; -\underset{\sim}{p}, \underset{\sim}{q}\right) \delta(\nu - \nu'')$$

$$\times\, G_{\beta j}\left(\tfrac{1}{2}\underset{\sim}{P} + \underset{\sim}{p}', \underset{\sim}{q}'; \underset{\sim}{p}, \underset{\sim}{q}''\right) \chi^{m}_{ij}\, \phi(p')\, \phi(p); \tag{29}$$

this is reduced to a longer but actually more manageable form in Appendix A.

Some comment on binding corrections is also in order. Using the techniques of M. Bander,[9] one can show that the binding corrections are of order $\frac{m_\pi}{m_N}$ compared to double scattering corrections, so they may be consistantly neglected.

III. RESULTS AND COMMENTS

Experimental data on the differential cross section for pion-deuteron elastic scattering is available for five values within the range that could be called the "intermediate" energy region. The kinetic energies of the incident pions in these five experiments are 61,[10] 87,[11] 142,[12] 180,[2] and 300[13] MeV/c^2. At low energies it is difficult to determine if the deuteron has remained intact after scattering the pion, so that data on purely elastic scattering is not available below 61 MeV/c^2. At high energies, one cannot treat the nucleus nonrelativistically, so that the calculation presented here is not applicable above 300 MeV/c^2. This is why we have restricted our attention to the experimental energies listed above.

Perhaps the most interesting data is that taken at 180 MeV. This is only ten of fifteen MeV/c^2 lower than the energy required to excite the 33 resonance at its peak. Since the individual scattering amplitudes are large near a resonance, one expects that the contribution of the double scattering terms will be largest here. Also, it has been suggested that the approximations that are made in the impulse approximation are for various reasons, not valid near a resonance region.

The results of this calculation and the experimental data is shown graphically in Figs. 4 through 8. The agreement with the experimental data is not good at 61 and 300 MeV/c^2; however the other curves are in agreement with experiment within statistical expectations.

The agreement at 180 MeV is reasonably good, although high in the backward direction, indicating that the impulse approximation is valid in this energy region. The table shows the effects of the various corrections at some angles at one of the energies, 180 MeV. Listed in the table are the results of calculating pion-deuteron elastic scattering first with a pure S-state deuteron and no double scattering; then including the D-state but not double scattering; then including double scattering but not the D-state; and finally both the D-state of the deuteron wave function and double scattering are included. The experimental results of J. Norem[2] are listed for comparison.

It is seen that the effect of the D-state part of the wave function is to increase the predicted cross section in the backward direction. This occurs because of the D-state radial wave function; the pure S-state form factor falls off rapidly with momentum transfer, while the form factors involving the D-state fall less rapidly.

The double scattering terms interfere destructively with the single scattering terms and also have their greatest effect at large angles. This is because of the rapid decrease of the form factors with momentum transfer. The form factors measure the probability that the deuteron will stay together when one of its constituents is given a certain momentum transfer. Attempting to deflect a pion through a large angle with just one scattering will probably knock the deuteron apart, and it becomes relatively easier to deflect the pion by scattering it twice through smaller angles.

The results here suggest that triple scattering, temporary binding of pion and nucleon to produce an N*, removing the pion-nucleon amplitude from the integral, and various other things which have been suggested as reasons for the invalidity of the impulse approximation at certain energies, do not in fact radically affect the result.[14] This is reasonable. The width of the 33 resonance is about 120 MeV while the width of the pion wave function, which determines the "width" of the integrals, is about 60 MeV. Thus it seems alright to remove the amplitude from the integrals. The impulse approximation assumes that the interaction takes place in a short time compared to other time scales within the deuteron, so that if the pion and nucleon bind together in an N* for an appreciable period the approximation is wrong. However, with the average size of the deuteron being 4.3 fermis and with the pions at our energies, any N* that is produced can travel only a small fraction of the distance between the nucleous before it decays. Finally, the triple scattering probably is small.[15]

Earlier calculations[16] of pion-deuteron elastic scattering near 140 MeV/c^2 gave a result that was much too large in the backward direction. The main difference between the present calculation and earlier calculations is not in the formal input, but in the wave function used. We have used here Moravcsik's[7] best analytic approximation to the Gartenhaus[17] wave function. This wave function was calculated by solving the deuteron bound state problem with a potential which was inferred from an analysis of nucleon-nucleon scattering data. The Moravcsik wave function, and other wave functions gotten by the same methods[18] fall off more quickly in momentum space than commonly available Hulthén or Hulthén with hard core wave functions. Thus the form factors or "overlap functions" decrease more quickly with increasing momentum transfer than if they were calculated with a Hulthén wave function, and thus the calculated cross section is more strongly surpressed in the backward direction.

Our knowledge of the deuteron wave function, though improved, is still a hindrance to calculating scattering at high momentum transfers. The momentum transfers involved in backward scattering at 140 or 180 MeV/c^2 are already high enough to put us in an area of marginal certainty for the wave function. Unfortunately the pion-deuteron scattering experiment probably cannot be used as a probe of the wave function because at higher momentum transfers multiple scattering becomes increasing important, for reasons already mentioned, and the calculation becomes muddled.

APPENDIX

Double Scattering Terms

Making the approximations stated in the text, the amplitude is

$$A^{(2)}_{m'm} = -i\pi \int d^3x\, \phi^2(x) \int \frac{d^3q''}{(2\pi)^3}\, e^{i(\underset{\sim}{q}''-\underset{\sim}{Q})\underset{\sim}{x}}\, \delta(\nu-\nu'')\, X^{+m'}_{\alpha\beta}\, F_{\alpha i} G_{\beta_d}\, X^{m}_{ij}$$

$$= -i\pi(2\pi)^{-3} \nu_q \sum_{\ell_1 m_1 \ell_2 m_2} X^{+m'}_{\alpha\beta}\, f^{\ell_1 m_1}_{\alpha i}\, g^{\ell_2 m_2}_{\beta_j}\, X^{m}_{ij} \int d^3x\, \phi^2(x)\, e^{-i\underset{\sim}{Q}\underset{\sim}{x}} \int d\Omega''\, e^{i\underset{\sim}{q}''\underset{\sim}{x}}\, Y_{\ell_1 m_1}(\Omega_1)\, Y_{\ell_2 m_2}(\Omega_2)$$

where (A.1)

$$\underset{\sim}{Q} = \frac{1}{2}(\underset{\sim}{q}+\underset{\sim}{q}'), \qquad \text{(A.2)}$$

Ω_1 represents the angle between q and q'', and Ω_2 represents the angles between q'' and q. Ω_3 represents the angle between $\underset{\sim}{x}$ and $\underset{\sim}{q}''$, then the exponential factor in the last integral may be expanded

$$e^{i\underset{\sim}{q}''\underset{\sim}{x}} = \sum_{\ell} i^{\ell} \sqrt{4\pi(2\ell+1)}\; j_\ell(qx)\, Y_{\ell 0}(\Omega_3)\,. \qquad \text{(A.3)}$$

This leaves integrals of the form

$$I = \int d\Omega''\, Y_{\ell_1 m_1}(\Omega_1)\, Y_{\ell_2 m_2}(\Omega_2)\, Y_{\ell 0}(\Omega_3)\,, \qquad \text{(A.4)}$$

which cannot be integrated immediately because the arguments of the spherical harmonics measure the direction of $\underset{\sim}{q}''$ in three different reference systems. One may, however, rotate each of these reference systems to some standard co-ordinate system. If the Euler angles for rotating the standard system to a system with its z axis along $\underset{\sim}{q}$ are ω_1, then

$$Y_{\ell_1 m_1}(\Omega_1) = \sum_{m'} D^{\ell_1}_{m_1' m_1}(\omega_1)\, Y_{\ell_1 m_1}(\Omega_0)\,, \text{ etc.} \qquad \text{(A.5)}$$

Now with the help of some formulas from Edmonds,[6] the integral can be done

$$I = \sum_{m' m_1' m_2'} D^{\ell}_{m'0}(\omega_3)\, D^{\ell_1}_{m_1' m_1}(\omega_1)\, D^{\ell_2}_{m_2' m_2}(\omega_2)$$

$$\times \left(\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{4\pi}\right)^{1/2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m' & m_1' & m_2' \end{pmatrix} \qquad \text{(A.6)}$$

Edmonds' convention is used for the 3-j symbols.

Choosing the z axis of the standard system parallel to $\underset{\sim}{Q}$ allows the angular part of the x integral to be done easily. One finds

$$A^{(2)}_{m'm} = -i(2\pi)^{-1} \nu q \sum_{\substack{\ell_1 m_1 \ell_2 m_2 \\ \ell m_1' m_2'}} (2\ell+1)(2\ell_1+1)^{1/2}(2\ell_2+1)^{1/2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & m_1' & m_2' \end{pmatrix}$$

$$\times\, \chi^{\dagger m'}_{\alpha\beta}\, f^{\ell_1 m_1}_{\alpha i}\, g^{\ell_2 m_2}_{\beta j}\, \chi^{m}_{ij}\, d^{\ell_1}_{m_1' m_1}(\theta_1)\, d^{\ell_2}_{m_2' m_2}(-\theta_2)\, \Phi_\ell\,, \qquad \text{(A.7)}$$

where θ_1 and θ_2 are the angles between $(\underset{\sim}{q}, \underset{\sim}{Q})$ and $(\underset{\sim}{q}', \underset{\sim}{Q})$ respectively, and Φ_ℓ is the radial integral,

$$\Phi_\ell = \int_0^\infty dx\, x^2\, j_\ell(qx)\, j_\ell(Qx)\, \phi^2(x)\,. \qquad \text{(A.8)}$$

If the wave function is of Hulthen type,

$$\phi(x) = N \Sigma C_i \frac{e^{-\alpha_i x}}{x}\,, \qquad \text{(A.9)}$$

then these radial integrals can be done analytically. One uses

$$j_\ell(qx) = \tfrac{1}{2}(-i)^\ell \int_{-1}^{1} d\zeta\; P_\ell(\zeta)\, e^{iqx\zeta} \qquad \text{(A.10)}$$

to show that

$$\Phi_\ell = \tfrac{1}{4}(-1)^\ell N^2 \sum_i \sum_j C_i C_j \int d\zeta\, d\eta\, P_\ell(\zeta) \frac{1}{\alpha_i + \alpha_j - q\zeta - iQ\eta} P_\ell(\eta) \qquad \text{(A.11)}$$

The last integrals are quite manageable.

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15. An interesting model is one in which the nucleon masses are infinite and there is only S-wave scattering and the scattering is the same for neutron as for proton. At resonance energy in the backward direction the various orders of multiple scattering cancel in pairs: i.e., single scattering and double scattering cancel, triple scattering and quadrupole scattering cancel, etc. The ratio of triple scattering to single scattering depends on the details of the situation; if other details of our model are made like our pion-deuteron scattering near the 3-3 resonances, triple scattering is about half single scattering. I was told of this model by Dr. Hugh Pendleton.

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TABLE 1

PION DEUTERON ELASTIC SCATTERING CROSS SECTIONS

AT 180 MeV/c^2 (mb/sr)

	45^o	75^o	105^o	135^o	175^o
Pure S-state single scattering	5.31	0.67	0.51	0.51	0.48
W/D-state single scattering	5.62	0.66	0.63	0.74	0.75
Pure S-state double scattering	6.07	0.94	0.69	0.62	0.56
W/D-state w/double scattering	5.32	0.82	0.74	0.76	0.75
Experiment (Ref. 2)	6.07 ± 0.48	0.87 ± 0.14	0.65 ± 0.13	0.41 ± 0.12	0.25 ± 0.36

FIGURE CAPTIONS

1. Pion-nucleon scattering kinematics.
2. Pion-deuteron single scattering processes.
3. Pion-deuteron kinematics in lab system.
4. Pion-deuteron double scattering processes.
5. π-d elastic scattering at 61 MeV/c^2 (Ref. 10).
6. π-d elastic scattering at 87 MeV/c^2 (Ref. 11).
7. π-d elastic scattering at 142 MeV/c^2 (Ref. 12).
8. π-d elastic scattering at 180 MeV/c^2 (Ref. 2).
9. π-d elastic scattering at 300 MeV/c^2 (Ref. 13).

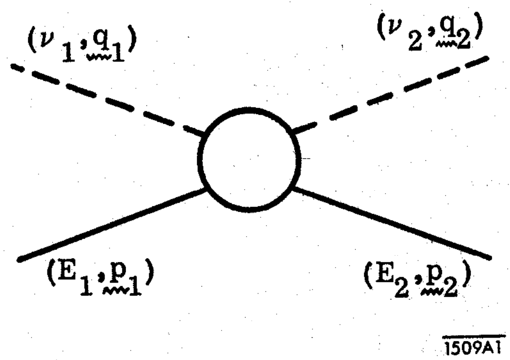

$(\nu_1, \underset{\sim}{q}_1)$
$(\nu_2, \underset{\sim}{q}_2)$
$(E_1, \underset{\sim}{p}_1)$
$(E_2, \underset{\sim}{p}_2)$
1509A1

Fig. 1

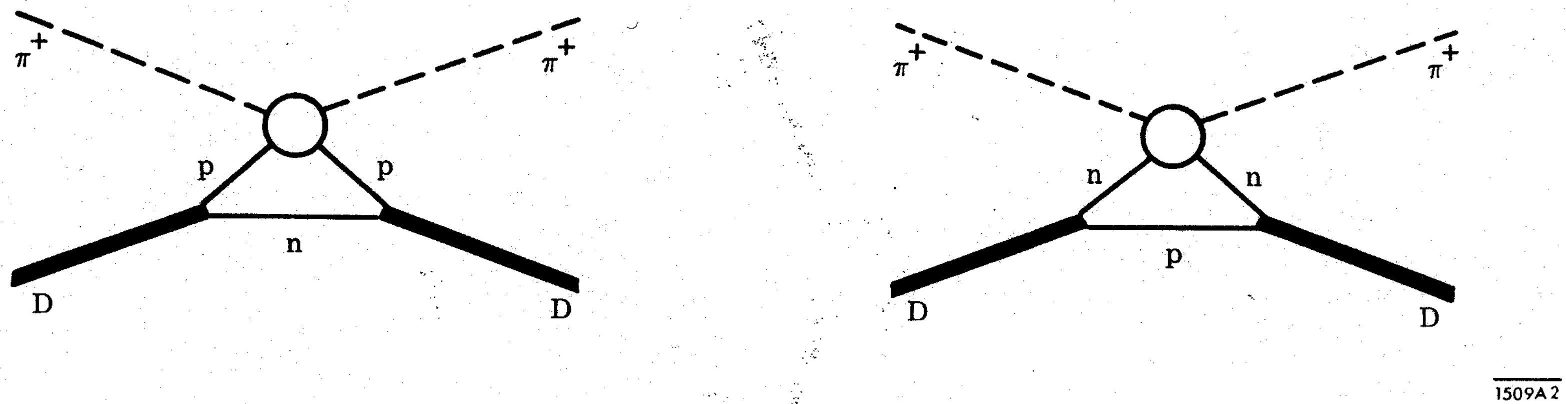
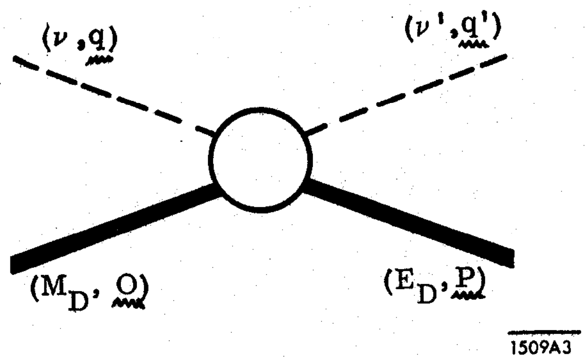
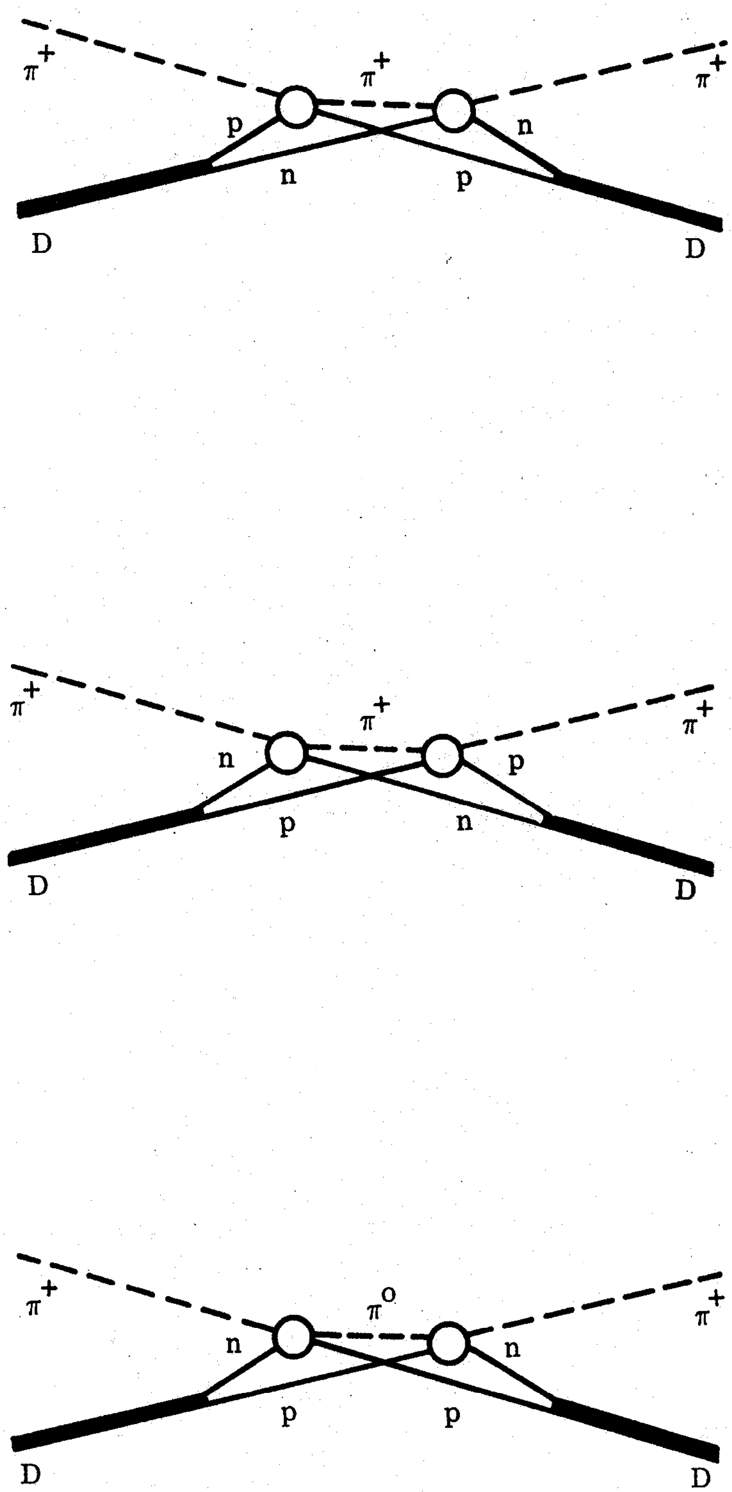


Fig. 2



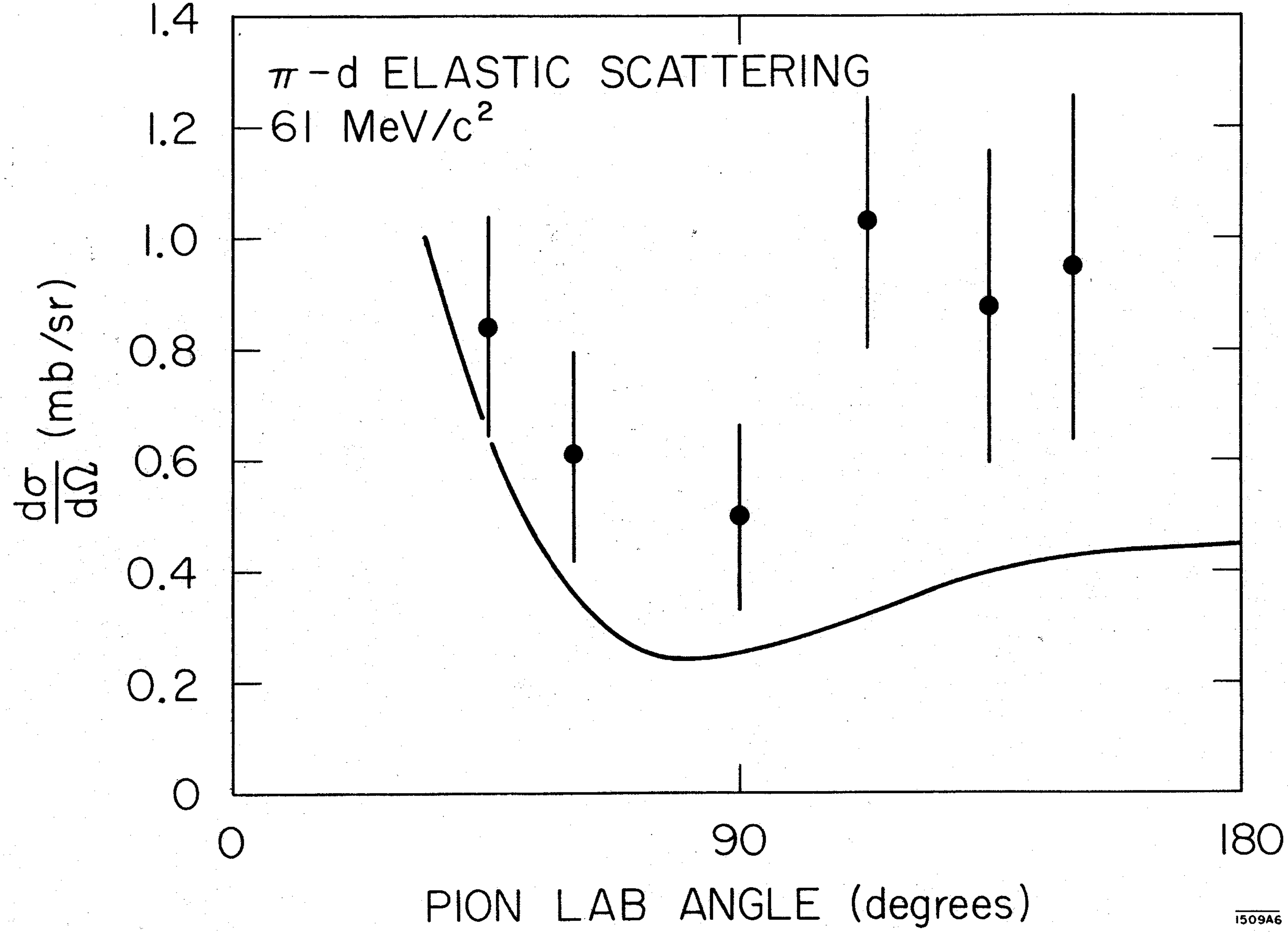
$(\nu, \underset{\sim}{q})$
$(\nu', \underset{\sim}{q}')$
$(M_D, \underset{\sim}{O})$
$(E_D, \underset{\sim}{P})$
1509A3

Fig. 3



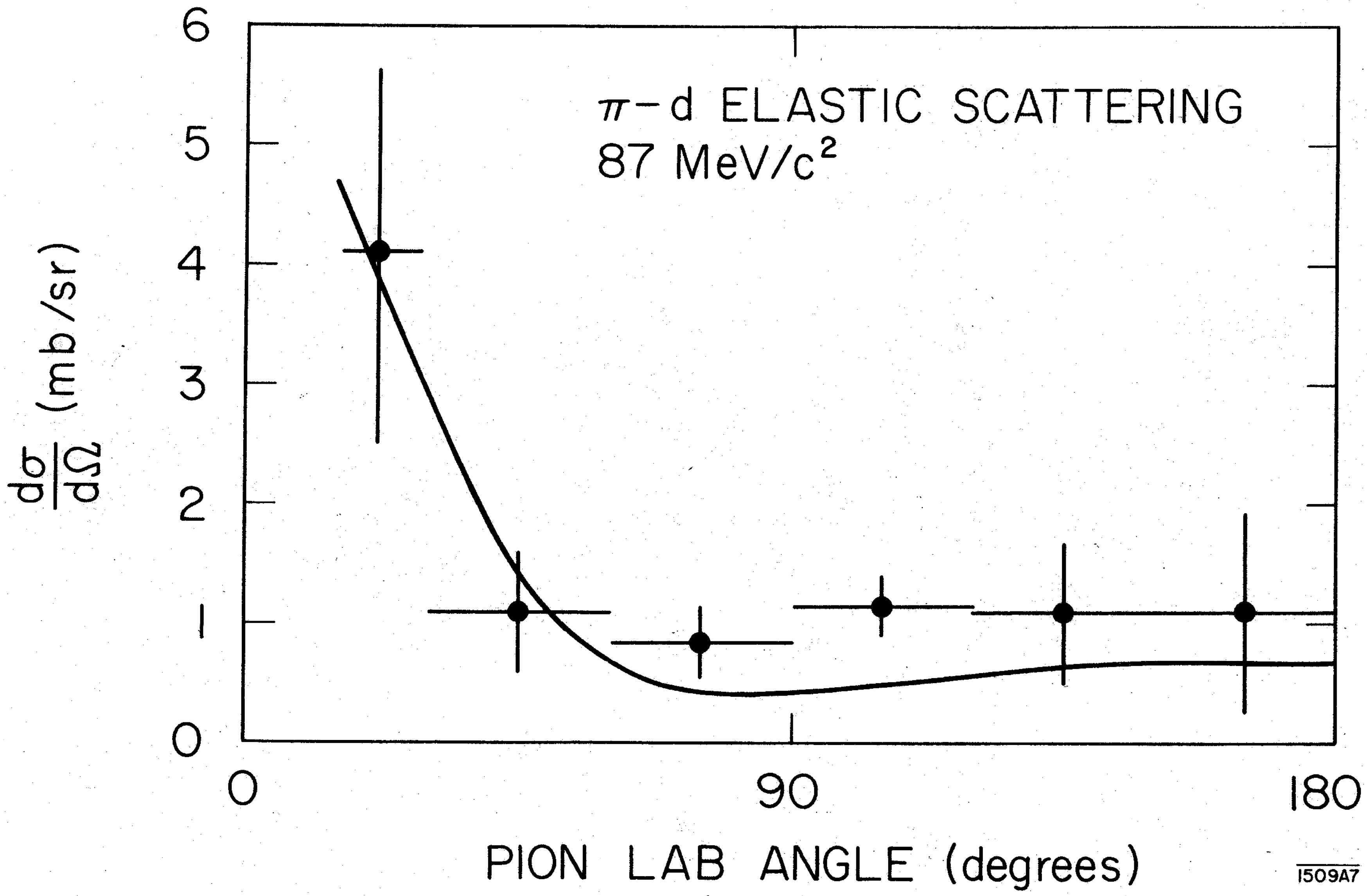
1509A4

Fig. 4



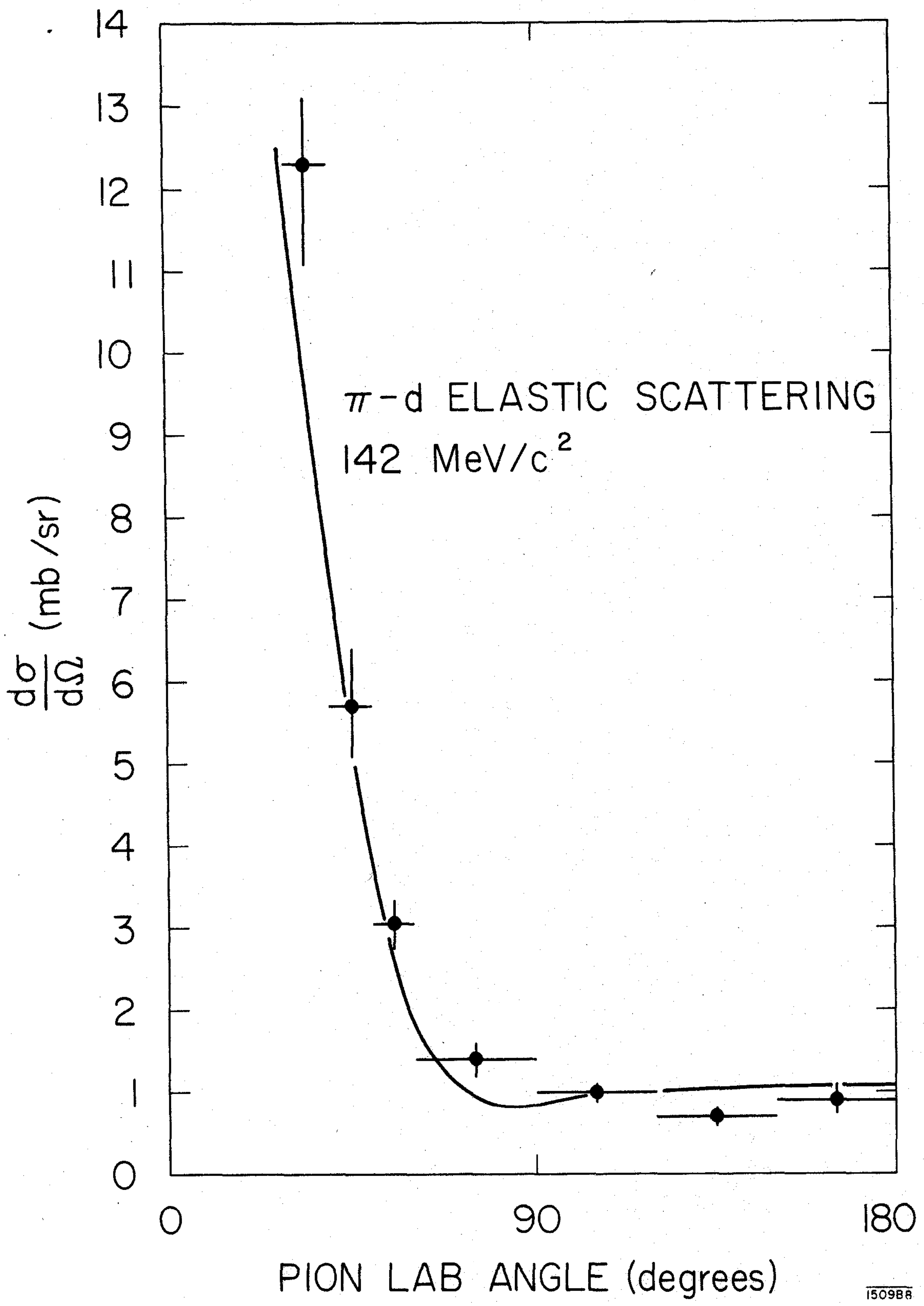
π-d ELASTIC SCATTERING
61 MeV/c²
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
dσ/dΩ (mb/sr)
0
90
180
PION LAB ANGLE (degrees)
1509A6

Fig. 5



$\pi-d$ ELASTIC SCATTERING
87 MeV/c^2
$\frac{d\sigma}{d\Omega}$ (mb/sr)
0
1
2
3
4
5
6
0
90
180
PION LAB ANGLE (degrees)
1509A7

Fig. 6



π - d ELASTIC SCATTERING
142 MeV/c^2
$\frac{d\sigma}{d\Omega}$ (mb/sr)
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
0
90
180
PION LAB ANGLE (degrees)
1509B8

Fig. 7

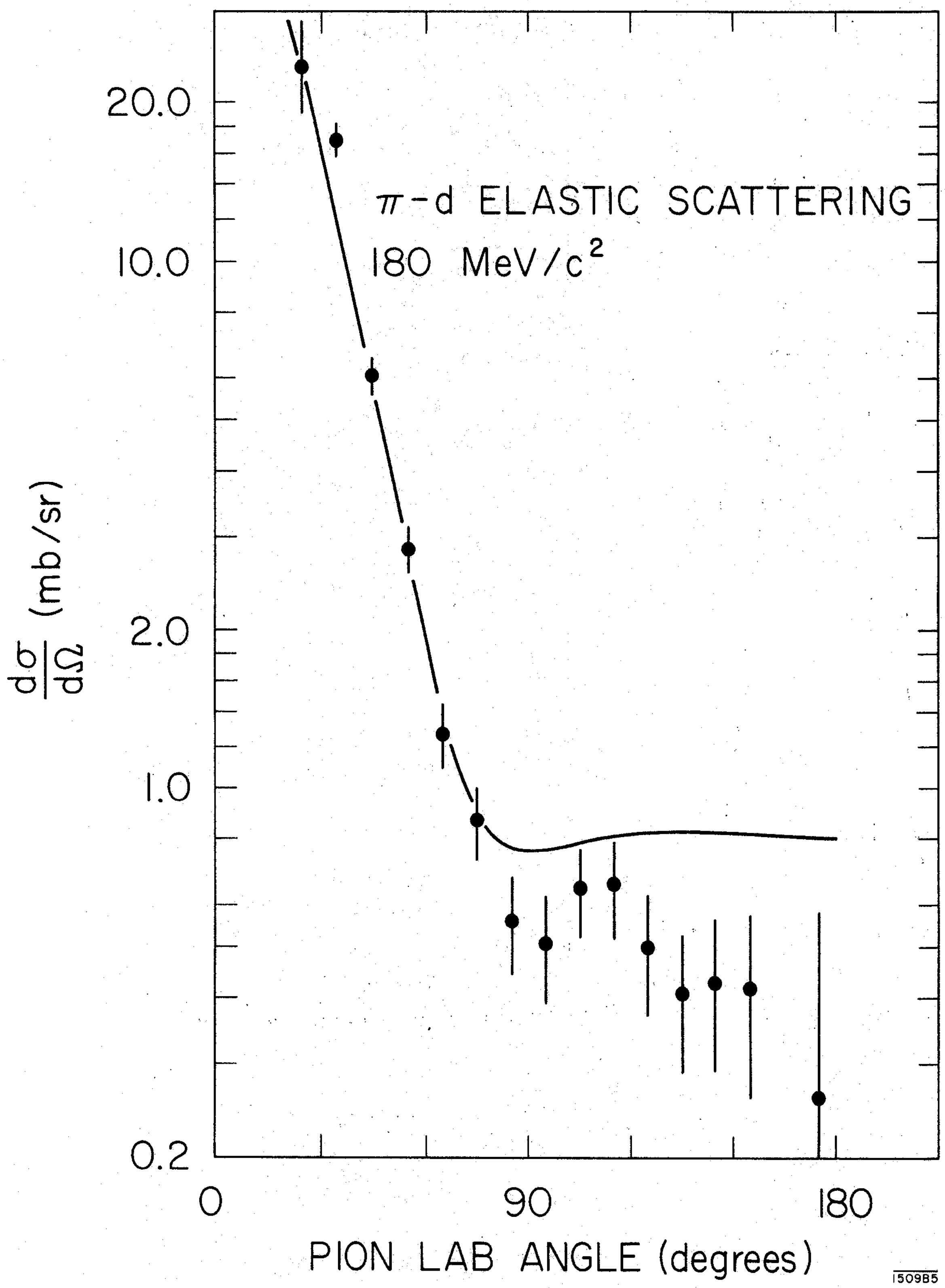

π-d ELASTIC SCATTERING
180 MeV/c²
$\frac{d\sigma}{d\Omega}$ (mb/sr)
20.0
10.0
2.0
1.0
0.2
0
90
180
PION LAB ANGLE (degrees)
1509B5

Fig. 8

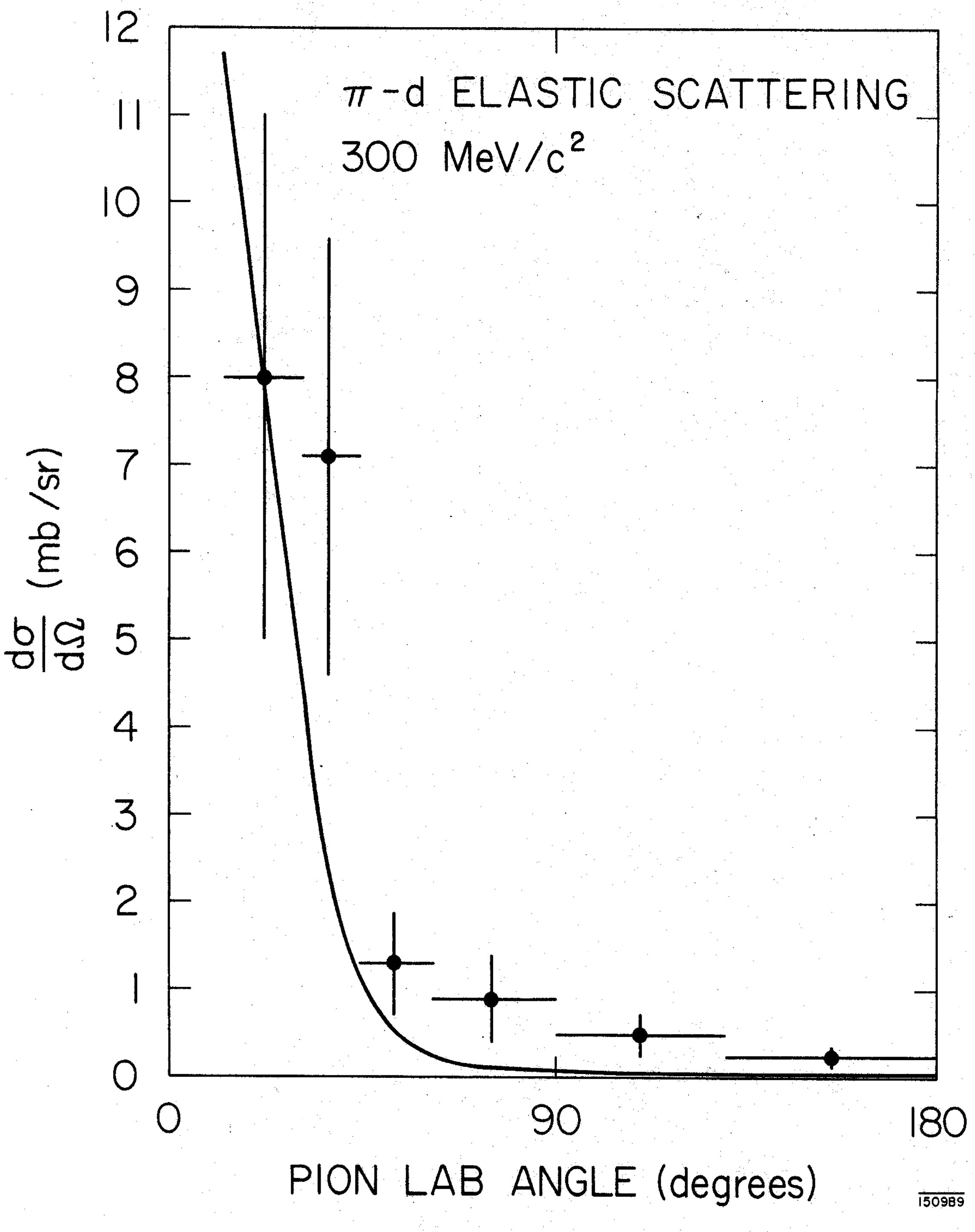

π-d ELASTIC SCATTERING
300 MeV/c²
$\frac{d\sigma}{d\Omega}$ (mb/sr)
0
1
2
3
4
5
6
7
8
9
10
11
12
0
90
180
PION LAB ANGLE (degrees)
150989

Fig. 9