SELF-CONSISTENT STRONG-COUPLING MODEL OF THE NUCLEON II*

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ABSTRACT

The static source of the old strong coupling theory has been replaced by a fermion of finite bare mass. The coupled equations of motion for the fermion and pion field are solved in a fully self-consistent way on the assumptions that the states are simple products of fermion and field states and the fermion is non-relativistic. A spectrum of nucleon excitations is found which contains negative parity states and Regge recurrences and which can be brought into rough agreement with the actual spectrum by the proper choices of fermion bare mass and fermion-field coupling constant. We also derive a representation for the strongly coupled pion field in terms of the pion number operator basis. This representation is not unlike that developed by Glauber for the treatment of coherent electromagnetic fields. A detailed analysis of the results of the model is presented which shows that several of the approximations we have used are inconsistent. A number of possibilities for improving the model are suggested.

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I. INTRODUCTION

In a recent paper[1] we presented a first attempt at incorporating a recoiling source into the old strong coupling model of the nucleon.[2] The basic constituents of the model were a heavy point fermion with a spin of 1/2 and the neutral pseudo-scalar meson field. The only free parameters in the system were the bare mass, m_0, of the fermion and the coupling constant g. The pion mass was taken to have its experimental value.

Solutions to the above model were obtained under the following set of assumptions:

1. The fermion remains non-relativistic both in the kinematic sense and in the sense that virtual fermion pair states can be consistently neglected.

2. The eigenstates of the Hamiltonian can be represented as simple products of fermion and field states (the "independent particle" assumption).

3. The strong coupling approximations in the form originally used by Pauli and Dancoff[2] are valid. This implies that the pion field can be constructed out of only p-wave pions and that the field strength executes only small zero point oscillations about some large constant value.

In addition to the above three assumptions which are quite important and whose consistency is essential to the validity of our results, three additional simplifications were made to make the model more tractable and its exposition more concise. These were as follows:

4. Isotopic spin was neglected and the assumption made that the basic result of Pauli and Dancoff (i.e., that T = J gives the bound states) would continue to hold.

5. Only spherically symmetric ($\ell = 0$) fermion wave functions were considered. This was a technical problem rather than a limitation in principle.

6. The basic interaction between the fermion and pion fields was taken to be

$$H_I = -g \int d^3x \ \psi^\dagger(x) \underset{\sim}{\sigma} \cdot \hat{r} \psi(x) \ \phi(x),$$

(hereafter referred to as the "non-gradient" model), as opposed to the probably more realistic

$$H_I = \frac{g}{2m_0} \int d^3x \ \psi^\dagger(x) \underset{\sim}{\sigma} \psi(x) \cdot \underset{\sim}{\nabla} \phi(x),$$

(i.e., the "gradient" model). Again this was done to simplify the algebra, but the implication was that there were no crucial differences between the two interactions. This is not true, and the physical implications of the two models are quite different.

It is the purpose of this paper to extend the calculations of I and examine in much greater detail the nature and validity of the first three assumptions. In particular we will show that the Pauli-Dancoff strong-coupling formulation is not successful in the fully self-consistent, non-gradient model and that the assumption of non-relativistic kinematics and the neglect of pair states are also unjustified.

On the positive side we will derive methods for overcoming the restriction to $\ell = 0$ core states (see #5 above), for expanding the self-consistency calculation to include the field strength as well as the core wave functions (see #3 above), and for calculation of matrix elements between different states of the system. This last technique will involve a derivation of the representation of the field eigenstates on the number operator basis which is quite instructive and highly reminiscent of similar representations derived by Glauber[3] for the description of the fields in a laser beam.

The result of the above development will be a nucleon resonance spectrum which for certain values of m_0 and g is a not too unreasonable facsimile of the actual spectrum. It is similar in that it contains both positive and negative parity resonances, it contains resonances in which the total angular momentum can be different from the isospin (i.e., "Regge recurrences") and the ordering of the states is not wildly different from that of the real spectrum.

However, the values of m_0 and g needed to get reasonable values for the energies are such that assumptions 1 - 3 are badly inconsistent. In other words, we find that in the non-gradient model g and m_0 must both be rather large to get the nucleon energy down below that of its excitations. This in turn leads to a "kinetic" energy in the field Hamiltonian which is comparable in magnitude with the field binding energy but opposite in sign. This results in the invalidation of the "small quantum oscillation" assumption. It also makes the energy of any given state the result of the subtraction of two large numbers of about the same size. This is highly inaccurate and very sensitive to small changes in the parameters.

In summary, the purpose of this paper is to demonstrate the inconsistencies of the non-gradient model introduced in I, but in so doing to expand and clarify the technical tools used for the model and to gain more physical insight into the nature of the states. These tools and insights will be valuable when we examine other formulations of the model in subsequent papers.

We will close this introduction by mentioning a point which is generally either ignored or only casually referred to as a potential problem in describing the spectrum of the nucleon. It is an unpleasant fact of life that the widths of nucleon resonances are in all cases comparable to or substantially larger than the spacings between levels (see Fig. 1). This is to be contrasted with the situation in atomic or nuclear physics where the width to spacing ratios are of the order of 10^{-5} or less. It seems quite naive to expect that we will find a neat and precise model of the nucleon spectrum. Such a model cannot be realistic. In fact a successful model of this spectrum will have to be, in a sense, as imprecise as the system itself seems to be. For obvious reasons, we hesitate to advertise our model as the most imprecise and therefore the most realistic of those proposed to date, but as we will see in the following, the strong coupling treatment of the pion field seems to have a certain amount of well defined "imprecision" built in.

This feature of the meson field has been examined in the past by Schiff[4] and Yennie[5] and recently by Walecka.[6] These authors have examined the problem of linearizing a non-linear boson field Hamiltonian to describe small quantum oscillations about a large, constant classical or c-number background field. This program is always plagued by the fact that for non-linear boson fields of large intensity the quantum fluctuations are also large. There seems to be no way to avoid this.

Our problem is non-linear in the sense that the pion field is coupled to itself through the self-consistency with the fermion field. In other words the fermion variables could in principle be eliminated from the pion field equations leaving an effective non-linear Hamiltonian. So we expect many of the same troubles encountered by the above mentioned authors. Our point is that these "troubles" may be just what are needed in the model to make it conform to reality. We do not minimize, however, the difficulty of doing believable calculations with such a model. This problem remains to be solved.

This paper will be organized as follows: In Section II we present the expanded self-consistency calculation and include core states in which $\ell \neq 0$. In Section III we give some results of this calculation and illustrate how the spectrum is affected by changes in the parameters g and m_0. We also discuss the various contributions to the energy and how they compare with what our approximations would lead us to expect.

In Section IV we will derive the representation of the field states on the pion number basis and show how this can be used to calculate matrix elements which are difficult to get in any other way.

In Section V we will examine in detail our approximations and show how and why they break down. Section VI will include a summary of conclusions and a

brief discussion of the possibilities of reformulating the strong coupling problem
to make it more realistic.

II. FULL SELF-CONSISTENCY

As in I we begin with the Hamiltonian:

$$H = \int d^3x\, \psi^\dagger(\underset{\sim}{x}) \left(m_0 + \frac{p^2}{2m_0} \right) \psi(\underset{\sim}{x}) + \frac{1}{2} \int d^3x \left[\pi^2(\underset{\sim}{x}) + \phi(\underset{\sim}{x})\, (-\nabla^2 + \mu^2)\, \phi(\underset{\sim}{x}) \right]$$

$$-g \int d^3x\ \psi^\dagger(\underset{\sim}{x})\, \underset{\sim}{\sigma} \cdot \hat{r}\ \psi(\underset{\sim}{x})\ \phi(\underset{\sim}{x}) \tag{1}$$

and we begin our variational calculation with the "independent particle" trial state vector

$$|\Psi> = |\psi_{n\ell m_\ell}> |\phi_{jm_j}> \tag{2}$$

These two statements involve several approximations, which we now enumerate:

1. Non-relativistic kinematics for the fermion.

2. Non-gradient psuedoscalar coupling.

3. One-particle fermion states.

4. The fermion spatial wave function and the field eigenstate (which includes the fermion spin) are uncorrelated in the sense that each is determined by averaging the other over time and space. This is in close analogy with ordinary Hartree-Fock methods for atoms and nuclei.

We will reserve comment on these approximations for Section V. We now consider the the set of coupled equations:

$$\frac{1}{2\ell+1} \sum_{m_\ell=-\ell}^{\ell} <\psi_{n\ell m_\ell}|H|\psi_{n\ell m_\ell}> |\phi_{jm_j}> = E|\phi_{jm_j}> \tag{3a}$$

$$\frac{1}{2j+1} \sum_{m_j=-j}^{j} <\phi_{jm_j}|H|\phi_{jm_j}> |\psi_{n\ell m_\ell}> = E|\psi_{n\ell m_\ell}> \tag{3b}$$

In both of the above we have taken spherical averages to ensure that the field has a spherically symmetric source (Eq. 3a) and that the fermion moves in a spherically symmetric potential (Eq. 3b). These assumptions lead to degenerate multiplets in $J = \ell + j$, and they can be checked for consistency by calculating the splittings in perturbation theory. (See Section V)

We begin with Eq. (3a). Using Eq. (1) for H and neglecting all fermion pair states we see that the eigenvalue problem for the field becomes:

$$\left\{ \frac{1}{2} \int d^3x \left\{ \pi^2 + \phi(-\nabla^2 + \mu^2)\phi \right\} - g\underset{\sim}{\sigma} \cdot \int d^3x \left[\frac{1}{2\ell+1} \sum_{m_\ell} \psi^*_{n\ell m_\ell}(\underset{\sim}{x}) \psi_{n\ell m_\ell}(\underset{\sim}{x}) \right] \hat{r} \, \phi(\underset{\sim}{x}) \right\} \left| \phi_{jm_j} \right> $$

$$= E_\phi \, \phi_{jm_j} > \tag{4}$$

where we have dropped the expectation value of the first term of H.

We define the source density for the field equation by

$$\rho(\underset{\sim}{x}) = \frac{1}{2\ell+1} \sum_{m_\ell} \psi^*_{n\ell m_\ell}(\underset{\sim}{x}) \psi_{n\ell m_\ell}(\underset{\sim}{x}) \tag{5}$$

and we see immediately that it is spherically symmetric. We can also see from Eq. (4) what happens if we do not make $\rho(\underset{\sim}{x})$ spherically symmetric. In this case the product $\psi^*_{\ell m_\ell}(\underset{\sim}{x}) \psi_{\ell m_\ell}(\underset{\sim}{x})$ causes all even values of angular momentum from 0 to 2ℓ to appear in $\rho(x)$. When this is multiplied by $\hat{r}$ and integrated with $\phi(x)$, portions of all odd multipoles from 1 to $2\ell + 1$ are projected out of $\phi(x)$.

There is nothing in principle preventing us from solving this larger problem. The self-consistency problem would have to be expanded to include f, h, ... wave pions and with sufficient labor this could be done. For a beginning, however, we will restrict ourselves to spherically symmetrized sources and therefore to only p-wave pions.

We now follow the same derivation as in I (see Eqs. (54-74) and arrive at the separated Hamiltonian:

$$H = \frac{R}{2} \left| \hat{q}(\hat{q} \cdot \underset{\sim}{\pi}) \right|^2 + \frac{L^2}{2Tq^2} + \frac{q^2}{2N} - g\underset{\sim}{\sigma} \cdot \underset{\sim}{q} + \frac{1}{2} \int d^3x \left\{ \pi'^2(\underset{\sim}{x}) + \phi'(\underset{\sim}{x})(-\nabla^2 + \mu^2) \phi'(\underset{\sim}{x}) \right\}$$

$$- \frac{1}{q^2} q(q \cdot \pi) \cdot \int d^3x \, \pi'(x) \, \hat{r} \, \rho(\underset{\sim}{x}) . \tag{6}$$

Still following the development of I we diagonalize the $\underset{\sim}{\sigma} \cdot \underset{\sim}{q}$ term and assume that we can measure the eigenvalue of H relative to the zero value of the "free" pion part. The last term is assumed to be small.

At this point we depart from the derivation in I and instead of assigning the value $q_0^2 = (gN)^2$ to the q^2 which appears in the centrifugal barrier term, we include this term in the eigenvalue problem as it stands. The "radial" part of the field equation is therefore:

$$\left\{ -\frac{R}{2} \left[\frac{d^2}{dq^2} + \frac{2}{q} \frac{d}{dq} \right] + \frac{(j+1/2)^2}{2Tq^2} + \frac{q^2}{2N} - gq \right\} f(q) = E_\phi f(q) . \quad (7)$$

This can also be written as follows:

$$\left\{ -\frac{R}{2} \left[\frac{d^2}{dq^2} + \frac{2}{q} \frac{d}{dq} - \frac{\delta(\delta+1)}{q^2} \right] + \frac{q^2}{2N} - gq \right\} f(q) = E_\phi f(q) \quad (8)$$

where

$$\delta(\delta+1) = \frac{(j+1/2)^2}{RT} \quad (9)$$

forms an effective centrifugal barrier for the field amplitude equation.

Equation (8) is solved numerically after the parameters R, N, T are calculated from the fermion source density (see Eqs. (67), (68), (71) of I). The resulting field wave function has the form[7]

$$< \underset{\sim}{q} |U| \phi_{jm_j} > = f_{n_q j}(q) \sqrt{\frac{2j+1}{4\pi}} D^{(j)}_{m_j + 1/2}(\alpha \beta 0) \quad (10)$$

where n_q is the radial quantum number for the field amplitude vibrations. The other result of this calculation is, of course, the eigenvalue E_ϕ.

This completes the solution of Eq. (3a). We now use this solution in Eq. (3b) to derive the fermion Schrödinger equation. Neglecting all fermion pair states we can reduce the fermion field equation to a single particle Schrödinger equation by

standard techniques and we get:

$$\left\{ m_0 + \frac{p^2}{2m_0} - g \frac{1}{2j+1} \sum_{m_j} \langle \phi_{jm_j} \Big| \underset{\sim}{\sigma} \cdot \hat{r} \, \phi(\underset{\sim}{x}) \Big| \phi_{jm_j} \rangle \right\} \psi(x) = E_\phi \psi(x). \quad (11)$$

This time we have dropped the expectation value of the free pion field Hamiltonian.

We now follow the derivation of Eq. (79-83) of I except that this time the value $q_0 = gN$ is replaced by $\langle q \rangle$ where

$$\langle q \rangle = \int dq \, q^3 \, f^2_{n_q j}(q) \quad (12)$$

So the fermion Schrödinger equation is

$$\left\{ - \frac{\nabla^2}{2m_0} - \frac{g \langle q \rangle}{3} \, \xi(\underset{\sim}{x}) \right\} \psi(\underset{\sim}{x}) = (E_f - m_0) \, \psi(\underset{\sim}{x}). \quad (13)$$

Using the $\xi(x)$ determined from Eq. (65) of I we solve Eq. (13) numerically for the fermion wave functions. These have the form

$$\langle \underset{\sim}{x} \Big| \psi_{n\ell m_\ell} \rangle = R_{n\ell}(r) \, Y_{\ell m_\ell}(\Omega) \quad (14)$$

where the $R_{n\ell}(r)$ are the solutions of

$$\left\{ \frac{1}{2m_0} \left[- \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} \right] - \frac{g \langle q \rangle}{3} \, \xi(r) \right\} R_{n\ell}(r) = (E_f - m_0) \, R_{n\ell}(r). \quad (15)$$

These fermion wave functions are then used to determine a new set of values of R, N, T and in turn these are used to get a new solution to the q-problem. This process is repeated until it converges.

The total energy of the product state is given by

$$E = \langle \psi | H | \psi \rangle \quad (16)$$

$$= \langle \phi, \psi \, | H | \, \phi, \psi \rangle. \quad (17)$$

From Eq.'s (1), (8), (13) we know that:

$$E_\phi = \langle \phi \, \Big| H_\phi + H_I \Big| \phi \rangle$$

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and

$$E_f = <\psi \left| H_f + H_I \right| \psi >$$

which gives

$$E_\phi + E_f = <\phi, \psi \left| H_\phi + H_f + 2H_I \right| \phi, \psi >. \tag{18}$$

This means that

$$E = E_\phi + E_f - <\phi, \psi \left| H_I \right| \phi, \psi > \tag{19}$$

which is easily shown to be

$$E = E_\phi + E_f + g <q>. \tag{20}$$

So as a result of our fully self-consistent solution we have the complete wave function (Eq.'s (10),(14)) and the energy (Eq. (20)) of a state with any given set of fermion and field quantum numbers. We now examine some numerical results.

III. SPECTRA

In Fig. 1 we show the known spectrum of nucleon resonances. We show only the "well established" ones as of August 1969.[8] Referring to Table I we see that the known $T = 1/2$ spectrum can almost entirely be accounted for by the set of quantum numbers shown.[9] The only discrepancy is the presence of a $(3/2)^+$ in Table I which does not appear in the spectrum. We note that Greenberg lists a possible $(3/2)^+$ with a very large width at 1900 MeV.

The $T = 3/2$ spectrum of 6 states is somewhat overdescribed by the states listed in Table I. The spectrum is lacking two $(3/2)^+$ states and a $(5/2)^-$. Again we note that there are candidates for all three of these listed in the "possible" category by Greenberg.

Finally there is the famous $T = 5/2$, $(5/2)^+$ resonance, one of the principle rocks upon which the strong coupling model has so far foundered. There is disagreement as to whether this state has or has not been seen,[10] but there are several reasons not to let the absence of this state discourage one from exploring the strong-coupling model:

1. The prediction of the static strong coupling model for the mass of this resonance may be greatly in error, and the mass may be large enough to have prevented its being seen as yet. Although, as we will see, our numbers are unreliable in many respects, we do consistently find the $T = 5/2$ resonance at very high masses.

2. The resonance may have a very large width causing it to be nearly indistinguishable from the background.

3. The resonance may be hard to produce by the standard techniques of photoproduction or pion–nucleon inelastic scattering.

In principle our theory enables us quantitatively to test the above possibilities. If we should find that, indeed, the $T = 5/2$ resonance has a mass within the low resonance region (i.e., around 1700 MeV as predicted by the static model), and if its width is reasonably narrow, and if there are no inhibitions to its production, then we might reasonably begin to worry about the absence of the $T = 5/2$ resonance as a challenge to the validity of the model.

Our assumption of spherical symmetry in the self-consistency calculation leads to the degeneracies indicated in Table I. It is amusing to compare these with the observed spectrum. For $T = 1/2$ we predict a $(1/2)^-$, $(3/2)^-$ doublet and there is a good candidate at (1525, 1515). Another $(1/2)^-$, $(3/2)^-$ doublet appears at (1715, 1755). We predict a $(3/2)^+$, $(5/2)^+$ doublet but our spectrum lacks one member. If we use the tentative 1900 MeV resonance we get (1900, 1690). Finally we predict a $(5/2)^-$, $(7/2)^-$ doublet and experimentally we have (1675, 2190). So if these are the correct assignments, then our model must account for only a small splitting in the first two doublets but a quite substantial one in the last two.

For $T = 3/2$ we predict a $(1/2)^-$, $(3/2)^-$, $(5/2)^-$ triplet for which the last member is missing. If we use the possible $(5/2)^-$ from Greenberg we get (1630, 1670, 1950). Our quartet of $(1/2)^+$, $(3/2)^+$, $(5/2)^+$, $(7/2)^+$ has three of its four members observed, and, depending on which of the tentative P_{33}'s we pick, we get (1905, 1690 or 2000, 1880, 1940).

As we will see in Section V our assumption of spherical symmetry is not at all borne out by our numerical results. In other words we predict multiplet splittings (for the $T = 3/2$ spectrum) which are as large as or greater than the spacings between multiplets. The data, however, seem to be quite suggestive of a system of multiplets, especially the $T = 3/2$ spectrum. We can only hope that our model when put into one of its other possible forms (e.g., gradient model) will give us back the near degeneracy within multiplets which the data suggests.

In Figs. 2 and 3 we show the behavior of some representative states as the parameters m_0 and g are changed. In Fig. 4 we show the spectrum for g = 33 and m_0 = 9 superimposed on the observed spectrum.

In Fig. 2 we have fixed m_0 at 9 pion masses and varied the coupling constant from 27 to 33. The most striking behavior is that of the ground state for each value of T. These fall dramatically as the coupling is increased. This drop is a result of the increasing concentration of the fermion wave function near the origin (see Fig. 5). This results in a highly concentrated probability density which in its role as source for the pion field produces a field which is very intense in the volume near the origin. This field is essentially the potential in which the fermion moves so the result is a particle lying very low in a deep, narrow potential well.

For the field equation (Eq. (8)) we see that the quantity R^{-1} acts like a mass and N^{-1} like a spring constant in the oscillator equation (see Fig. 5b). As the fermion wave function becomes more concentrated at the center all three of the parameters R, N, T increase, but R (the integral of the squared probability density of the fermion) increases most rapidly. Notice also that as T increases the centrifugal barrier is reduced. Finally g is being increased so the binding term becomes more and more negative.

The result of all this is a balancing of competing effects with the net tendency being to lower the field energy. The lowering of the centrifugal barrier and the increased binding are the lowering effects. The decreasing mass and increasing spring constant contribute to increasing the field energy. The increase in the spring constant turns out to be a rather mild effect so the combination of the centrifugal and binding effects is enough to overcome the decreasing mass (especially since it is the square root of the mass which is relevant) and decrease the field energy as g increases.

So we see that as g increases the total energy of the states should decrease, and it should decrease most for those states which allow the fermion to have a concentrated probability distribution. The states with $\ell > 0$ have an additional centrifugal barrier which forces the wave function to be more spread out (see Fig. 6a, b), but there is still some lowering of the energy as g increases. The states with $n > 0$, however, seem to be very insensitive to changes in g because the combined constraints of normalization and n radial nodes always force the higher n wave functions to spread out considerably (see Fig. 7a, b). Finally, in Fig. 8a, b we show the results for the $\Delta_{3/2, 3/2}$ resonance. The extra centrifugal barrier in the field Hamiltonian causes the field energy to increase and because the effective mass in the field equations is so small, the effect is too large and the Δ lies much too high in energy.

It is interesting to compare the graphs for the field problems in the order, Figs. 8b, 5b, 6b, 7b, where we have listed the figures in the order of increasing extension of the fermion wave function. As the fermion spreads out the constants R, N, T all decrease and the field potential becomes narrower and steeper (T and N) and the effective mass becomes larger (R). The eigenvalue does not change nearly as fast as the separate parameters do because, as we have mentioned above, these changes in the parameters have opposing effects on the eigenvalue. The value of $\langle q \rangle$ is substantially affected, however.

The set of solutions shown in Fig. 4 comes as close as we can reasonably get to the actual spectrum. It is quite apparent that there are still serious discrepancies between the calculated and observed spectra for this model. In order to get the ground state to lie below the excited states, g must be quite large. But no matter how large we make g we cannot get the first radial excitation (i.e., the Roper 1460) to lie below the rotational excitations of the core fermion.

There is one possibility of fixing this, which depends on the fact that the

n = 1 and n = 2 radial excitations are close together (indeed the n = 2 state lies

below the n = 1). Since our states are not orthogonal to each other (in effect they

are solutions to different Hamiltonians) there will be mixing between levels with

the same angular wave functions and they will repel each other. In the next section

we will develop the techniques necessary for calculating off-diagonal matrix elements

of the Hamiltonian so that in principle this mixing can be calculated. However, as

we will see below, the results of this model are bad in so many other respects that

it seems rather pointless to add such a refinement now.

One possibility remains to be discussed. There is no reason to exclude

"radial" excitations in the q variable. Referring to Figs. 5b, 6b, etc., we see

that this would imply finding an $f(q)$ with one radial node. We have looked for and

found such solutions, but because of the very tight binding (in particular the large

value of R which corresponds to a small mass in the oscillator equation) this

radially excited state has an extremely large energy. These states are therefore

far above the region which is depicted in Figs. 1 and 4 and not of great interest as

yet. There are also reasons to expect that these states will be very broad (see Ref. 1).

There are more details of the spectra to be examined, but these are better

treated in a discussion of the validity of the approximations, and we put this off

until Section V.

IV. NUMBER OPERATOR REPRESENTATION

Before examining in greater detail the results of Section III we will derive a representation of the field states which will allow us to calculate matrix elements between different states and also estimate the errors we have made in performing the separation of the field Hamiltonian into bound and free parts. In fact this technique will lead us to an entirely new approach to the self-consistency problem which shows some promise of being more realistic and reliable than the method described in this paper. This new approach will be explored in a later paper.

We begin with the interaction part of the Hamiltonian after the fermion coordinates have been spherically averaged and integrated out. The interaction term is then

$$H_I = -g \underset{\sim}{\sigma} \cdot \int d^3x \, \rho(x) \, \hat{r} \, \phi(x) , \tag{21}$$

and as usual we define

$$\underset{\sim}{q} = \int d^3x \, \rho(x) \, \hat{r} \, \phi(x) , \tag{22}$$

where $\underset{\sim}{q}$ is an axial-vector operator representing the strength of the overlap of the p-wave part of the pion field with the source density.

We are trying to solve the eigenvalue problem

$$(H_\phi + H_I) \left| \phi_{jm_j} \right> = E_\phi \left| \phi_{jm_j} \right> , \tag{23}$$

and we now assume that the eigenstates of $\underset{\sim}{q}$ form a complete set, i.e., span the entire space of possible pion field states. This, of course, is not true. There are pions of all partial waves besides $\ell = 1$ which can be created and destroyed, and there are p-wave pions whose radial wave functions are orthogonal to $\rho(x)$. The strong-coupling approach neglects these states, and previously we have done this by splitting the field and its conjugate momentum into two pieces, viz

$$\phi(x) = \underset{\sim}{q} \cdot \hat{r} \, \xi(x) + \phi'(x)$$

$$\pi(x) = \underset{\sim}{\pi} \cdot \hat{r} \, \rho(x) + \pi'(x) \tag{24}$$

and then assuming that all terms containing $\phi'(x)$ and $\pi'(x)$ could be neglected. The alternative we propose now is to avoid separations like those of Eq. (24) and assume directly that the states $|q'\rangle$ form a complete basis for the description of the pion field.

With this assumption we can write Eq. (23) as

$$\int d^3 q' \langle q | (H_\phi + H_I) | q' \rangle \langle q' | \phi_{jm_j} \rangle = E_\phi \langle q | \phi_{jm_j} \rangle , \qquad (25)$$

which is now an eigenvalue equation for the q-space wave function of the field

$$\int d^3 q' \, H(q, q') \, \phi_{jm_j}(q') = E \, \phi_{jm_j}(q) . \qquad (26)$$

If we know $H(q, q')$ then we can solve this as we would any Schrödinger equation in coordinate space.

In the model discussed in the first three sections we actually had such a representation (see Eq. (8)):

$$\langle q | H_\phi + H_I | q' \rangle = \delta^3(q - q') \left\{ \frac{R}{2} \left[-\frac{d^2}{dq^2} - \frac{2}{q} \frac{d}{dq} + \frac{\delta(\delta+1)}{q^2} \right] + \frac{q^2}{2N} - gq \right\} \qquad (27)$$

which was the result after all terms containing $\phi'(x)$ and $\pi'(x)$ had been dropped. We will now show how an analogous, but quite different, expression for $H(q, q')$ can be derived without ever making the bound-free separation.

We begin with the requirement that the states $|q'\rangle$ be eigenstates of the operator q:

$$q | q' \rangle = q' | q' \rangle . \qquad (28)$$

This represents, of course, three separate and independent eigenvalue equations for the three components of q, e.g.,

$$q_x | q' \rangle = q_x' | q \rangle . \qquad (29)$$

Henceforth we will deal with the q_x problem alone, noting that obviously the q_y and q_z derivations will be identical to it. Referring to Eq. (22) we have

$$q_x = \int d^3x \; \hat{r}_x \rho(x) \; \phi(x) \tag{30}$$

and if we expand $\phi(x)$ in partial waves

$$\phi(x) = \sum_{\ell, m} \int \frac{dk \; k}{\sqrt{\pi\omega}} \; j_\ell(kr) \left[a_{\ell m}(k) \; Y_{\ell m}(\Omega) + a_{\ell m}^\dagger(k) \; Y_{\ell m}^*(\Omega) \right] \tag{31}$$

we can do the spatial integration and get:

$$q_x = \int dk \, \rho(k) \left[a_x^\dagger(k) + a_x(k) \right] , \tag{32}$$

where

$$\rho(k) = \frac{2k}{\sqrt{3\omega}} \int dr \; r^2 \rho(r) j_1(kr) \tag{33}$$

and

$$a_x(k) = \frac{1}{\sqrt{2}} \left(-a_{1+1}(k) + a_{1-1}(k) \right) . \tag{34}$$

Except for the factor of $\omega^{-1/2}$ in Eq. (33) we see that $\rho(k)$ is proportional to the coefficient in an expansion of $\rho(r)$ on the complete set of $j_1(kr)$ (i.e., complete on the interval $0 \leq k \leq \infty$). The factor of $\omega^{1/2}$ comes from the expansion of $\phi(x)$ and ensures that the $a(k)$ operators have the simple commutation properties

$$\left[a_i(k), \; a_j^\dagger(k') \right] = \delta_{ij} \delta(k-k') . \tag{35}$$

The next step is to expand the state $|q_x'\rangle$ in terms of the number operator representation. Let

$$|q_x'\rangle = \sum_{n=0}^{\infty} \int dk_1 \ldots dk_n \; \alpha_n(k_1 \ldots k_n) |k_1 \ldots k_n\rangle \tag{36}$$

where

$$|k_1 \ldots k_n\rangle = \frac{1}{(n!)^{1/2}} \; a_x^\dagger(k_1) \, a_x^\dagger(k_2) \ldots a_x^\dagger(k_n) |0\rangle . \tag{37}$$

Bose statistics tell us that the $\alpha_n(k_1 \ldots k_n)$ must be completely symmetric in their arguments. The number states in Eq. (37) are clearly an orthonormal and complete set on the subspace of p-wave pions polarized in the x-direction.

Now inserting Eqs. (36) and (32) into (29) we get

$$\sum_{n=0}^{\infty} \int dk_1 \ldots dk_n \int dk \rho(k) \alpha_n(k_1 \ldots k_n) \left(a_x^{\dagger}(k) + a_x(k)\right) \big| k_1 \ldots k_n \big>$$

$$= q_x' \sum_{n=0}^{\infty} \int dk_1 \ldots dk_n \alpha_n(k_1 \ldots k_n) \big| k_1 \ldots k_n \big> . \tag{38}$$

Using the orthonormality of the states in Eq. (37) and the symmetry of the α's we can project out of (38) the coefficient of the general state with n pions:

$$(n+1)^{1/2} \int dk \rho(k) \alpha_{n+1}(k, k_1 \ldots k_n) + \frac{1}{n^{1/2}} \sum_{i=1}^{n} \rho(k_i) \alpha_{n-1}(k_1 \ldots (i) \ldots k_n)$$

$$= q_x' \alpha_n(k_1 \ldots k_n) \tag{39}$$

where the symbol "(i)" in the second term means that k_i is missing from the argument.

A solution to this equation is easily obtained by induction:

$$\underline{n=0} \quad \int dk \, \rho(k) \alpha_1(k) = q_x' \alpha_0 \, , \tag{40}$$

$$\underline{n=1} \quad \sqrt{2} \int dk \rho(k) \alpha_2(k, k_1) = q_x' \alpha_1(k_1) - \alpha_0 \rho(k_1) \, , \tag{41}$$

$$\underline{n=2} \quad \sqrt{3} \int dk \rho(k) \alpha_3(k, k_1, k_2) = q_x' \alpha_2(k_1, k_2) - \frac{1}{\sqrt{2}} \left[\rho(k_1) \alpha_1(k_2) + \rho(k_2) \alpha_1(k_1) \right] . \tag{42}$$

From these equations, and the requirement of symmetry of the α's we can see by inspection that the simplest solution has the form:[11]

$$\alpha_n(k_1 \ldots k_n) \sim \rho(k_1) \rho(k_2) \ldots \rho(k_n) \, . \tag{43}$$

Using this in Eq. (40) we find that

$$\alpha_1(k) = \frac{\alpha_0}{\overline{R}} \, q_x' \rho(k), \tag{44}$$

where

$$\tilde{R} = \int dk \, \rho^2(k).$$ (45)

Equation (41) now becomes

$$\sqrt{2} \int dk \, \rho(k) \, \alpha_2(k, k_1) = \alpha_0 \left(\frac{q_x'^2}{\tilde{R}} - 1 \right) \rho(k_1) \,,$$ (46)

and if $\alpha_2(k, k_1)$ has the form (43) it must be

$$\alpha_2(k, k_1) = \frac{\alpha_0}{\sqrt{2} \, \tilde{R}} \left(\frac{q_x'^2}{\tilde{R}} - 1 \right) \rho(k) \, \rho(k_1) \,.$$ (47)

If we now define the parameter

$$x' = \frac{q_x'}{\tilde{R}^{1/2}}$$ (48)

we can write the general α_n in the form

$$\alpha_n(k_1 \ldots k_n) = \frac{\alpha_0}{(n! \, \tilde{R}^n)^{1/2}} \, \rho(k_1) \ldots \rho(k_n) \, P_n(x'),$$ (49)

where the $P_n(x')$ are polynomials of order n in x' which can be shown by means of Eq. (39) to satisfy the recursion relation

$$P_{n+1}(x') = x' P_n(x') - n P_{n-1}(x').$$ (50)

But these are closely related to the Hermite polynomials which satisfy

$$H_{n+1}(x) = 2x \, H_n(x) - 2n H_{n-1}(x),$$ (51)

and it is well known that[12]

$$P_n(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}} \right) \,.$$ (52)

Therefore, if we define

$$x = \frac{q_x'}{(2 \, \tilde{R})^{1/2}}$$ (53)

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we can write (49) as

$$\alpha_n(k_1 \ldots k_n) = \frac{\alpha_0}{(n! \, 2^n \widetilde{R}^n)^{1/2}} \, H_n(x) \, \rho(k_1) \ldots \rho(k_n) \, . \tag{54}$$

We now insert (54) into (36) and get the full expansion for the basis states q'_x:

$$|q'_x> = \alpha_0 \sum_{n=0}^{\infty} \frac{H_n(x)}{(2^n n!)^{1/2}} \frac{\left(a^{\dagger}_{\rho_x}\right)^n}{(n!)^{1/2}} |0> \, , \tag{55}$$

where we have defined

$$a^{\dagger}_{\rho_x} = \frac{1}{\widetilde{R}^{1/2}} \int dk \, \rho(k) \, a^{\dagger}_x(k) \, . \tag{56}$$

Finally we determine α_0 by demanding that

$$<q''_x | q'_x> \; = \delta(q''_x - q'_x) \, . \tag{57}$$

This leads directly to the requirement that

$$\alpha_0^2 \sum_n \frac{H_n(x'')H_n(x')}{2^n n!} = \frac{1}{(2\widetilde{R})^{1/2}} \, \delta(x'' - x') \, , \tag{58}$$

where we have used (53) to convert the δ-function. But the sum in (58) can be done explicitly because we know the expansion of the δ-function in terms of the complete set of harmonic oscillator wave functions. A one-dimensional oscillator wave function is[13]

$$<x|n> = \psi_n(x) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\beta}{\pi}\right)^{1/4} e^{-1/2\beta x^2} H_n(\sqrt{\beta} x), \tag{59}$$

where $\beta = m\omega$. If we let $\beta = 1$ and use the completeness property of these wave functions we get

$$\sum_n \frac{H_n(x) \, H_n(x')}{2^n n!} = \pi^{1/2} \, e^{x^2} \, \delta(x - x') \, . \tag{60}$$

Applying this to Eq. (58) we have

$$\alpha_0 = \frac{e^{-x^2/2}}{(2\pi\widetilde{R})^{1/4}} \, . \tag{61}$$

Our final result is therefore

$$|q'> = |q'_x> \, |q'_y> \, |q'_z>$$ (62)

where each of the factors has the form of Eq. (55).

We can now proceed to use the expansion (55) to calculate the matrix element of H in (25). The matrix element of H_I is, of course, trivial since the entire representation was built around this term. We have

$$<q' \, |H_I|q> = -g\sigma \cdot q \, \delta^3(q - q').$$ (63)

The matrix element of H_ϕ is more complicated but reduces to a simple form. We first express H_ϕ as a sum of partial wave terms

$$H_\phi = \frac{1}{2} \int d^3x \left\{ \pi^2(x) + |\nabla\phi(x)|^2 + \mu^2 \phi^2(x) \right\} = \sum_{\ell,m} \int dk \, \omega \, a^\dagger_{\ell m}(k) \, a_{\ell m}(k)$$ (64)

where we have dropped the zero point energy as usual. Forming the bracket $<q'|H_\phi|q>$ and using the commutation relation

$$\left[a_{\ell i}(k), \, a^\dagger_{\rho_j} \right] = \frac{\rho(k)}{\widetilde{R}^{1/2}} \, \delta_{\ell 1} \delta_{ij}$$ (65)

we find that

$$<q'|H_\phi|q> = Q \sum_{n_x, n_y, n_z = 0}^{\infty} \frac{n H_{n_x}(x) H_{n_x}(x') \dots H_{n_z}(z) H_{n_z}(z')}{2^n \, n_x! \, n_y! \, n_z!} \exp\left\{ -\frac{q^2 + q'^2}{4\widetilde{R}} \right\}$$ (66)

where

$$n = n_x + n_y + n_z$$

and

$$Q = \frac{1}{(2\pi)^{3/2} \widetilde{R}^{5/2}} \int dk \, \omega \, \rho^2(k).$$ (67)

A straightforward calculation shows that the integral in Eq. (67) is proportional to the quantity R defined in Eq. (67) of I. In fact

$$R = 2 \int dk \, \omega \, \rho^2(k),$$ (68)

so

$$Q = \left(\frac{R}{2\widetilde{R}}\right) \frac{1}{(2\pi\widetilde{R})^{3/2}} \quad . \tag{69}$$

Referring to Eq. (66) we see that the presence of the n in the numerator of the sum means that (66) is the sum of three terms, each of which has two factors which are precisely of the form of Eq. (60), and one factor which differs by a factor of n in each term of the sum:

$$\langle \underset{\sim}{q'} | H_\phi | \underset{\sim}{q} \rangle$$

$$= Q \exp\left\{-\frac{q^2+q'^2}{4\widetilde{R}}\right\} \left\{ \sum_{n_x} \frac{H_{n_x}(x) H_{n_x}(x')}{2^{n_x}(n_x-1)!} \left(\pi^{1/2} e^{y^2/2} \delta(y-y')\right) \left(\pi^{1/2} e^{z^2/2} \delta(z-z') + \text{cycl.}\right) \right\}. \tag{70}$$

Converting the arguments of the δ-functions back to q_y and q_z we get

$$\langle \underset{\sim}{q'} | H_\phi | \underset{\sim}{q} \rangle$$

$$= \frac{1}{(2\pi\widetilde{R})^{1/2}} \left(\frac{R}{2\widetilde{R}}\right) \left\{ \exp\left[-\frac{q_x^2+q'^2_x}{4\widetilde{R}}\right] \sum_n \frac{H_n(x) H_n(x')}{2^n(n-1)!} \delta(q_y-q'_y) \delta(q_z-q'_z) + \text{cycl.} \right\}. \tag{71}$$

So the problem reduces to the summation of the series in (71).

For this we use the following identity.[14] Let

$$f(\eta, x, x') = \frac{1}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{\eta^n H_n(x) H_n(x')}{2^n n!} \exp\left[-\frac{1}{2}(x^2+x'^2)\right]. \tag{72}$$

Then

$$f(\eta, x, x') = \frac{1}{\pi^{1/2}(1-\eta^2)^{1/2}} \exp\left\{\frac{2xx'\eta - \frac{1}{2}(x^2+x'^2)(1+\eta^2)}{1-\eta^2}\right\} . \tag{73}$$

It is relatively straightforward to show by taking limits in the proper order that

$$\lim_{\eta \to 1} \int_a^b dx' \, F(x') \, f(\eta, x, x') = \begin{cases} F(x) & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases} .$$

Therefore, as we expect

$$\lim_{\eta \to 1} f(\eta, x, x') = \delta(x - x') \tag{74}$$

We now notice that the sum in Eq. (71) can be written as

$$\frac{1}{\pi^{1/2}} \exp\left(-\frac{x^2 + x'^2}{2}\right) \sum_n \frac{H_n(x) H_n(x')}{2^n (n-1)!} = \lim_{\eta \to 1} \frac{\partial}{\partial \eta} f(\eta, x, x'). \tag{75}$$

The same technique can be used to prove

$$\lim_{\eta \to 1} \frac{\partial}{\partial \eta} f(\eta, x, x') = \left(2x^2 - \frac{1}{2} \frac{\partial^2}{\partial x^2}\right) \delta(x - x'). \tag{76}$$

Finally, using (53) again to convert the argument and noting that the three terms of the matrix element of H_ϕ can trivially be added we have

$$\langle \underset{\sim}{q'} | H_\phi | \underset{\sim}{q} \rangle = \frac{R}{2} \left[-\nabla_q^2 + \frac{q^2}{\tilde{R}^2} \right] \delta^3(\underset{\sim}{q} - \underset{\sim}{q'}). \tag{77}$$

The similarity in form with Eq. (27) is striking, but we see that the coefficients are quite different. The constants N and T which appear in (27) no longer appear here, and only the old constant R and the new one $\tilde{R}$ remain.

The implications of Eq. (77) are great. First it means that we can check the validity of the separation performed in the derivation of Eq. (27). But even more important we see that the ability to evaluate this matrix element exactly opens the way to an entirely new self-consistency problem in which the bound-free separation need never be made. We will explore this new formulation in a later paper.

We now demonstrate one more very useful property of the expansion (55). We will in the future wish to calculate matrix elements of the Hamiltonian and other operators between different states. But each state is characterized by its own core density $\rho(r)$, so the $\underset{\sim}{q}$ operators for each state span different spaces. These spaces overlap, however, and we now calculate the overlap. The transformation

function is given by

$$\langle q_{N'} | q_N \rangle$$

where N' and N stand for different nucleon states. Using Eq. (55) we find

$$\langle q_{N'} | q_N \rangle = \langle q_{N'_x} | q_{N_x} \rangle \; \langle q_{N'_y} | q_{N_y} \rangle \; \langle q_{N'_z} | q_{N_z} \rangle \tag{78}$$

which follows from the fact that the three directions are independent.

Concentrating on one of the above factors we have

$$\langle q_{N'_x} | q_{N_x} \rangle = \frac{\exp\left(-\frac{x_N^2 + x_{N'}^2}{2}\right)}{\left(4\pi^2 \widetilde{R}_N \widetilde{R}_{N'}\right)^{1/2}} \sum_n \frac{H_n(x_{N'}) H_n(x_N)}{2^n \, n!} \; \frac{\langle 0 | \left(a_{\rho'_x}\right)^n \left(a_{\rho_x}^\dagger\right)^n | 0 \rangle}{n!} . \tag{79}$$

But it is easy to show that

$$\left[a_{\rho'}, a_\rho^\dagger \right] = \frac{1}{\left(\widetilde{R}_N \widetilde{R}_{N'}\right)^{1/2}} \int dk \rho_{N'}(k) \rho_N(k) \equiv B_{N'N}. \tag{80}$$

Note that $B_{NN} = B_{N'N'} = 1$. Using (80) we get for (79)

$$\langle q_{N'_x} | q_{N_x} \rangle = \frac{\exp\left(-\frac{x_N^2 + x_{N'}^2}{2}\right)}{(4\pi^2 \widetilde{R}_N \widetilde{R}_{N'})} \sum_n \frac{B_{N'N}^n H_n(x_{N'}) H_n(x_N)}{2^n \, n!} \tag{81}$$

which is exactly the form we saw before in Eqs. (72) – (73). The series can therefore be summed to give

$$\langle q_{N'_x} | q_{N_x} \rangle = \frac{1}{(4\pi^2 \widetilde{R}_N \widetilde{R}_{N'})^{1/2}} \; \frac{1}{(1 - B_{NN'}^2)^{1/2}} \exp\left\{ \frac{2 x_N x_{N'} B_{NN'} - \frac{1}{2}\left(x_N^2 + x_{N'}^2\right)\left(1 + B_{NN'}^2\right)}{1 - B_{NN'}^2} \right\}. \tag{82}$$

Of course there are similar solutions for the other two factors in Eq. (78).

This is a rather complicated expression, but since all our calculations must

be done by computer anyway, it is not as unpleasant as it looks. For any general

matrix element we have

$$\langle \phi_{N'} | \theta | \phi_N \rangle = \int d^3 q_{N'} d^3 q_N d^3 q'_N \langle \phi_{N'} | \underset{\sim}{q}_{N'} \rangle \langle \underset{\sim}{q}_{N'} | \underset{\sim}{q}_N \rangle \langle \underset{\sim}{q}_N | \theta | \underset{\sim}{q}'_N \rangle \langle \underset{\sim}{q}'_N | \phi_N \rangle , \quad (83)$$

and if we know the $\underset{\sim}{q}$ representation of the operator θ we have all we need to calculate the off-diagonal matrix element.

We close this section with some remarks concerning the generality of the solution (54). If we return to Eq. (40) we note that Eq. (44) is not its most general solution. We could have written

$$\alpha_1(k) = \frac{\alpha_0}{R} q'_x \rho(k) + \eta(k) \quad (84)$$

where the only requirement on $\eta(k)$ is

$$\int dk \, \rho(k) \, \eta(k) = 0 . \quad (85)$$

A straightforward calculation then shows that the most general form for $\alpha_2(k_1, k_2)$ is

$$\alpha_2(k_1, k_2) = \frac{\alpha_0}{\sqrt{2} \, R} \left(\frac{q'^2_x}{R} - 1 \right) \rho(k_1) \rho(k_2) + \frac{q'_x}{\sqrt{2} \, R} \left(\eta(k_1) \rho(k_2) + \eta(k_2) \rho(k_1) \right) + \eta_2(k_1, k_2) , \quad (86)$$

where the only requirements on $\eta_2(k_1, k_2)$ are that it be symmetric and orthogonal to $\rho(k)$.

Since the requirement of being orthogonal to $\rho(k)$ is quite unrestrictive we see that there are an infinite number of possible solutions for the α's, and that we have written down only the simplest one in Eq. (54). It seems clear that this ambiguity is analogous in some way to the ambiguity inherent in the separation of the field operators into bound and free parts. Presumably with the proper choices of the functions $\eta_1(k)$, $\eta_2(k_1, k_2)$, ..., etc., we could reproduce the result of Eq. (27), but we do not intend to attempt this here.

In conclusion, the result of this section has been to transform the ambiguity of the bound-free separation into a different form. The advantage of this method over the Pauli-Dancoff method is that we can derive explicit expressions for matrix

elements and therefore evaluate the effects of different $\eta(k)$, whereas we had no reliable way to calculate the matrix elements of $\pi'(x)$ and $\phi'(x)$ in the Pauli-Dancoff representation (see Section V. C).

V. CHECKS OF APPROXIMATIONS

In this section we return to our results of Section III and examine the degree to which they are consistent with the approximations we have made. We will find some rather disturbing inconsistencies which will lead us to the conclusion that the non-gradient model presented in this paper is inadequate in its present form. In the last section we will present some possible alternatives.

A. Non-Relativistic Kinematics

Before we do any quantitative calculations we can see very quickly that we are in trouble on this assumption. Referring to Fig. 5a which gives the ground state wave function and recalling that the bare mass of the fermion in this case is 9 pion masses we note that the binding energy is nearly 8 times as large as the mass. The average height above the bottom of the well is about 17 pion masses meaning that the average kinetic energy is roughly twice the mass.

Another relevant observation is the radius of the probability density. We notice that in the ground state the fermion is confined almost completely within a region of radius .1 pion Compton wavelength. But this is almost exactly the Compton wavelength of the fermion itself, so we expect that the formation of virtual pairs will not be negligible.

The approximation of non-relativistic kinematics has been checked quantitatively. We have calculated the expectation value of the third and fourth terms in the expansion

$$E = (p^2 + m^2)^{1/2} = m + \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3} + \frac{1}{16}\frac{p^6}{m^6} - \cdots \tag{87}$$

and compared their sum to the bare mass of the fermion and the kinetic energy of the fermion. The results for four of our states are given in Table II. One look at this table is enough to convince us that our assumption of non-relativistic kinematics

is patently ridiculous. Even in the best case, the Roper excitation, the fact that the relativistic correction is positive shows that the fourth term of (87) is larger than the third term.

There is very little more to be said about this problem. We must either find a formulation of the model in which the binding need not be so strong or go to a Dirac equation for the fermion.

B. Spherical Averaging

We have been working with states which are products of fermion states with a given ℓ and field states with a given j. Since the solution of each separate eigenvalue problem is obtained by spherically averaging over the coordinates of the other system the net result is an energy which depends only on the values of ℓ and j and not on their vector sum. We now proceed to test this assumption by returning to the basic Hamiltonian of Eq. (1) and calculating the "fine-structure." If the splittings are small we can feel safe with our procedure, but if they are large, as indeed they will be, we are led to question the consistency of this approximation and look for a more valid one.

Even at the outset we should be apprehensive. We have seen that the expectation value of the interaction term in Eq. (1) is very large in our results. Indeed it is the essence of a strong coupling approximation that this term shall dominate the energy. But all the contribution to the fine structure splitting comes from this term, so it seems almost a priori inconsistent to enforce spherical symmetry for the basic equations. An alternative approach which incorporates the fine structure into the self-consistency problem from the beginning is possible and will be briefly mentioned at the end of this subsection.

We now consider the matrix element of

$$H_I = -g \int d^3x \; \psi^\dagger(x) \; \underset{\sim}{\sigma} \cdot \hat{r} \; \psi(x) \; \phi(x) \tag{88}$$

in a state with given values of ℓ, j, and J. We evaluate this matrix element in the "body-fixed frame"[15] so that the transformation U (see I Eqs. (24) – (27)) must be applied to H_I first. As we have seen in I the effect of this transformation is to change $\underset{\sim}{\sigma}$ to $\hat{q}$ and reduce the two-component spinor equation to a one-component equation. Our matrix element is now

$$-g \ \langle \Psi_{n\ell jJM} | \int d^3x \ \psi^\dagger(x) \ \hat{q} \cdot \hat{r} \ \psi(x) \ \phi(x) | \Psi_{n\ell jJM} \rangle \ , \tag{89}$$

where

$$\langle \underset{\sim}{q}, \underset{\sim}{r} | \Psi_{n\ell jJM} \rangle = R_{n\ell}(r) \ f_j(q) \sum_{m,m'} \langle \ell m j m' | \ell j JM \rangle Y_{\ell m}(\Omega) (-1)^{m'-1/2} \ C^{(j)}_{m'+1/2}(\Gamma) \ .$$

The field angular wave function is[7]

$$C^{(j)}_{m'+1/2}(\alpha\beta 0) = \sqrt{\frac{2j+1}{4\pi}} \ D^{(j)}_{m'+1/2}(\alpha\beta 0) \ . \tag{90}$$

The next step is to assume that the expectation value of $\phi'(x)$ (see Eq. (24)) is zero and that the expectation value of $\phi(x)$ in the state in question is given by $\underset{\sim}{q} \cdot \hat{r} \xi(x)$. With this assumption, the matrix element (89) breaks up into three distinct factors:

$$\langle \Psi_{n\ell jJ} | H_I | \Psi_{n\ell jJ} \rangle = -g \left[\int r^2 dr \ R^2_{n\ell}(r) \xi(r) \right] \times \left[\int dq \ q^3 f^2_j(q) \right] \times \langle \Omega_{\ell jJ} | (\hat{q} \cdot \hat{r})^2 | \Omega_{\ell jJ} \rangle \ , \tag{91}$$

where we have suppressed the obviously irrelevant index M. The first factor in (91) is nothing more than

$$\int r^2 dr \ \rho(r) \ \xi(r) = 3 \ , \tag{92}$$

which can be seen by using Eqs. (5) and (14) of this paper and Eq. (62) of I. The second factor of (91) is just $\langle q \rangle$ (Eq. (12)) so we are left with

$$\langle \Psi_{n\ell jJ} | H_I | \Psi_{n\ell jJ} \rangle = -3g\langle q \rangle \langle \Omega_{\ell jJ} | (\hat{q} \cdot \hat{r})^2 | \Omega_{\ell jJ} \rangle \ . \tag{93}$$

The problem has now reduced to the evaluation of the angular matrix element, and this is done using standard Wigner-Eckhart theorem techniques. We write

$$(\hat{q} \cdot \hat{r})^2 = \frac{1}{3} + Q_{ij} R_{ij} \ , \tag{94}$$

where

$$Q_{ij} = \hat{q}_i \hat{q}_j - \frac{1}{3} \delta_{ij} \ , \tag{95}$$

and R_{ij} is the analogous tensor made from $\hat{r}$. Eq. (93) becomes

$$\langle \Psi_{n\ell jJ} | H_I | \Psi_{n\ell jJ} \rangle = -g\langle q \rangle - 3g \langle q \rangle \langle \Omega_{\ell jJ} | Q_{ij} R_{ij} | \Omega_{\ell jJ} \rangle$$
$$\tag{96}$$
$$= -g\langle q \rangle - 3g\langle q \rangle \ \gamma_{\ell jJ} \quad ,$$

where we recognize the first term as our spherically averaged interaction energy and the second term as the "perturbation" which will break the degeneracy.

To evaluate the second term we first note that

$$Q_{ij} R_{ij} = \frac{2}{3} \underset{\sim}{Q} \cdot \underset{\sim}{R} \tag{97}$$

where Q and R are the properly normalized irreducible tensor operators. Then using a standard formula (Ref. 7, Eq. (7.1.6)) involving the 6-j symbol we find:

$$\gamma_{\ell jJ} = \frac{2}{3} (-1)^{\ell+j+J} \begin{Bmatrix} J & j & \ell \\ 2 & \ell & j \end{Bmatrix} \langle \ell \| \underset{\sim}{R} \| \ell \rangle \langle j \| \underset{\sim}{Q} \| j \rangle \ . \tag{98}$$

The reduced matrix elements can in turn be evaluated (Ref. 7, Eq. (7.1.1)) in terms of the reduced matrix elements of $\hat{r}$ and $\hat{q}$. All of this is quite straightforward and we simply give the results:

$$<\ell \parallel \underset{\sim}{R} \parallel \ell> = - \left[\frac{\ell(\ell+1)(2\ell+1)}{3(2\ell-1)(2\ell+3)} \right]^{1/2} , \tag{99}$$

$$<j \parallel \underset{\sim}{Q} \parallel j> = -\frac{1}{2\sqrt{6}} \frac{\left(j+\frac{1}{2}\right)^2}{\left[j(j+1)\right]^{3/2}} \left[(2j+3)(2j+1)(2j-1)\right]^{1/2} . \tag{100}$$

After inserting Eqs. (99) and (100) into Eq. (98) and evaluating the 6-j symbol (Ref. 7, p. 132) we have

$$\gamma_{\ell j J} = \frac{1}{72\sqrt{2}} \left[\frac{2j+1}{j(j+1)} \right]^2 \frac{3X(X-1) - 4j(j+1)\,\ell(\ell+1)}{(2\ell-1)(2\ell+3)} , \tag{101}$$

where

$$X = j(j+1) + \ell(\ell+1) - J(J+1) . \tag{102}$$

It is easy to verify that when either $\ell=0$, or $j=1/2$, $\gamma_{\ell j J}=0$. So we predict zero splitting for all the $T=1/2$ core excitations. As we have seen in Section III this is not a bad prediction for two of the observed doublets, but it cannot account for the sizable splitting of two others. Of course our choice of states could be wrong and there might be as yet unobserved resonances which fit the predictions much better.

The only splittings occur in the $T=3/2$ spectrum (and of course also when $T > 3/2$). The $\gamma_{\ell j J}$'s for these multiplets can be expressed in the form

$$\gamma = \frac{1}{3\sqrt{2}} \left(\frac{16}{15} \right)^2 \alpha, \tag{103}$$

and the α's are given in Table III. We note that all of the above γ's are reasonably small compared to 1/3 so that the splitting is only a small fraction of the binding energy. However, as we can see from Fig. 9, where the splittings have been

incorporated into the T=3/2 spectrum, the splittings are not small compared to the separations between multiplets. This is because $g<q> = 50.0$ for the $\ell=1$ state and 28.9 for the $\ell=2$ state. In fact it is only luck which keeps the lowest lying levels from having negative energy. If the coupling were only a little stronger this would occur.

As we examine Fig. 9 we see that we can draw very little comfort from the comparison of our model with the real spectrum. It is not only the fact that the fine structure splittings are very large which is disturbing but the directions of the splittings are not consistent either. For the $\ell=1$ set the $(1/2)^-$ is pushed down, the $(3/2)^-$ up and the $(5/2)^-$ slightly down. We therefore predict the ordering $\left(\frac{1^-}{2} , \frac{5^-}{2} , \frac{3^-}{2} \right)$ whereas we observe the order $\left(\frac{1^-}{2} , \frac{3^-}{2} , \frac{5^-}{2} \right)$ if we accept the reality of the $(5/2)^-$ at 1950.

The $\ell=2$ set is even worse. Our theory predicts the order $\left(\frac{1^+}{2} , \frac{7^+}{2} , \frac{3^+}{2} , \frac{5^+}{2} \right)$ and we observe either $\left(\frac{3^+}{2} , \frac{5^+}{2} , \frac{1^+}{2} , \frac{7^+}{2} \right)$ or $\left(\frac{5^+}{2} , \frac{1^+}{2} , \frac{7^+}{2} , \frac{3^+}{2} \right)$ depending on which of the two possible $(3/2)^+$ states we use. Both of these suffer from having the $(5/2)^+$ state at the wrong end.

We conclude that our procedure of calculating the degenerate energy levels by spherically averaging the potentials and then calculating the splittings in perturbation theory is not valid. The fine structure must somehow be incorporated into the zeroth order solution. What this means in practice is that we cannot really treat ℓ and j as good quantum numbers since strong matrix elements exist which mix different ℓ, j states to give the same J. This is analogous to the trouble one encounters with Russell-Saunders coupling in atoms when the individual spin-orbit forces are beginning to become appreciable.

The only solution we can offer to this problem is to begin with states which are eigenstates of J and sums of products of ℓ and j eigenstates. These of course will be infinite sums in general, but one might hope to truncate them and get better

answers than we presently have. This will involve considerably more work since it means the abandonment of assumption 2 (Section I) and the introduction of correlations. Relativistic kinematics would also introduce a spin-orbit interaction for the fermion which would split the J-states. If in addition g were smaller, then there might be a chance that the orderings of the levels and the magnitudes of the splittings would be more realistic. As of now this is only a hope.

C. Bound-Free Separation

The results of Section IV have been used to test the validity of the Pauli-Dancoff separation of the Hamiltonian into bound and free parts. We recall that in evaluating the energies of the nucleon states we have neglected the last two terms in Eq. (6). We have done this under the following assumptions:

1. There is a zero-free-pion state with energy equal to zero which is an eigenstate of the fifth term of 6.

2. The expectation value of $\pi'(x)$ is zero in our states and the second order effect of the last term is negligible.

We can now test these assumptions by taking the field wave functions we have found from the first four terms of 6, and using the techniques developed in Section IV to calculate the expectation value of the full field Hamiltonian $H_\phi + H_I$ (see Eq. (23)). To the extent that this latter result agrees with the field energy derived from 6, our assumptions are valid.

We will not give the details of the calculation here but will outline the procedure used and summarize the results. From Eqs. (63) and (77) we have that

$$\langle q' | H_\phi + H_I | q \rangle = H(q)\ \delta^3(q-q'), \tag{104}$$

where

$$H(q) = \frac{R}{2}\left[-\nabla_q^2 + \frac{q^2}{\tilde{R}^2} \right] - g\,\sigma \cdot q. \tag{105}$$

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We use the wave functions of Eq. (10) to evaluate the matrix element

$$\langle \phi_{jm_j} | U^\dagger H(q) U | \phi_{jm_j} \rangle = \int d^3q \, \phi^*(q) \, H(q) \, \phi(q) \quad , \tag{106}$$

and we compare the result of Eq. (106) with the result of a similar calculation using Eq. (27) for $H(q)$.

The results can be summarized quite briefly. In virtually every case the difference between the two calculations of E_ϕ is comparable to the total energy of the state. Again we have found that the intense binding required to give realistic energy levels has taken us into a region of solutions in which the initial assumptions are inconsistent. However, in this case the problem may be even more fundamental. The "free" fields $\phi'(x)$ are actually very complicated and non-local, and the assumption that there exists a zero energy ground state of the Hamiltonian $H_{\phi'}$ made up of $\phi'(x)$ and $\pi''(x)$ seems to be not at all justifiable no matter what the binding energy.

We have made some attempts at analyzing in detail the nature of the eigenstates of $H_{\phi'}$, but without great success. It is clear, however, that they cannot be eigenstates of any number operator and that to speak of the state with "no free pions" (i.e., no free quanta of the ϕ' field) is incorrect. Indeed it was to make this concept more precise that the representation of Section IV was developed.

We can summarize this section by noting that each of the approximations or assumptions we tested has turned out to be grossly in error. We are forced to the conclusion that the model in its present form is unacceptable. In the next section we propose some changes in the model which might rectify some of these difficulties.

VI. CONCLUSION

We have extended the strong coupling model begun in Ref. I to include a full self-consistency calculation for the field and core eigenstates and a mechanism for including core excitations with $l > 0$. We have not changed the assumptions of non-relativistic kinematics, no virtual pairs, and independent particle wavefunctions.

We have found that in order to get a rough agreement with experiment on the magnitudes of the energies we need a very large coupling constant and a fairly large mass. An inevitable result of this is that the kinetic energy stored in the field oscillations is large and comparable to the field binding energy. The energy of a state is therefore the small difference between two large numbers and this leads to many difficulties, three of which are:

a. The assumption of non-relativistic kinematics is demonstrably inconsistent and the neglect of virtual pairs almost certainly unjustified.

b. The spherical averaging technique which makes both the core and field angular momenta good quantum numbers is demonstrably inconsistent.

c. The separation into bound and free fields originally used by Pauli and Dancoff is not reliable in our self-consistent model.

These difficulties have led us to reject the self-consistent strong coupling model in the form in which it is presented in this paper. There is still some hope, however, that the model may prove useful when suitably modified.

In subsequent papers we hope to explore the consequence of using relativistic kinematics (i.e., the Dirac equation) for the fermion, a gradient coupling between the fermion density and the field, the number operator representation for

the field states, and trial wavefunctions which include some correlations. There is also the possibility of introducing explicitly non-linear terms into the pion field Hamiltonian.

At the present writing we cannot predict whether these changes will save the strong-coupling model. In fact we cannot even predict whether self-consistent solutions will exist, since each change in the equations of motion re-opens this question.

We do believe, however, that these possibilities ought to be explored before the strong-coupling model is totally abandoned.

Finally, we must face up to the problem mentioned in the introduction. It may be true that by postulating a strong-coupling between source and field we are building in at the outset the result that the field energy will be a delicate balance between the binding energy and the energy stored in the quantum fluctuations. This is known to happen in simple non-linear theories,[4-6] and to the extent that the non-linearity in our model is similar to that in these other models, we can expect the same troubles.

As we mentioned in the introduction this result is not necessarily bad. We have already left behind the necessity for assuming an essentially classical pion field. Our field equation (either Eq. (27) or Eq. (77)) is a bona-fide quantum mechanical equation and we have solved it exactly. It may be that the large quantum fluctuations in the pion field are just the mechanism we need to account for the large widths of the resonances.

ACKNOWLEDGEMENTS

FOOTNOTES AND REFERENCES

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7. As in Reference I we use the notation of A. R. Edmonds, Angular Momentum in Quantum Mechanics, (Princeton University Press, Princeton, New Jersey 1957).

8. The data for Fig. 1 is taken from the rapporteur's review given by O. W. Greenberg at the Lund Conference on Elementary Particles, July 1969, Technical Report No. 70-071, Center for Theoretical Physics, University of Maryland, Collage Park, Maryland.

9. We are, of course, assuming that the prediction $T = j$, where j is the field angular momentum, will continue to hold. This result emerged from the old strong coupling theory and was a property of the meson field solutions. We see no reason to doubt that it will also be true in our self-consistent version.

10. A. Benvenuti, E. Marquit, and F. Oppenheimer, Phys. Rev. Letters 22, 970 (1969); J. S. Danburg et al., Phys. Rev. Letters 23, 41 (1969).

11. We will return to the question of uniqueness at the end of this section.

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13. See for example L. I. Schiff, Quantum Mechanics, III Ed. (McGraw-Hill Book Co., New York 1968); p. 71.

14. Erdelyi et al., Higher Transcendental Functions II, Bateman Manuscript Project (McGraw-Hill Book Co., New York 1953); p. 194.

15. C. J. Goebel, Non-Compact Groups in Particle Physics, Y. Chow (Ed.) (W. A. Benjamin Co., New York 1966).

LIST OF TABLES

1. Quantum number assignments for the observed nucleon resonances in the strong coupling model. The quantum numbers are defined as follows: n = radial fermion quantum number (i.e., number of radial nodes in fermion wave function); ℓ = fermion orbital angular momentum (determines parity of state); j = field angular momentum which is equal to the isospin T of the state. The last two columns give the standard spin-parity and partial wave assignments and the degeneracies have been made explicit.

2. The first three columns give the quantum numbers of the states in question and the next three list respectively, the fermion bare mass m_0, the fermion kinetic energy (i.e., the expectation value of the second term of Eq. 87), and the relativistic correction to the fermion energy (the third and fourth terms of Eq. 87). The energies are measured in pion masses.

3. The quantity α in Eq. (103) of the text is given for each of the J values possible for a given set of ℓ, j.

TABLE I

$\underline{n}$	$\underline{\ell}$	$\underline{j}$	$\underline{T}$	$\underline{J}^P$	Partial Wave
0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^+$	P_{11}
1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^+$	P_{11}
2	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^+$	P_{11}
0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-$	S_{11}, D_{13}
1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-$	S_{11}, D_{13}
0	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}^+, \frac{5}{2}^+$	P_{13}, F_{15}
0	3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}^-, \frac{7}{2}^-$	D_{15}, G_{17}
0	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}^+$	P_{33}
1	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}^+$	P_{33}
0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$	S_{31}, D_{33}, D_{35}
0	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$	$P_{31}, P_{33}, F_{35}, F_{37}$
0	0	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}^+$	not applicable

TABLE II

n	ℓ	j	m_0	KE	ΔE_{rel}	J^P
0	0	$\frac{1}{2}$	9	17.4	308.	$\frac{1}{2}^+$ (N)
0	0	$\frac{3}{2}$	9	19.3	435.	$\frac{3}{2}^+$ (Δ)
1	0	$\frac{1}{2}$	9	5.1	1.4	$\frac{1}{2}^+$
0	1	$\frac{1}{2}$	9	11.4	21.6	$\frac{1}{2}^-$, $\frac{3}{2}^-$

TABLE III

ℓ, j	$J = \frac{1}{2}$	$J = \frac{3}{2}$	$J = \frac{5}{2}$	$J = \frac{7}{2}$
1, $\frac{3}{2}$	$\frac{1}{4}$	$-\frac{1}{5}$	$\frac{1}{20}$	—
2, $\frac{3}{2}$	$\frac{1}{4}$	0	$-\frac{5}{28}$	$\frac{1}{14}$

FIGURE CAPTIONS

1. Known spectrum of $T = 1/2$ and $T = 3/2$ nucleon resonances. The widths of several of the states are shown for comparison with the level spacings.

2. Plots of the masses of some states vs the coupling constant g for fixed bare mass equal to 9 pion masses. The energies are all in MeV. We have included the $T = 5/2$ state in the $T = 3/2$ column.

3. Plots of the masses of some states vs the fermion bare mass m_0 for a fixed coupling constant $g = 31$. The energies of the states are in MeV.

4. The spectrum of states resulting from the choice of $g = 33$, $m_0 = 9$ is shown on the right in each column and the actual spectrum on the left.

5. (a) Plots of the final self consistent ground state radial wave function $\psi(r)$, probability density $\rho(r)$, and potential $V(r)$, vs r, where r is measured in pion Compton wavelengths. The energy values of $V(r)$ are measured in pion masses.

    (b) Plots of the effective potential for the field problem $V(q)$ and the final q-space wave function $f(q)$ for the ground state (i. e., nucleon) problem.

6. (a) Plots of $\psi(r)$, $\rho(r)$, and $V(r)$ for the state with $T = 1/2$ and $J^P = (1/2^-, 3/2^-)$.

    (b) Plots of $V(q)$, $f(q)$ for the same state.

7. (a) Plots of $\psi(r)$, $\rho(r)$, and $V(r)$ for the state with $T = 1/2$ and $J^P = 1/2^+$ which has energy equal to 2220 MeV in Fig. 4. This is the "Roper" resonance in the strong coupling model.

    (b) Plots of $V(q)$ and $f(q)$ for the same state.

8. (a) Plots of $\psi(r)$, $\rho(r)$ and $V(r)$ for the state with $T = 3/2$ and $J^P = 3/2^+$, the Δ_{33} resonance.

    (b) Plots of $V(q)$ and $f(q)$ for the same state.

9. Spectrum of $T=3/2$ resonances. The observed states are on the left, and on the right is the calculated spectrum in which the fine structure splitting has

been taken into account. We note that the vertical scale of this figure is expanded by a factor of two from that of Fig's. 1 and 4.
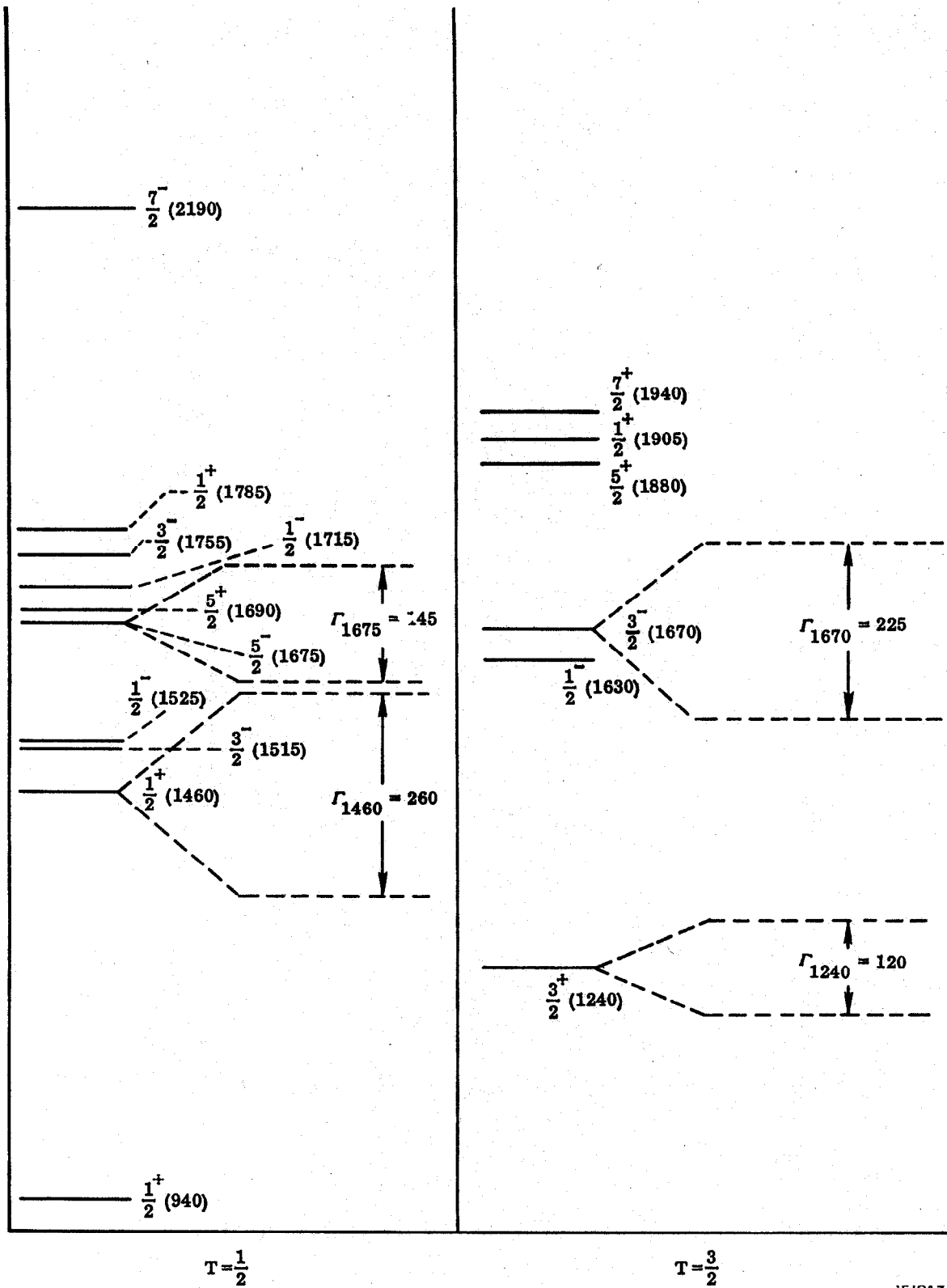
Fig. 1

Fig. 2
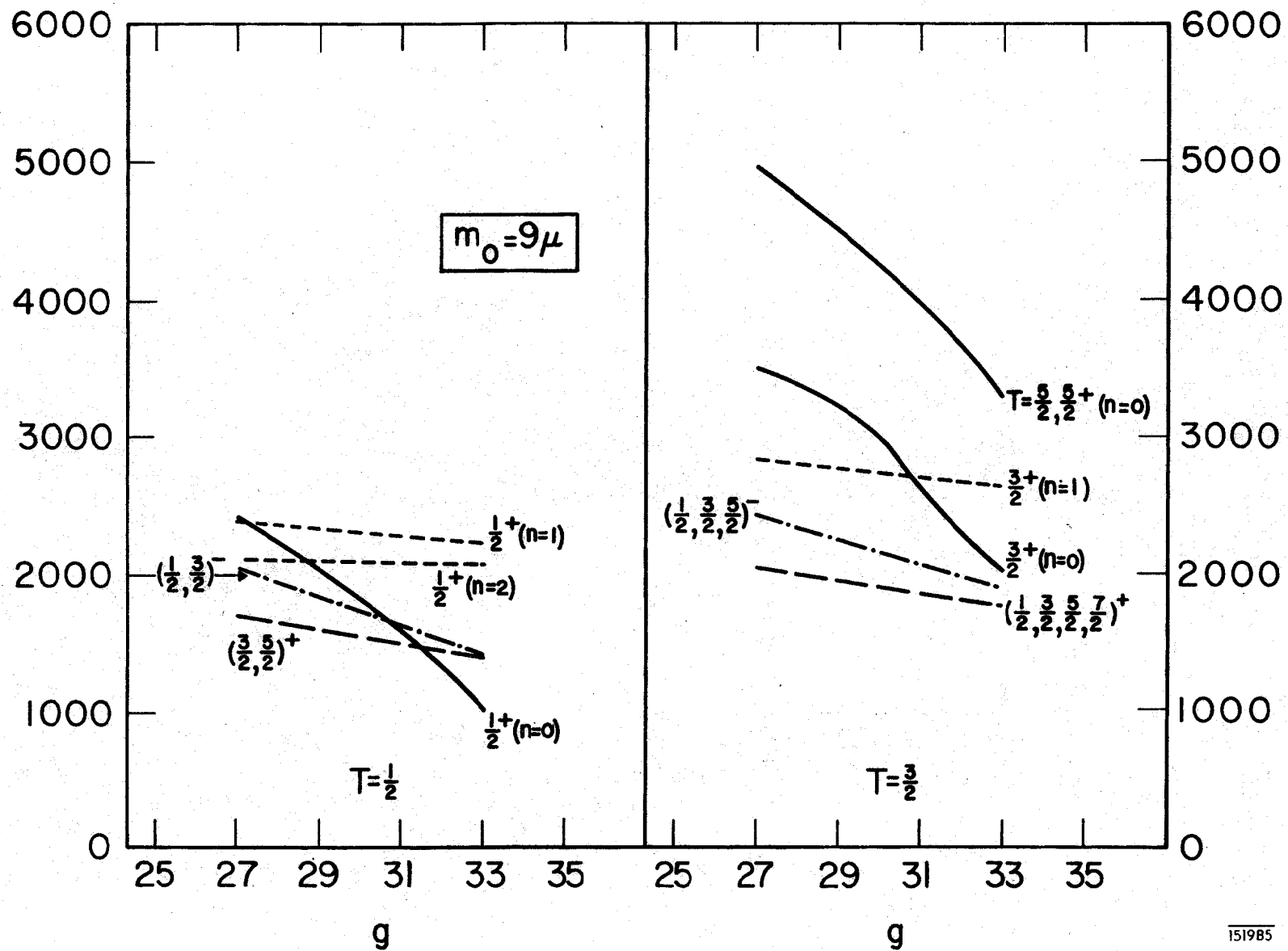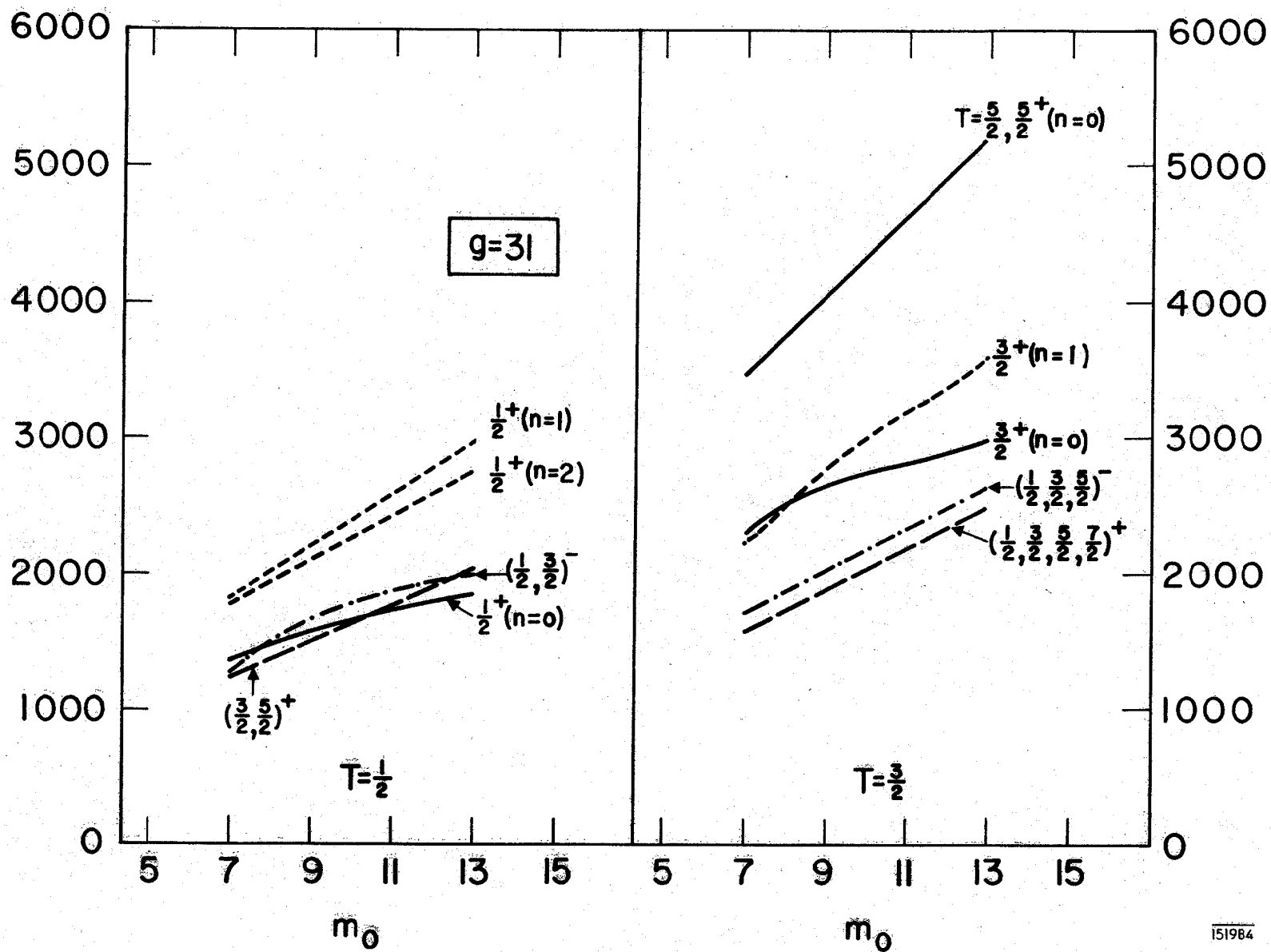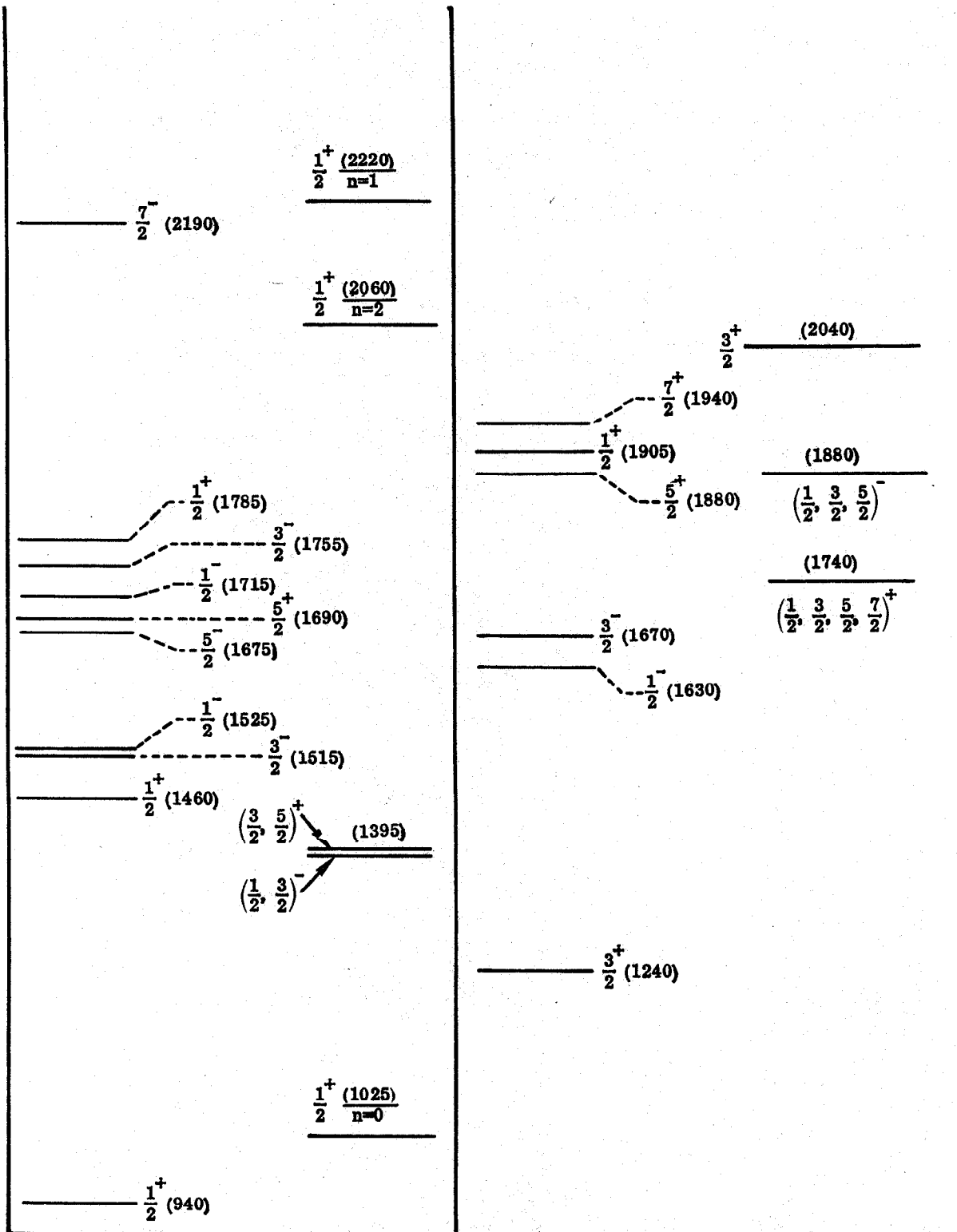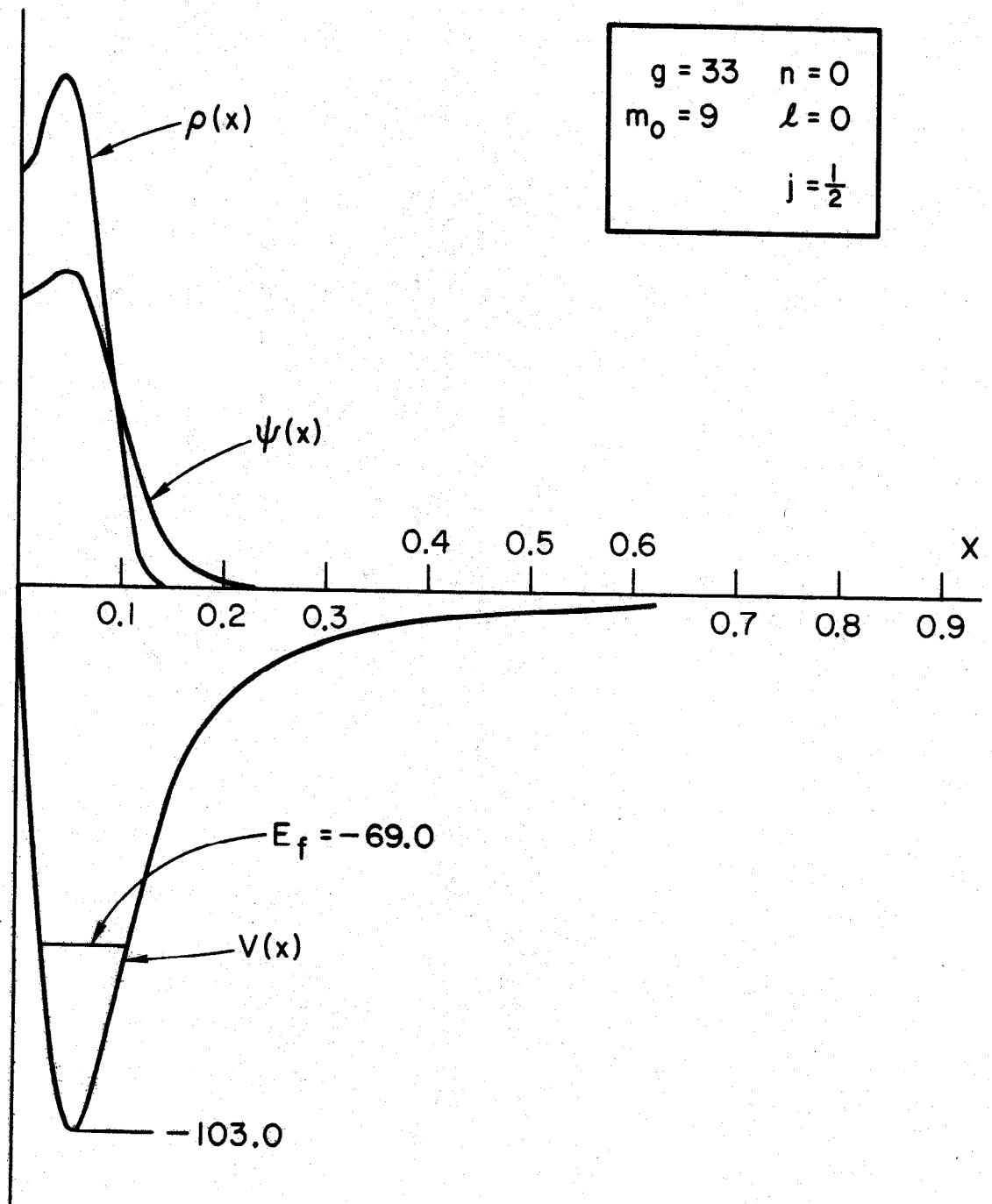
Fig. 3

Fig. 4

Fig. 5a

Fig. 5b

Fig. 6a

$\langle q \rangle = 1.35$

f(q)

V(q)

q

$E_q = -10.3$

Fig. 6b

Fig. 7a

V(q)

<q> = 0.63

f(q)

q

$E_q = -0.06$

Fig. 7b

1519C8

Fig. 8a

$V(q)$

$<q> = 2.95$

$f(q)$

$E_q = -12.6$

1519C9

Fig. 8b

$\frac{3}{2}^-$ (3000)

$\frac{3}{2}^+$ (2660)

$\frac{5}{2}^+$ (2320)

$\frac{1}{2}^+$ (1905)

$\frac{7}{2}^+$ (1940)

$\frac{3}{2}^+$ (2040)

(1880)

$\frac{5}{2}^+$ (1880)

$\frac{3}{2}^+$ (1740)

$\frac{3}{2}^-$ (1670)

$\frac{5}{2}^-$ (1600)

$\frac{1}{2}^-$ (1630)

$\frac{7}{2}^+$ (1505)

$\frac{3}{2}^+$ (1240)

$\frac{1}{2}^+$ (940)

$\frac{1}{2}^-$ (485)

$T = \frac{3}{2}$

Fig. 9