SLAC-PUB-684
November 1969
(TH)

TWISTED PROPAGATOR IN THE OPERATORIAL DUALITY FORMALISM*

L. Caneschi**

University of California
Santa Barbara, California 93601

A. Schwimmer** ††

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

G. Veneziano** † ††

Lawrence Radiation Laboratory
University of California, Berkeley, California 94720

ABSTRACT

We find an explicit operatorial form which represents the "twisted" lines in the duality diagrams. Some applications, and in particular the derivation of the symmetric three-resonance vertices, are discussed.

* Work supported in part through funds provided by the U.S. Atomic Energy Commission and in part by the National Science Foundation.

** Fulbright Fellow.

† Permanent address: Massachusetts Institute of Technology, Cambridge, Mass.

†† On leave of absence from the Weizmann Institute of Science, Rehovoth, Israel.

A convenient operatorial formalism which allows to recast in a compact form the factorization properties[1,2] of the n-point dual amplitudes[3] has been recently obtained.[4,5]

The results of Ref. 4 and Ref. 5 are sufficient in order to express all tree-diagrams and single-loop planar diagrams in terms of vertices and propagators. The calculation of a general Feynman-like dual diagram[6] demands the additional knowledge of the three-resonance vertices and of the "twisted" propagator introduced by Kikkawa et al.[7]

The first problem has been studied by Sciuto,[8] who was able to write an explicit form for the three-resonance vertex. However, he had to treat the three resonances in an asymmetric way, this being reflected in the asymmetry of his final result.

In this letter we find the explicit form of the "twisted" propagator, or, more generally, of a "twisting" operator and discuss some of its properties. In particular we are able to obtain the three-resonance vertex in a symmetric form.

Let us now define more precisely the twisted propagator. We use the notations and the results of Ref. 4, and we define:

$$|\bar{p}_0, \bar{p}_1 \ldots \bar{p}_r\rangle \equiv |\bar{p}_i\rangle = G(\bar{P}, a^\dagger)\,|0\rangle = \int d\bar{x}\; \phi(\bar{x}, \bar{p}_i)\, \exp\left(\sum_{n=1}^{\infty} \bar{P}^{(n)}_{(\bar{x}_i, \bar{p}_i)}\, a^{\dagger(n)}/\sqrt{n}\right)|0\rangle \tag{1}$$

where $\phi(\bar{x}, \bar{p}_i)$ is the usual[3] integral of the r+2 point function and the vectors $\bar{P}^{(n)}$ are defined in Ref. 1. Then we can write the amplitude for the (r+s+2)-point function of Fig. 1a as

$$A_{r+1,\,s+1} = \langle q_i|D(R, \Pi^2)|\bar{p}_i\rangle, \qquad \Pi = -\sum_{i=0}^{s} q_i = \sum_{j=0}^{r} p_j$$

The states $|p_i\rangle$ are clearly a superposition of the so-called coherent states of the harmonic oscillator problem[9] defined as

$$|\alpha\rangle = e^{\alpha a^\dagger}\,|0\rangle$$

where α is any complex number. The properties of these states are fully discussed in Ref. 9 and will be used extensively in the following. In our case, of an infinite set of harmonic oscillator the general coherent state will be defined

$$|\beta\rangle = |\beta_1 \ldots \beta_n \ldots\rangle = \exp\left(\sum_n \beta_n \, a^{\dagger(n)}\right)|0\rangle \tag{2}$$

The twisted propagator (Fig. 1c), as it is clear from a comparison with Fig. 1b, could be obtained from an untwisted tree diagram (Fig. 1b). On the other hand the graph 1a is equal to 1b because of duality. Therefore the twisted propagator can be obtained from the normal graph (Fig. 1a) provided we can reverse the order of the external lies in the left vertex. We look then for an operator Ω which transforms the state $|\bar{p}_i\rangle = |\bar{p}_0, \ldots \bar{p}_r\rangle = |p_r \ldots p_0\rangle$ into the state $|p_i\rangle = |p_0 \ldots p_r\rangle$. Using the relation of Ref. 1:

$$\bar{P}^{(n)}(\bar{x}_i \bar{p}_i) = (1 - P(x_i p_i))^n = \sum_i p_i + \sum_{\ell=1} (-)^\ell \binom{n}{\ell} P^{(\ell)}(x_i p_i) \tag{3}$$

it is easy to show that the operator Ω is given by

$$\Omega(\Pi, a^\dagger, a) = e^{a^\dagger \cdot \Pi} T(a^\dagger, a) \qquad \left(a^\dagger \cdot \Pi = \Pi \cdot \sum_n a^{\dagger(n)}/\sqrt{n}\right)$$

$$T = \left[\prod_{i=1}^{\infty} \exp\left(\sum_{j>i} c_{ji} \, a^{\dagger(j)} a^{(i)}\right)\right] (-)^R =: \exp\left(\sum_{i,j} d_{ji} \, a^{\dagger(j)} a^{(i)}\right) : \tag{4}$$

$$c_{ji} = \sqrt{\frac{i}{j}} \binom{j}{i}, \quad d_{ji} = c_{ji} (-)^i - \delta_{ji} \tag{5}$$

In Eq. (4) the column means a normal ordering of $a^\dagger, a$, which allows to compute easily[9] how Ω acts on $|p_i\rangle$. Formally we can also write:

$$T =: \exp\left[\sum_{n=1}^{\infty} \frac{\tilde{a}^{\dagger(n)}}{n} (1 - \tilde{a})^n - \left(1 + \tilde{a}^{(n)}\right)\right] :, \quad \tilde{a}^{(n)} = a^{(n)} \sqrt{n}$$

where the power n has the same formal meaning as in Eq. (3).

In fact we can easily check that:

$$\Omega|p_i\rangle = \int dx\ \phi(x_i p_i)\ \Omega \exp\left(\sum_n\ P^{(n)}(x_i p_i)\ a^{\dagger(n)}/\sqrt{n}\right)|\ 0\rangle$$

$$= \int d\bar{x}\ \phi(\bar{x}_i \bar{p}_i) \exp\left(\sum_n\ \bar{P}^{(n)}(\bar{x}_i, \bar{p}_i)\ a^{\dagger(n)}/\sqrt{n}\right)|\ 0\rangle = |\bar{p}_i\rangle \tag{6}$$

when in the last step we have used the invariance property[1] $dx\ \phi(x_i p_i) = d\bar{x}\ \phi(\bar{x}_i \bar{p}_i)$.

Guided by the equivalence of Fig. 1a and Fig. 1c, we define the "twisted" propagator $\tilde{D}_R$ requiring that

$$A_{r+1,\, s+1} = \langle q_j | D(R, \Pi^2) | \bar{p}_i\rangle = \langle q_j | \tilde{D}_R(a, a^\dagger, \Pi) | p_i\rangle \tag{7a}$$

An alternative form of the twisted propagator is provided by twisting the q-momenta, i.e.,

$$\langle q_j | D(R, \Pi^2) | \bar{p}_i\rangle = \langle \bar{q}_j | \tilde{D}_L(a, a^\dagger, \Pi) | \bar{p}_i\rangle \tag{7b}$$

Comparing with Eq. (6) we find:

$$\tilde{D}_R(a, a^\dagger, \Pi) = D(R, \Pi^2)\ \Omega(\Pi, a^\dagger, a); \qquad \tilde{D}_L = D_R^\dagger(a, a^\dagger, -\Pi) \tag{8}$$

Notice that the identification of $\tilde{D}$ with a propagator rests crucially on the fact that Ω depends only on Π and the operators $a^\dagger, a,$.

Direct computation shows that

$$T^2 = 1; \qquad T\, e^{a^\dagger \cdot p} = e^{a^\dagger \cdot p}\, T$$

which implies

$$\Omega^2(\Pi, a^\dagger, a) = 1 \tag{9}$$

as expected.

Let us discuss shortly the two different forms $\tilde{D}_R$ and $\tilde{D}_L$ obtained for the twisted propagators. It is known[1,2] that the amplitude $A_{r+1,\, s+1}$ can be written in two different forms:

$$A_{r+1,\, s+1} = \langle q_j | D | \bar{p}_i \rangle = \langle \bar{q}_j | D | p_i \rangle \tag{10}$$

The equivalence of the two forms is related to the existence of Ward-like identities for the coupling of the resonances to the external particles.[1] In our formalism these identities can be expressed as

$$\langle q_j | D | \bar{p}_i \rangle = \langle \bar{q}_j | D | p_i \rangle = \langle q_j | \Omega^{+}(-\Pi)\, D\Omega(\Pi) | \bar{p}_i \rangle$$

Therefore the equivalence of the two forms (10) is translated into the statement that the two (different) operators D and $\Omega^{+}(-\Pi)\, D\Omega(\Pi)$ have the same matrix elements between the external states $\langle q_j |$ and $| \bar{p}_i \rangle$. Using the definitions (8) of the twisted propagators and Eq. (9) also the two twisted propagators $\tilde{D}_R$ and $\tilde{D}_L$ will have the same matrix elements.

The twisting operator Ω enables us to pass from an ordering of the external lines of a tree diagram to all the others. The operator Ω can be used more than once, still allowing a clear interpretation: in particular we can use it twice in the form suggested in Figs. 2a and 2b, in order to obtain what can be called a "twisted vertex"*:

$$\tilde{V}(\Pi, \Pi') = \Omega(\Pi')\; V(k)\; \Omega^{\dagger}(-\Pi) = e^{\Pi \cdot a^{\dagger}}\, TT^{\dagger}\, e^{-\Pi' \cdot a} \tag{11}$$

In order to write $TT^{\dagger}$ in a compact form, it is convenient to work with the coherent states in Eq. (2). Then

$$T^{\dagger} | \beta \rangle = | \bar{\beta} \rangle, \quad \text{with} \quad \bar{\beta}_n = (-)^n \sum_{i=n}^{\infty} \beta_i c_{in} \tag{12}$$

Following Ref. 8 we define now:

$$[\alpha, \beta]_0 \equiv \sum_n \alpha_n \beta_n ; \quad [\alpha, \beta]_{\pm} = \sum_{n,m=1}^{\infty} \frac{\alpha_n \beta_m}{\sqrt{nm}\; B(\pm n, m)} \tag{13}$$

---

*We use for the vertex the form $V(k) = e^{a^{\dagger} \cdot k}\, e^{a \cdot k}$, where k is the incoming momentum.

The following identities are easily checked

$$\sum_{n=1}^{\infty} \beta_n/\sqrt{n} = - \sum_{n=1}^{\infty} \bar{\beta}_n/\sqrt{n} \tag{14}$$

$$[\alpha, \bar{\beta}]_0 = [\beta, \alpha]_- = [\bar{\alpha}, \beta]_+ = [\bar{\alpha}, \bar{\beta}]_- \tag{15}$$

Using (15) and the completeness of the coherent states, we can write the compact form for $\widetilde{V}$:

$$\widetilde{V}(\Pi, \Pi') = e^{\Pi \cdot a^\dagger} : e^{[a^\dagger, a]_+} ; e^{-\Pi' \cdot a} \tag{16}$$

It is amusing to see explicitly on some simple examples how the "twisted" vertex works: let us consider the three forms of the five-point function represented by Figs. 2c, d, and e, which in our formalism are:

$$A_{5c} = \langle 0 | V(p_3) D((p_3+p_4)^2, R) \; V(p_2) D((p_1+p_5)^2, R) \; \widetilde{V}(+p_1, +p_1+p_5) | 0 \rangle$$

$$A_{5d} = \langle 0 | V(p_3) D((p_3+p_4)^2, R) \; \widetilde{V}(p_1+p_2, -p_3-p_4) D((p_1+p_2)^2, R) V(p_2) | 0 \rangle$$

$$A_{5e} = \langle 0 | \widetilde{V}(-p_4-p_5, -p_4) D((p_4+p_5)^2, R) \; V(p_3) D((p_1+p_2)^2, R) \; V(p_2) | 0 \rangle \tag{17}$$

We expect all the three amplitudes to be equal to the usual

$$A_5 = \langle 0 | V(p_2) \; D((p_1+p_2)^2, R) \; V(p_3) \; D((p_4+p_5)^2, R) \; V(p_4) | 0 \rangle$$

The equality of A_5, A_{5c} and A_{5e} is trivial, if we note that

$$\widetilde{V}(p_1, p_1+p_5) | 0 \rangle = V(p_1) | 0 \rangle$$

and

$$\langle 0 | V(-p_4-p_5, -p_4) = \langle 0 | V(p_4)$$

To compute A_{5d} we introduce the standard integral representation for the D:

$$A_{5d} = \int dx \, dy \, x^{-(p_3 \cdot p_4)-1} (1-x)^{-1} \, y^{-(p_1 \cdot p_2)-1} (1-y)^{-1}$$

$$\langle 0 | e^{a \cdot p_1} x^R \, e^{+(p_1+p_2) \cdot a^\dagger} : e^{[a^\dagger, a]_+} : e^{+(p_3+p_4) \cdot a} \, y^R \, e^{a^\dagger \cdot p_3} | 0 \rangle$$

when, for the sake of simplicity, we have assumed $\alpha(0) = 0$, and we have absorbed a factor $\sqrt{2\alpha'}$ in the definition of the momenta, as in Ref. 1. We easily obtain using the standard commutation relations[4]:

$$A_{5b} = \int dx\, dy\, x^{-p_3\cdot p_4-1}(1-x)^{-p_3\cdot p_1-1}\, y^{-p_1\cdot p_2-1}(1-y)^{-p_2\cdot p_4-1}(1-x-y)^{-p_2\cdot p_3}$$

The identification of (1-x) and (1-y) with u_4 and u_5 of Bardakci, Ruegg and Virasoro[3] yields the desired result.

We now consider the problem of the general three-resonance vertices W. Sciuto[8] has factorized W out of the tree graph of Fig. 3a, which is clearly non-symmetric. We want to obtain the symmetric vertex starting from the graph 3b. For this purpose we write the process corresponding to the configuration of Fig. 3c as $\langle k_i, q_i | \Omega^\dagger(-P_1)\, D\left(P_1^2, R\right) | \bar{p}_i \rangle$. Following the procedure of Sciuto,[8] we carry out a factorization of the resonances in the variable P_2^2 and perform the change of variables which allows W to pass to the configuration 3b. As a result of this straightforward procedure we find that the three-resonances vertex $\widetilde{W}$, which we factor out of 3b, is given by the W of Sciuto multiplied on the right by $\Omega^\dagger(-P_1)$, namely

$$\begin{aligned}\widetilde{W}(P_1,P_2,P_3,\lambda',\nu,\lambda) &= \langle \nu, c | \langle \lambda', a | \exp\left(P_3\cdot a^\dagger + \left[a^\dagger, c^\dagger\right]_- + P_1\cdot c^\dagger\right) \exp\left(\left[a, c^\dagger\right]_+ + P_3\cdot a\right) T^+ \exp(-P_1\cdot a) | \lambda a \rangle | 0, c \rangle \\ &= \langle \nu, c | \langle \lambda', a | \exp\left(P_3\cdot a^\dagger + \left[a^\dagger, c^\dagger\right]_- + P_1\cdot c^\dagger\right) T^+(a^\dagger, a) \exp\left(\left[c^\dagger, a\right]_- + P_2\cdot a\right) | \lambda, a \rangle | 0, c \rangle\end{aligned} \tag{18}$$

When the identities (15) have been used to perform the last step, we received that, as in (8), a and c are sets of commuting operators each acting on the states carrying the correspondent label. In terms of the three commuting operators a, $b^\dagger$, $c^\dagger$, which act on λ, λ' and ν, respectively,[8] we can write $\widetilde{W}$ in a form that exhibits the cyclic symmetry:

$$\widetilde{W}(P_1, P_2, P_3, a^\dagger, b^\dagger, c^\dagger) = \langle 0, a| \langle \nu, c| \langle \lambda', b| \exp\left[P_3 b^\dagger + P_1 c^\dagger + P_2 a + \left[b^\dagger, c^\dagger\right]_- + \left[a, b^\dagger\right] + \left[c^\dagger, a\right]_-\right] |\lambda, a\rangle |0, b\rangle |0, c\rangle$$

$$= \langle \lambda, a| \langle \lambda', b| \langle \nu, c| \exp\left[P_1 \cdot c^\dagger + P_2 \cdot a^\dagger + P_3 \cdot b^\dagger + \left[a^\dagger, b^\dagger\right]_- + \left[b^\dagger, c^\dagger\right]_- + \left[c^\dagger, a^\dagger\right]_-\right] |0, a\rangle |0, b\rangle |0, c\rangle \tag{19}$$

where at the last step we have used the reality of the matrix elements and of the external states. The fact that $\widetilde{W} = W\Omega^+$ represents a new vertex with cyclic symmetry can be understood in a symbolic way as suggested in Fig. 3d. The dots show[8] how the external lines should be joined to the resonances. Analogously we expect that

$$\overline{\overline{W}}(P_1, P_2, P_3, \lambda, \lambda', \nu) = \langle \lambda, \lambda', \nu| \Omega(-P_2, b^\dagger, b)\, \Omega(-P_3, c^\dagger, c)\; W(P_1, P_2, P_3, a^\dagger, b^\dagger, c^\dagger)|0\rangle \tag{20}$$

represents itself a cyclic vertex (see Fig. 3e), but with the opposite ordering. In fact direct calculation yields:

$$\overline{\overline{W}}(P_1, P_2, P_3, \lambda, \lambda', \nu) = \langle \lambda, \lambda', \nu| \exp\left[P_3 \cdot a^\dagger + P_1 \cdot b^\dagger + P_2 \cdot c^\dagger + \left[a^\dagger, c^\dagger\right]_- + \left[c^\dagger, b^\dagger\right]_- + \left[b^\dagger, a\right]_-\right] \tag{21}$$

The explicit knowledge of the general three-particle vertices can be very important for the construction of a Lagrangian, by means of which it could be possible to reproduce the results of the duality theory in a field-theoretical formalism. The always incumbent ghost of double-counting and the additional complication forced by the existence of several different vertices, identified graphically in Ref. 8 and in Fig. 3d, e by the position of the dots, look however, like non-trivial difficulties to be faced in the realization of this program.

Another very interesting application of the twisting operator is the calculation of non-planar closed loops from factorization, in analogy to the planar case.[10]

The main questions to be answered are the following:

1. Does the singularity structure of the non-planar loops obtained by factorization exhibit duality, as in the planar case?
2. If this is the case, is also here the multiplicative function $F(x_i)$, independent of the kinematical variables, badly behaved at some point of the integration volume?

The explicit calculation of the four-points non-planar loops is now being performed, and will be the subject of a forthcoming paper.

One of us, L. Caneschi, would like to thank the theoretical group at Lawrence Radiation Laboratory for the hospitality extended to him during the summer.

REFERENCES

1. S. Fubini and G. Veneziano, Report No. CTP-81, preprint, Massachusetts Institute of Technology, Cambridge, Mass. (1969) (to be published in Nuovo Cimento).
2. K. Bardakci and S. Mandelstam, "Analytic solution of the linear trajectory bootstrap," University of California, Berkeley, California (1969) (to be published in Phys. Rev.).
3. For a review of the subject and a complete list of references see M. M. Chan, "The generalized Veneziano model," Report No. TH-1057, preprint, CERN, Geneva, Switzerland (1969).
4. S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 679 (1969).
5. Y. Nambu, Report No. COO-264-507, "Quark model and the factorization of the Veneziano amplitude," Enrico Fermi Instituto, University of Chicago, Chicago, Illinois (1969).
6. K. Kikkawa, B. Sakita and M. Virasoro, Report No. COO-224, preprint, "Feynman like diagrams compatable with duality I," University of Wisconsin, Madison, Wisconsin (1969) (to be published in Phys. Rev.).
7. K. Kikkawa, S. Klein, B. Sakita and M. Virasoro, Report No. COO-881-248, preprint, "Feynman like diagrams compatable with duality II," University of Wisconsin, Madison, Wisconsin (1969);
   K. Kikkawa, Report No. COO-230, preprint, "Regge cut from non-planar duality amplitude," University of Wisconsin, Madison, Wisconsin (1969).
8. S. Sciuto, "The general vertex function in dual resonance models," Instituto di Fisica dell Universita, Torino, Italy (1969), (to be published in Nuovo Cimento).

9. J. R. Klauder and E. C. G. Sudarshan, Fundamentals of Quantum Optics, (W. A. Benjamin Co., New York, 1968).

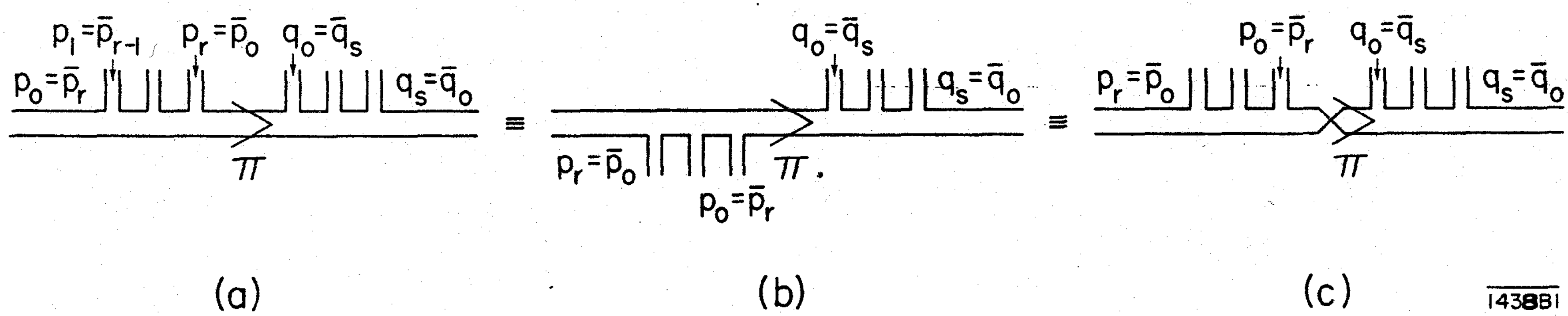
10. See note in Ref. 1 and K. Bardakci, M. Halpern and J. Shapiro, Report No. 69-1417, preprint, "Unitary closed loops in Reggeized Feynman theory," University of California, Berkeley, California (1969).

FIGURE CAPTIONS

1. Graphical interpretation of the "twisted propagator."

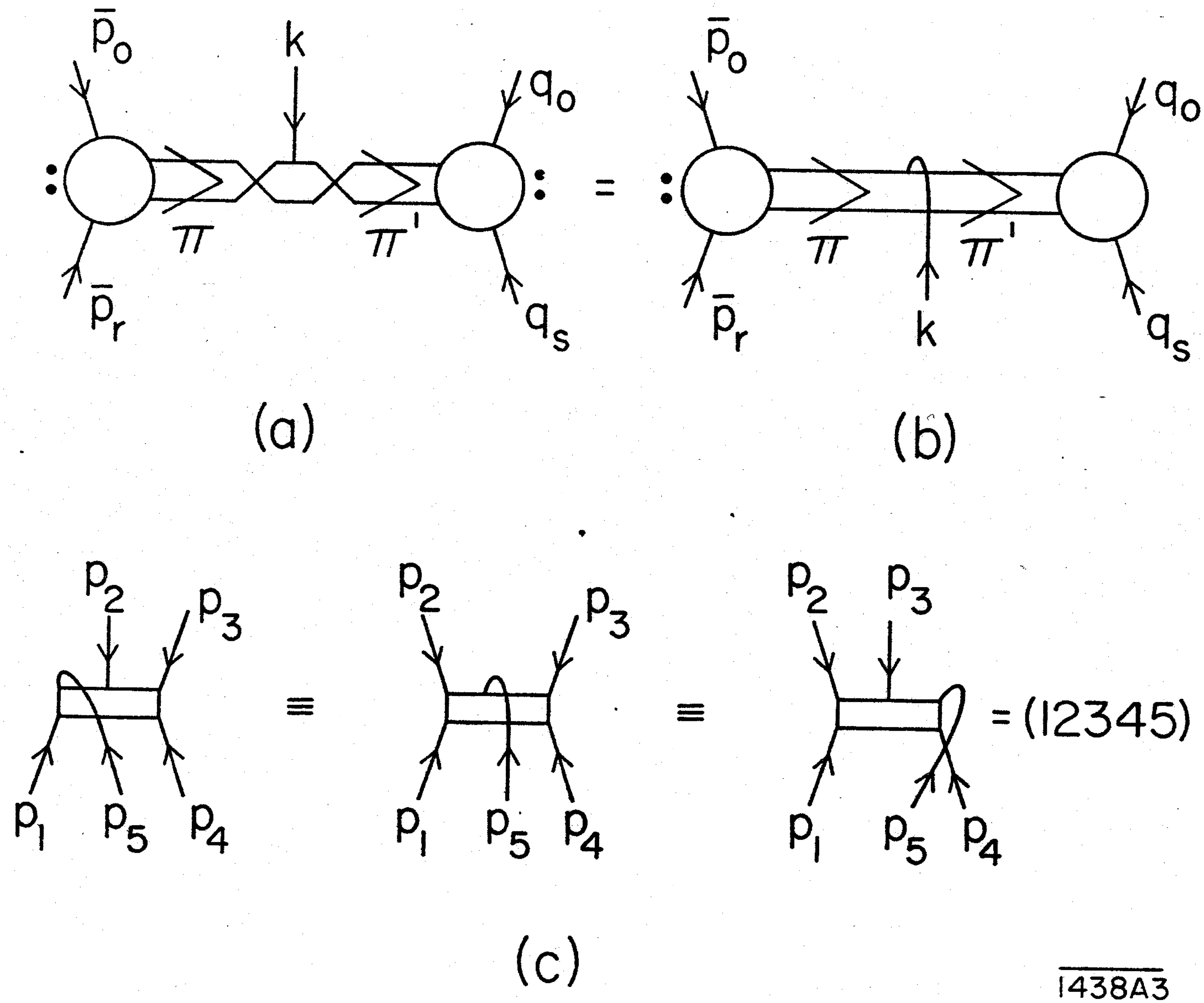
2. Graphical intepretation of the twisted vertex (a,b) and three different equivalent forms of the five point amplitude in the ordering (12345) (c,d,e).

3. Factorization of the non-symmetric (a) and symmetric (b) three-resonance vertices, (c) the three diagrams that can be transformed into (b) [graphical interpretation of Eqs. (19(d)) and (21(e))].



$p_1=\bar{p}_{r-1}$
$p_r=\bar{p}_o$
$q_o=\bar{q}_s$
$p_o=\bar{p}_r$
$q_s=\bar{q}_o$
π
≡
$q_o=\bar{q}_s$
$q_s=\bar{q}_o$
$p_r=\bar{p}_o$
π.
$p_o=\bar{p}_r$
≡
$p_o=\bar{p}_r$
$q_o=\bar{q}_s$
$p_r=\bar{p}_o$
$q_s=\bar{q}_o$
π
(a)
(b)
(c)
1438B1

Fig. 1



$\bar{p}_o$
k
q_o
π
π'
$\bar{p}_r$
q_s
=
$\bar{p}_o$
q_o
π
π'
$\bar{p}_r$
k
q_s
(a)
(b)
p_2
p_3
p_1
p_5
p_4
≡
p_2
p_3
p_1
p_5
p_4
≡
p_2
p_3
p_1
p_5
p_4
= (12345)
(c)
1438A3

Fig. 2