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ON THE ANALYSIS OF VECTOR MESON PRODUCTION BY POLARIZED PHOTONS*

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ABSTRACT

The formalism necessary to analyze production of vector mesons with polarized photons ($\gamma p \rightarrow pV$) is presented in detail. The decay angular distribution of the vector mesons is parameterized by the density matrices ρ^{α}, $\alpha = 0, 1, 2, 3$. Restrictions on the numerical values of the density matrix elements are derived. From the symmetry properties of the helicity amplitudes, it is shown that at high energies the combinations $\rho^{0}_{\lambda\lambda'} \mp (-1)^{\lambda} \rho^{1}_{-\lambda\lambda'}$ to leading order in energy receive only contributions from natural (unnatural) parity exchange in the t-channel. It is shown that this is true in any coordinate system which can be reached from the vector meson helicity system by a rotation around the normal to the production plane. The values of the density matrices as predicted by various models: elementary $0^{\pm}$ exchange, spin independence, helicity conservation, are given.

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I. INTRODUCTION

In view of current experiments at DESY[1] and SLAC[2] on the production of vector mesons by polarized photons

$$\gamma + N \to V + N \tag{1}$$

we investigate which information on the production amplitudes can be obtained from the decay distributions of the vector mesons. Our aim is to provide the theoretical tools necessary for a maximal exploitation of experiments with polarized photons, where the target nucleon is unpolarized and the polarization of the recoiling nucleon is not detected. Part of the material presented here can be found at various places in the literature[3, 4, 5, 6, 7] and, to some extent, is an application of the general work on polarization experiments by Cohen-Tannoudji _et al._[8]

After a rather pedestrian excursion into the spin problem of reaction (1), we shall write the decay angular distribution of the vector mesons in terms of their spin density matrices. This will show explicitly that experiments of the type discussed here yield at most 18 real and independent bilinear forms of the 12 complex helicity amplitudes. From the common decay modes ($\rho \to 2\pi, \omega \to 3\pi, \phi \to K\bar{K}$), however only 11 of these bilinears can be measured. The range of their values is not unlimited but restricted by a set of inequalities.

With linearly polarized photons, at high energies 8 out of 12 measurable bilinears can be separated into contributions from natural and unnatural parity exchange in the t-channel. Experiments with circularly polarized photons do not yield any information on the nature of the t-channel exchanges. To leading order in energy no interference terms between natural and unnatural parity exchanges in the t-channel can be observed in these experiments.

Finally, the predictions of various models ($J^P = 0^{\pm}$ exchange, spin independence, helicity conservation) for the spin density matrix of the vector meson are given.

II. FORMALISM

In this section the formalism for describing the polarization of the photon (γ) and the vector meson (V) in reaction (1) is developed. It will be assumed that the target nucleon is unpolarized and that the polarization of the recoiling nucleon is not observed.

A. Notations

The four-momenta of the incoming photon and the outgoing vector meson in the CMS will be denoted by k and q. We use the corresponding three-momentum vectors $\underset{\sim}{k}$ and $\underset{\sim}{q}$ to define a right-handed coordinate system:

$$\underset{\sim}{Z} = \frac{\underset{\sim}{k}}{|\underset{\sim}{k}|}$$

$$\underset{\sim}{Y} = \frac{\underset{\sim}{k} \times \underset{\sim}{q}}{|\underset{\sim}{k} \times \underset{\sim}{q}|}$$

$$\underset{\sim}{X} = \frac{(\underset{\sim}{k} \times \underset{\sim}{q}) \times \underset{\sim}{k}}{|(\underset{\sim}{k} \times \underset{\sim}{q}) \times \underset{\sim}{k}|}$$

The polarization states of the photon and the vector meson are expressed in terms of their spin space density matrices $\rho(\gamma)$ and $\rho(V)$. These density matrices are connected by the production amplitudes T

$$\rho(V) = T\, \rho(\gamma)\, T^{+} \tag{2}$$

which we write in the CMS helicity representation of Jacob and Wick[9]:

$$\rho(V)_{\lambda_V \lambda_V'} = \frac{1}{N} \sum_{\lambda_{N'} \lambda_\gamma \lambda_N \lambda_\gamma'} T_{\lambda_V \lambda_{N'}, \lambda_\gamma \lambda_N}\, \rho(\gamma)_{\lambda_\gamma \lambda_\gamma'}\, T^*_{\lambda_V' \lambda_{N'}, \lambda_\gamma' \lambda_N} \tag{3}$$

The λ's denote the helicities of the respective particles of reaction (1); N is the normalization factor:

$$N = \frac{1}{2} \sum_{\lambda_V \lambda_{N'} \lambda_\gamma \lambda_N} \left| T_{\lambda_V \lambda_{N'}, \lambda_\gamma \lambda_N} \right|^2 \tag{4}$$

The normalization of the amplitudes T can be chosen such that the production cross section for unpolarized photons is given by

$$\left(\frac{d\sigma}{d\Omega}\right)^{\mathrm{unpol}} = \left(\frac{2\pi}{k}\right)^2 \frac{1}{4} \sum_{\lambda_V \lambda_{N'} \lambda_\gamma \lambda_N} \left| T_{\lambda_V \lambda_{N'}, \lambda_\gamma \lambda_N} \right|^2 \tag{5}$$

The decay distribution of the vector meson will be discussed in its helicity system: The z direction is chosen opposite to the direction of the outgoing nucleon in the V rest system (i.e., equal to the direction of flight of the vector meson in the overall c.m. system). The y direction is the normal to the production plane, defined by the cross product $\underset{\sim}{k} \times \underset{\sim}{q}$ of the three-momenta of the vector meson and the photon. The x direction is given by $\underset{\sim}{x} = \underset{\sim}{y} \times \underset{\sim}{z}$. The decay angles θ, ϕ are defined as the polar and azimuthal angles of the unit vector $\underset{\sim}{\pi}$, which, in case of a two-particle decay of the vector meson, denotes the direction of flight of one of the decay particles in the V rest frame. (For a three-particle decay, $\underset{\sim}{\pi}$ is equal to the normal to the decay plane in the V rest frame.)

$$\cos\theta = \underset{\sim}{\pi} \cdot \underset{\sim}{z}$$

$$\cos\phi = \frac{\underset{\sim}{y} \cdot (\underset{\sim}{z} \times \underset{\sim}{\pi})}{|\underset{\sim}{z} \times \underset{\sim}{\pi}|}$$

$$\sin\phi = -\frac{\underset{\sim}{x}\,(\underset{\sim}{z} \times \underset{\sim}{\pi})}{|\underset{\sim}{z} \times \underset{\sim}{\pi}|}$$

The Gottfried-Jackson system and the Adair system which will be used in connection with the predictions of various models differ from the helicity system only in the choice of the z axis. In the Gottfried-Jackson system, the z axis is equal to the direction of flight of the incoming photon in the V rest frame. In the Adair system the z axis is equal to the direction of flight of the incoming photon in the c.m. system.

B. General Decay Angular Distribution of $V \rightarrow 2$ Pseudoscalar Mesons and $V \rightarrow 3$ Pseudoscalar Mesons

The decay angular distribution of the vector meson in its rest frame reads:

$$\frac{dN}{d\cos\theta\, d\phi} \equiv W(\cos\theta, \phi) = M\,\rho(V) M^{+}$$

$$= \sum_{\lambda_V \lambda'_V} \langle\theta, \phi | M | \lambda_V\rangle\, \rho(V)_{\lambda_V \lambda'_V} \langle\lambda'_V | M^{+} | \theta, \phi\rangle \tag{6}$$

where M is the decay amplitude:

$$\langle\theta, \phi | M | \lambda_V\rangle = C \sqrt{\frac{3}{4\pi}}\, D^{1*}_{\lambda_V 0}(\phi, \theta, -\phi) \tag{7}$$

Note that we consider V decays into spinless particles only. The quantity $|C|^2$ is proportional to the V decay width.[10] Due to rotation invariance C is independent of λ_V. Because we consider a normalized decay angular distribution, we have set C equal to one. The Wigner rotation functions D are given by (sign convention of Rose[11]):

$$D^{1}_{10}(\phi, \theta, -\phi) = -\frac{1}{\sqrt{2}} \sin\theta\, e^{-i\phi}$$

$$D^{1}_{00}(\phi, \theta, -\phi) = \cos\theta \tag{8}$$

$$D^{1}_{-10}(\phi, \theta, -\phi) = \frac{1}{\sqrt{2}} \sin\theta\, e^{i\phi}.$$

With the help of Eq. (7), the decay distribution (6) can be written:

$$W(\cos\theta, \phi) = \frac{3}{4\pi} \sum_{\lambda_V \lambda'_V} D^{1*}_{\lambda_V 0}(\phi, \theta, -\phi)\, \rho(V)_{\lambda_V \lambda'_V}\, D^{1}_{\lambda'_V 0}(\phi, \theta, -\phi) \tag{9}$$

Using the fact that $\rho(V)$ is hermitian $\left(\rho(V)_{\lambda_V \lambda'_V} = \rho^*(V)_{\lambda'_V \lambda_V}\right.$, see Eq. (3) and Eq. (12) below$\left.\right)$ one obtains from (8) and (9):

$$W\left(\cos\theta, \phi, \rho(V)\right) = \frac{3}{4\pi}\left(\frac{1}{2}(\rho_{11}+\rho_{-1-1}) \cdot \sin^2\theta + \rho_{00}\cos^2\theta \right.$$
$$+ \frac{1}{\sqrt{2}}(-\mathrm{Re}\,\rho_{10} + \mathrm{Re}\,\rho_{-10}) \sin 2\theta \cos\phi$$
$$+ \frac{1}{\sqrt{2}}(\mathrm{Im}\,\rho_{10} + \mathrm{Im}\,\rho_{-10}) \sin 2\theta \sin\phi \tag{10}$$
$$\left. - \mathrm{Re}\,\rho_{1-1}\sin^2\theta\cos 2\phi + \mathrm{Im}\,\rho_{1-1}\sin^2\theta \sin 2\phi\right)$$

where on the r.h.s. the label V has been omitted from the $\rho(V)_{ik}$. This general form of W will be simplified in subsection E by using the symmetries of $\rho(V)$ which follow from the properties of $\rho(\gamma)$ and T.

C. Density Matrix of the Photon

The density matrix $\rho^{pure}(\gamma)$ of pure photon states can be constructed from the photon wave function $|\gamma>$ in the helicity basis.

$$|\gamma> = a_+|\lambda_\gamma = +1> + a_-|\lambda_\gamma = -1> \tag{11}$$

where

$$\left||\lambda_\gamma = \pm 1>\right|^2 = 1, \quad |a_+|^2 + |a_-|^2 = 1$$

The result is:

$$\rho^{pure}(\gamma) = |\gamma><\gamma| = \begin{pmatrix} |a_+|^2 & a_+a_-^* \\ a_-a_+^* & |a_-|^2 \end{pmatrix} \tag{12}$$

In the case of circular polarization and $\lambda_\gamma = +1, -1$, one obtains:

$$\rho^{pure}(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

For linearly polarized photons Eq. (4) reads:

$$|\gamma> = -\frac{1}{\sqrt{2}}\left(e^{-i\Phi}|\lambda_\gamma = +1> - e^{i\Phi}|\lambda_\gamma = -1>\right) \tag{14}$$

where Φ is the angle between the polarization vector of the photon, $\underset{\sim}{\epsilon} = (\cos\Phi,$ $\sin\Phi,\ 0)$, and the production plane (x, z plane) (note: our definition of Φ differs by a sign from that of Ref. 4). The density matrix is:

$$\rho^{\text{pure}}(\gamma) = \frac{1}{2}\begin{pmatrix} 1 & -e^{-2i\Phi} \\ -e^{2i\Phi} & 1 \end{pmatrix} \tag{15}$$

For elliptically polarized photons, Eq. (11) reads:

$$|\gamma> = \frac{1}{\sqrt{2(a^2+b^2)}}\left\{-(a+b)\,e^{-i\Phi}|\lambda_\gamma = +1> + (a-b)e^{i\Phi}|\lambda_\gamma = -1>\right\} \tag{16}$$

where a and b are the lengths of the principal axes of the ellipse and Φ is the azimuthal angle of the principal axis a. The corresponding density matrix is given by:

$$\rho^{\text{pure}}(\gamma) = \frac{1}{2}\begin{pmatrix} 1+2a\sqrt{1-a^2} & e^{-2i\Phi}(1-2a^2) \\ e^{2i\Phi}(1-2a^2) & 1-2a\sqrt{1-a^2} \end{pmatrix} \tag{17}$$

with a, b normalized to $a^2 + b^2 = 1$. Obviously the cases of circularly or linearly polarized photons can be obtained by specializing Eq. (17) to $a = \pm 1/\sqrt{2}$ or $a = 1$ respectively.

On the other hand, it follows from Eqs. (13), (15), (17) that $\rho^{\text{pure}}(\gamma)$ for elliptically polarized photons can be written as a linear combination of the density matrices for photons of linear and circular polarization. Therefore, experiments with elliptical polarization do not yield more information on the helicity amplitudes than a set of experiments with linear and circular polarization. The elliptical case will not be pursued any further.

We generalize these results to the case of partially polarized photons and put them into a standard form by writing $\rho(\gamma)$ as a linear combination of the matrices I, σ_i (i = 1, 2, 3), which form a complete set in the space of 2 × 2 hermitian matrices:

$$\rho(\gamma) = \frac{1}{2} I + \underset{\sim}{P}_\gamma \cdot \frac{\underset{\sim}{\sigma}}{2} \tag{18}$$

I is the 2 × 2 unit matrix, σ_i are the three Pauli matrices. The length P_γ of the three-vector $\underset{\sim}{P}_\gamma$ is equal to the degree of polarization. The direction of $\underset{\sim}{P}_\gamma$ depends on the kind of polarization, e.g. (from Eqs. (13), (15)):

$$\begin{aligned} \underset{\sim}{P}_\gamma &= P_\gamma(0, 0, \pm 1) \\ \underset{\sim}{P}_\gamma &= P_\gamma(-\cos 2\Phi, -\sin 2\Phi, 0) \end{aligned} \tag{19}$$

for circular polarization with $\lambda_\gamma = \pm 1$ and for linear polarization respectively, where the degree of polarization is denoted by P_γ, $0 \leq P_\gamma \leq 1$.

D. Symmetry Properties of the Helicity Amplitudes

The symmetry properties of the helicity amplitudes imply symmetry relations for the density matrix $\rho(V)$. With our choice of coordinate system parity conservation leads for reaction (1) to[9]:

$$T(\Theta^*)_{-\lambda_V -\lambda_{N'}, -\lambda_\gamma -\lambda_N} = (-1)^{(\lambda_V - \lambda_{N'}) - (\lambda_\gamma - \lambda_N)} \, T(\Theta^*)_{\lambda_V \lambda_{N'}, \lambda_\gamma \lambda_N} \tag{20}$$

with Θ^* being the CMS production angle. If only natural $\left(P = (-1)^J\right)$ or only unnatural parity $\left(P = -(-1)^J\right)$ exchanges in the t-channel contribute, one has to leading order in the energy of the incoming photon the additional symmetry[8]:

$$\begin{aligned} T(\Theta^*)_{-\lambda_V \lambda_{N'}, -\lambda_\gamma \lambda_N} &= \pm (-1)^{\lambda_V - \lambda_\gamma} \, T(\Theta^*)_{\lambda_V \lambda_{N'}, \lambda_\gamma \lambda_N} \\ &= \mp (-1)^{\lambda_V} \, T(\Theta^*)_{\lambda_V \lambda_{N'}, \lambda_\gamma \lambda_N} \end{aligned} \tag{21}$$

where the upper (lower) sign applies to natural (unnatural) parity exchanges.

Let $T^N(T^U)$ be that part of the helicity amplitude which receives contributions only from natural (unnatural) parity exchanges in the t-channel.

$$T = T^N + T^U \tag{22}$$

Using Eq. (20) one can project out T^N, T^U :

$$T^{\binom{N}{U}}(\Theta^*)_{\lambda_V\lambda_{N'},\lambda_\gamma\lambda_N} = \frac{1}{2}\left(T(\Theta^*)_{\lambda_V\lambda_{N'},\lambda_\gamma\lambda_N} \mp (-1)^{\lambda_V} T(\Theta^*)_{-\lambda_V\lambda_{N'},-\lambda_\gamma\lambda_N}\right) \tag{23}$$

E. Standard Decomposition of $\rho(V)$

The density matrix $\rho(V)$ can be written in a form showing explicitly the dependence on the polarization vector $\underset{\sim}{P}_\gamma$. Defining[10]

$$(\rho^o, \rho^\alpha) = T(\frac{1}{2} I, \frac{1}{2}\sigma^\alpha)\, T^+, \qquad \alpha = 1, 2, 3 \tag{24}$$

we find from Eqs. (2) and (18)

$$\rho(V) = \rho^o + \sum_{i=1}^{3} P_\gamma^\alpha \rho^\alpha \tag{25}$$

The four hermitian matrices ρ^α, $\alpha = 0, 1, 2, 3$ read explicitly:

$$\rho^o_{\lambda_V\lambda'_V} = \frac{1}{2N} \sum_{\lambda_\gamma\lambda_{N'}\lambda_N} T_{\lambda_V\lambda_{N'},\lambda_\gamma\lambda_N}\, T^*_{\lambda'_V\lambda_{N'},\lambda_\gamma\lambda_N}$$

$$\rho^1_{\lambda_V\lambda'_V} = \frac{1}{2N} \sum_{\lambda_\gamma\lambda_{N'}\lambda_N} T_{\lambda_V\lambda_{N'},-\lambda_\gamma\lambda_N}\, T^*_{\lambda'_V\lambda_{N'},\lambda_\gamma\lambda_N}$$

$$\rho^2_{\lambda_V\lambda'_V} = \frac{i}{2N} \sum_{\lambda_\gamma\lambda_{N'}\lambda_N} \lambda_\gamma\, T_{\lambda_V\lambda_{N'},-\lambda_\gamma\lambda_N}\, T^*_{\lambda'_V\lambda_{N'},\lambda_\gamma\lambda_N} \tag{26}$$

$$\rho^3_{\lambda_V\lambda'_V} = \frac{1}{2N} \sum_{\lambda_\gamma\lambda_{N'}\lambda_N} \lambda_\gamma\, T_{\lambda_V\lambda_{N'},\lambda_\gamma\lambda_N}\, T^*_{\lambda'_V\lambda_{N'},\lambda_\gamma\lambda_N}$$

Parity conservation (Eq. (20)) reduces the number of independent matrix elements

$$\rho^{\alpha}_{\lambda\lambda'} = (-1)^{\lambda-\lambda'} \rho^{\alpha}_{-\lambda-\lambda'} , \quad \alpha = 0, 1 \tag{27}$$

and

$$\rho^{\alpha}_{\lambda\lambda'} = -(-1)^{\lambda-\lambda'} \rho^{\alpha}_{-\lambda-\lambda'} , \quad \alpha = 2, 3 \tag{28}$$

From these equations and the hermiticity of the $\rho^{\alpha}(\alpha = 0, \ldots 3)$ follows that ρ^{0}_{1-1}, ρ^{1}_{1-1} are real and ρ^{2}_{1-1}, ρ^{3}_{1-1} purely imaginary. Because the decay distribution W in Eq. (10) is linear in $\rho(V)$, the representation Eq. (25) may be used to decompose W as well:

$$W(\cos\theta, \phi, \rho) = W^{0}(\cos\theta, \phi) + \sum_{\alpha=1}^{3} P^{\alpha}_{\gamma} W^{\alpha}(\cos\theta, \phi) \tag{29}$$

where W^{α} $(\alpha = 0, \ldots 3)$ is defined by Eq. (10) with ρ replaced by ρ^{α} :

$$W^{\alpha}(\cos\theta, \phi) = W(\cos\theta, \phi, \rho^{\alpha}) , \quad \alpha = 0, \ldots 3 \tag{30}$$

Because of the symmetries of the ρ^{α} (Eqs. (27), (28)), the W^{α} reduce to:

$$\begin{aligned} W^{0}(\cos\theta, \phi) = \frac{3}{4\pi}\Big(&\frac{1}{2}(1-\rho^{0}_{00}) + \frac{1}{2}(3\rho^{0}_{00} - 1)\cos^{2}\theta \\ &- \sqrt{2}\,\mathrm{Re}\,\rho^{0}_{10}\sin 2\theta\cos\phi \\ &- \rho^{0}_{1-1}\sin^{2}\theta\cos 2\phi\Big) \end{aligned}$$

$$\begin{aligned} W^{1}(\cos\theta, \phi) = \frac{3}{4\pi}\,(&\rho^{1}_{11}\sin^{2}\theta + \rho^{1}_{00}\cos^{2}\theta - \sqrt{2}\,\rho^{1}_{10}\sin 2\theta\cos\phi \\ &-\rho^{1}_{1-1}\sin^{2}\theta\cos 2\phi) \end{aligned} \tag{31}$$

$$W^{2}(\cos\theta, \phi) = \frac{3}{4\pi}\,(+\sqrt{2}\,\mathrm{Im}\,\rho^{2}_{10}\sin 2\theta\sin\phi + \mathrm{Im}\,\rho^{2}_{1-1}\sin^{2}\theta\sin 2\phi)$$

$$W^{3}(\cos\theta, \phi) = \frac{3}{4\pi}\,(+\sqrt{2}\,\mathrm{Im}\,\rho^{3}_{10}\sin 2\theta\sin\phi + \mathrm{Im}\,\rho^{3}_{1-1}\sin^{2}\theta\sin 2\phi)$$

Because ρ^2, ρ^3 have the same symmetries the structures of W^2, W^3 are the same. W^o, W^1 differ only insofar as for our choice of normalization tr $\rho^o = 1$, whereas there is no trace condition for ρ^1. For easy reference we list here the explicit forms of the decay angular distributions for the various photon polarizations by inserting $\underset{\sim}{P}_\gamma$ of Eq. (19) into Eq. (31):

1. Unpolarized photons

   From $\underset{\sim}{P}_\gamma = 0$, one has

$$W^{unpol}(\cos\theta, \phi) = W^o(\cos\theta, \phi) \tag{32}$$

2. Circular polarization of helicity $\lambda_\gamma = \pm 1$:

$$W^{\pm}(\cos\theta, \phi) = W^o(\cos\theta, \phi) \pm P_\gamma W^3(\cos\theta, \phi) \tag{33}$$

3. Linear polarization

$$\begin{aligned} W^L(\cos\theta, \phi, \Phi) = W^o(\cos\theta, \phi) &- P_\gamma \cos 2\Phi\, W^1(\cos\theta, \phi) \\ &- P_\gamma \sin 2\Phi\, W^2(\cos\theta, \phi) \end{aligned} \tag{34}$$

The Eqs. (27), (28) hold in any coordinate system that can be reached from the helicity system by a rotation R around the normal to the production plane, due to the symmetry properties of the rotation matrices d(R). We sketch the proof, e.g., for ρ^o. In the rotated system, ρ^o is given by:

$$\tilde{\rho}^o_{mm'} = \sum_{\mu\mu'} d_{m\mu}(R)\, \rho^o_{\mu\mu'}\, d_{\mu'm'}(R)$$

The following calculation shows the symmetry property Eq. (27) to hold in the rotated coordinate system:

$$\begin{aligned} \tilde{\rho}^o_{-m-m'} &= \sum_{\mu\,\mu'} d_{-m\mu}(R)\, \rho^o_{\mu\mu'}\, d_{\mu'-m'}(R) \\ &= \sum_{\mu\,\mu'} (-1)^{m-\mu}\, d_{m-\mu}(R)\, \rho^o_{\mu\mu'}\, d_{-\mu'm'}(R)(-1)^{\mu'-m'} \end{aligned}$$

$$= (-1)^{m-m'} \sum_{\mu\,\mu'} d_{m-\mu}(R)\, \rho^{o}_{-\mu-\mu'}\, d_{-\mu' m'}(R)$$

$$= (-1)^{m-m'}\, \tilde{\rho}^{o}_{m\,m'} \qquad \text{q.e.d.}$$

Hence, the structures of the decay angular distributions given by Eq. (32) remain unchanged under such a rotation.

F. Restrictions on the Values of the Density Matrix Elements

When extracting the density matrix elements from experimental data by means of fits, one should keep in mind that their numerical values are restricted by the following inequalities:

$$\left|\rho^{\alpha}_{\lambda\lambda'}\right|^2 \leq \rho^{o}_{\lambda\lambda}\, \rho^{o}_{\lambda'\lambda'} \quad \alpha = 0, 1, 2, 3 \tag{35}$$

$$\det \rho(V) = \prod_{i=1}^{3} \mu_i \geq 0 \tag{36}$$

$$\mathrm{Tr}\ \rho(V) = \sum_{i=1}^{3} \mu_i \geq 0 \tag{37}$$

$$\sum_i \det R(V)_{ii} = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \geq 0 \tag{38}$$

where the μ_i, $i = 1, 2, 3$, are the eigenvalues of $\rho(V)$, and $R(V)_{ik}$ denotes the adjoint of $\rho(V)_{ik}$ (the matrix obtained by deleting the ith row and the kth column of $\rho(V)$). Equations (36)-(38) lead to the conditions

$$\rho(V)_{\lambda\lambda} \geq 0 \tag{39}$$

Equation (35) is obtained by applying the Schwarz inequality to the bilinear expression in the helicity amplitudes for the ρ^{α} (Eqs. (26)). Equations (36)-(38) are the necessary and sufficient conditions for positive definiteness of $\rho(V)$ which is a consequence of the defining Eq. (3). The inequalities following from Eqs. (35), (36) for the set of measurable parameters are given in Table II. The results for ρ^{o} have been obtained before.[13]

III. SEPARATION OF NATURAL AND UNNATURAL PARITY EXCHANGE CONTRIBUTIONS

How much information can we gain on reaction (1) from experiments with polarized photons, unpolarized target and recoil polarization not being detected? This question can now be answered by simple parameter counting: The results of these experiments can be described in terms of $\rho(V)$ and $d\sigma^{unpol}$, the production cross section with unpolarized photons. From the standard decompositon of $\rho(V)$ into ρ^{α}, we find that $\rho(V)$ is described by 17 independent functions, of which eleven can actually be measured from the common decay modes $\rho \to 2\pi$, $\omega \to 3\pi$, $\phi \to K\bar{K}$ (see Table I). Hence one measures altogether 12 independent quantities (one for $d\sigma^{unpol}$). On the other hand, reaction (1) is described by 12 independent complex amplitudes, i.e., 23 real functions. Nevertheless experiments with linearly polarized photons provide an important new insight into the production process of reaction (1) because they allow to measure the contributions of natural and unnatural parity exchange in the t-channel to the matrix elements of Re ρ^{0}, Re ρ^{1}, as will be shown in the following. However, experiments with circularly polarized photons do not yield any information on the parity of the t-channel exchanges when the polarization of the recoiling nucleon is not measured.

At high energies the density matrix elements of $\rho^{\alpha}(V)$, $(\alpha = 0, 1, 2, 3)$ can be written as a sum of two terms which receive contributions from natural or unnatural parity exchanges in the t-channel. These two terms are themselves linear combinations of the ρ^{α} (the label V will be omitted from now on). The separation is achieved by using the symmetry property (23). As an example we outline the proof for ρ^{0}. Inserting Eq. (22) into the definition of ρ^{0} (Eq. (26)) one finds:

$$\rho^{0}_{\lambda\lambda'} = \frac{1}{2N} \sum_{\lambda_\gamma \lambda_N \lambda_{N'}} \left[T^{N}_{\lambda\lambda_{N'}, \lambda_\gamma\lambda_N} + T^{U}_{\lambda\lambda_{N'}, \lambda_\gamma\lambda_N}\right]\left[T^{N*}_{\lambda'\lambda_{N'}, \lambda_\gamma\lambda_N} + T^{U*}_{\lambda'\lambda_{N'}, \lambda_\gamma\lambda_N}\right] \tag{40}$$

The interference term between natural and unnatural parity exchanges <u>vanishes</u> in the limit of high energies:

$$\frac{1}{2N}\sum_{\lambda_\gamma\lambda_N\lambda_{N'}}\left(T^{N}_{\lambda\lambda_{N'},\lambda_\gamma\lambda_N}\,T^{U*}_{\lambda'\lambda_{N'},\lambda_\gamma\lambda_N}+T^{U}_{\lambda\lambda_{N'},\lambda_\gamma\lambda_N}\,T^{N*}_{\lambda'\lambda_{N'},\lambda_\gamma\lambda_N}\right)$$

$$=\frac{1}{8N}\sum_{\lambda_\gamma\lambda_N\lambda_{N'}}\Bigg[\left(T_{\lambda\lambda_{N'},\lambda_\gamma\lambda_N}-(-1)^{\lambda}T_{-\lambda\lambda_{N'},-\lambda_\gamma\lambda_N}\right)$$

$$\times\left(T^{*}_{\lambda'\lambda_{N'},\lambda_\gamma\lambda_N}+(-1)^{\lambda'}T_{-\lambda'\lambda_{N'},-\lambda_\gamma\lambda_N}\right)$$

$$+\left(T_{\lambda\lambda_{N'},\lambda_\gamma\lambda_N}+(-1)^{\lambda}T_{-\lambda\lambda_{N'},-\lambda_\gamma\lambda_N}\right) \tag{41}$$

$$\times\left(T^{*}_{\lambda'\lambda_{N'},\lambda_\gamma\lambda_N}-(-1)^{-\lambda'}T^{*}_{-\lambda'\lambda_{N'},-\lambda_\gamma\lambda_N}\right)\Bigg]$$

$$=\frac{1}{2}\left[\rho^{o}_{\lambda\lambda'}-(-1)^{\lambda-\lambda'}\rho^{o}_{-\lambda-\lambda'}\right]=0$$

Here Eqs. (23) and (27) were used. For ρ^{o} one obtains therefore:

$$\rho^{o}=\rho^{o(N)}+\rho^{o(U)} \tag{42}$$

where:
$$\rho^{o\binom{N}{U}}_{\lambda\lambda'}=\frac{1}{2N}\sum_{\lambda_\gamma\lambda_N\lambda_{N'}}T^{\binom{N}{U}}_{\lambda\lambda_{N'},\lambda_\gamma\lambda_N}\;T^{\binom{N}{U}^*}_{\lambda'\lambda_{N'},\lambda_\gamma\lambda_N} \tag{43}$$

For the contributions $\rho^{o(N)}$ and $\rho^{o(U)}$ of natural and unnatural parity exchanges to the density matrix ρ^{o}, one gets by a calculation similar to that of Eq. (41):

$$\rho^{o\binom{N}{U}}_{\lambda\lambda'}=\frac{1}{2}\left(\rho^{o}_{\lambda\lambda'}\mp(-1)^{\lambda}\rho^{1}_{-\lambda\lambda'}\right) \tag{44}$$

Defining $\rho^{1\binom{N}{U}}$, $\rho^{2\binom{N}{U}}$, $\rho^{3\binom{N}{U}}$ analogous to $\rho^{o\binom{N}{U}}$ (Eq. (43)) one can show that:

$$\rho^{1} = \rho^{1(N)} + \rho^{1(U)}$$

$$\rho^{1\binom{N}{U}}_{\lambda\lambda'} = \frac{1}{2}\left(\rho^{1}_{\lambda\lambda'} \mp (-1)^{\lambda}\rho^{o}_{-\lambda\lambda'}\right) \tag{45}$$

$$\rho^{3} = \rho^{3(N)} + \rho^{3(U)}$$

$$\rho^{3\binom{N}{U}}_{\lambda\lambda'} = \frac{1}{2}\left(\rho^{3}_{\lambda\lambda'} \pm i\,(-1)^{\lambda}\rho^{2}_{-\lambda\lambda'}\right) \tag{46}$$

$$\rho^{2} = \rho^{2(N)} + \rho^{2(U)}$$

$$\rho^{2\binom{N}{U}}_{\lambda\lambda'} = \frac{1}{2}\left(\rho^{2}_{\lambda\lambda'} \mp i\,(-1)^{\lambda}\rho^{3}_{-\lambda\lambda'}\right) \tag{47}$$

Like the symmetries following from parity conservation the relations (43)-(47) hold in all coordinate systems that can be reached from the helicity system by a rotation around the normal of the production plane.

From Table I it is evident that all elements of ρ^{o} and ρ^{1} which are measurable in the type of experiments discussed here can be split into their natural and unnatural parity contributions, as listed in Table III. This separation is not possible for ρ^{2} and ρ^{3} because relations (46), (47) connect measurable elements of ρ^{2} with unmeasurable elements of ρ^{3} and vice versa. It is worth noting that the (Lorentz invariant) eigenvalue $(\rho^{o}_{11}+\rho^{o}_{1-1})$ of $\rho^{o(13)}$ can be directly decomposed into its two t-channel parity parts by an experiment with linearly polarized photons measuring W^{L} (Eq. (34)) at the angles $\theta = \pi/2$, $\phi = \pi/2$, $\Phi = 0,\ \pi/2$ (see Ref. 1):

$$\frac{\sigma_{\perp}}{\sigma_{\parallel}} \equiv \frac{(P_{\gamma}-1)\,W^{L}(0,\pi/2,\pi/2) + (P_{\gamma}+1)\,W^{L}(0,\pi/2,0)}{(P_{\gamma}+1)\,W^{L}(0,W/2,\pi/2) + (P_{\gamma}-1)\,W^{L}(0,\pi/2,0)} = \frac{\rho^{o(U)}_{11} + \rho^{o(U)}_{1-1}}{\rho^{o(N)}_{11} + \rho^{o(N)}_{1-1}} \tag{48}$$

The asymmetry P_σ (parity asymmetry) in the contributions σ^N, σ^U of natural and unnatural parity exchanges to the total cross section is given by $\left(\text{Eq. (44)}\right)$:

$$P_\sigma = \frac{\sigma^N - \sigma^U}{\sigma^N + \sigma^U} = 2\rho^1_{1-1} - \rho^1_{00} \tag{49}$$

For completeness we also give the quantity $\sum$ of Ref. 1 in terms of the density matrix elements:

$$\begin{aligned} \sum = \frac{\sigma_\parallel - \sigma_\perp}{\sigma_\parallel + \sigma_\perp} &= \frac{1}{P_\gamma} \frac{W^L(0,\pi/2,\pi/2) - W^L(0,\pi/2,0)}{W^L(0,\pi/2,\pi/2) + W^L(0,\pi/2,0)} \\ &= \frac{\rho^1_{11} + \rho^1_{1-1}}{\rho^o_{11} + \rho^o_{1-1}} \end{aligned} \tag{50}$$

IV. MODEL PREDICTIONS FOR THE DENSITY MATRIX ELEMENTS

In this section we review the predictions of various models for reaction (1). Roughly speaking, these models may be divided into two classes:

(1) t-channel exchange models of elementary or reggeized particles;

(2) Models inspired by the idea that vector-meson photoproduction proceeds via diffraction: the spin independence model (SIM)[14] and the helicity conserving model (HCM).

The $J^P = 0^\pm$ exchange models and SIM and HCM have in common that in a reference frame characteristic to the particular model, they predict that

(a) the matrices ρ^o, ρ^1, ρ^2, ρ^3 are independent of photon energy and production angle;

(b) ρ^o, ρ^3 are diagonal, ρ^1, ρ^2 antidiagonal.

These properties are a consequence of the simple spin structure of the CMS production amplitudes in these models:

$$T_{m_V m_{N'}, m_\gamma m_N} = t_{m_N m_\gamma} \delta_{m_{N'} m_N} \delta_{m_V m_\gamma}$$

The m's are the spin projections in a reference system appropriate to the models, i.e.,:

for elementary or reggeized particle system the t-channel cm helicity system. The vector-meson decay is analyzed in its Gottfried-Jackson system (GJ);

for SIM the s-channel cm system with quantization axis along the direction of the photon. The vector meson decay is analyzed in its Adair system (A);

for HCM the s-channel cm helicity system. The vector meson decay is analyzed in its helicity system (H).

The density matrices in these coordinate systems read:

$$\rho^0 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \rho^3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

$$\rho^1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \qquad \rho^2 = \begin{pmatrix} 0 & 0 & -ia \\ 0 & 0 & 0 \\ ia & 0 & 0 \end{pmatrix} \tag{51}$$

For $J^P = 0^+(0^-)$ exchange one has $a = \frac{1}{2}$ $(a = -\frac{1}{2})$,[15] for SIM one has $a = \frac{1}{2}$. In the spirit of the diffraction idea we assume only natural parity exchange contributions for HCM and therefore set $a = \frac{1}{2}$.

The density matrices given above in the characteristic systems of the models can be transformed into the other systems by rotations around the normal to the production plane:

$$\rho^{A} = d^{1}(\alpha_{A\rightarrow B})\, \rho^{B}\, d^{1}(\alpha_{A\rightarrow B})^{+} \tag{52}$$

The rotation angles $\alpha_{A\rightarrow B}$ are:

$$\begin{aligned} \alpha_{H\rightarrow GJ} &= \arccos \left(\frac{\beta - \cos\Theta^{*}}{\beta\cos\Theta^{*} - 1}\right) \\ \alpha_{H\rightarrow A} &= \Theta^{*} \end{aligned} \tag{53}$$

where Θ^{*} is the production angle and β the velocity of the vector meson both evaluated in the cms.

With the simple form of the density matrices (Eq. (51)) in mind, one can ask for conditions under which the density matrix ρ^{o} can be diagonalized by a rotation through some angle α around the normal to the production plane:

$$\rho^{o}_{\lambda\lambda'} = d_{\lambda\mu}(-\alpha)\, a^{o}_{\mu\mu}\, \delta_{\mu\mu'}\, d_{\mu'\lambda'}(-\alpha) \tag{54}$$

Evaluating Eq. (54) one finds that such an angle α exists if:

$$\left|\rho^{o}_{10}\right| = \sqrt{\rho^{o}_{1-1}\,(3\rho^{o}_{11} + \rho^{o}_{1-1} - 1)} \tag{55a}$$

$$\rho^{o}_{11} + \rho^{o}_{1-1} \leq \frac{1}{2} \tag{55b}$$

$$0 \leq \frac{2\rho^{o}_{1-1}}{3\rho^{o}_{11} + \rho^{o}_{1-1} - 1} \leq 1 \tag{55c}$$

The values of α and $a^{o}_{\mu\mu}$ are

$$\mathrm{tg}\ \alpha = -\frac{1}{\sqrt{2}}\ \frac{\rho^{o}_{1-1}}{\mathrm{Re}\ \rho^{o}_{10}}$$

$$a^{o}_{11} = a^{o}_{-1-1} = \rho^{o}_{11} + \rho^{o}_{1-1} \tag{56}$$

$$a^{o}_{00} = 1 - 2(\rho^{o}_{11} + \rho^{o}_{1-1})$$

In order that ρ^{1} can be antidiagonalized by the same rotation $R_y(\alpha)$, the following conditions have to be satisfied in addition:

$$\rho^{1}_{10} = \frac{\rho^{1}_{00} + 3\rho^{1}_{11} + \rho^{1}_{1-1}}{1 - 3(\rho^{o}_{11} + \rho^{o}_{1-1})}\ \rho^{o}_{10} \tag{57a}$$

$$\rho^{1}_{1-1} = \rho^{1}_{11} + \rho^{1}_{1-1} - \rho^{o}_{1-1} \cdot \frac{\rho^{1}_{00} + 3\rho^{1}_{11} + \rho^{1}_{1-1}}{1 - 3(\rho^{o}_{11} + \rho^{o}_{1-1})} \tag{57b}$$

The elements $a^{1}_{-\mu\mu}$ of ρ^{1} in its antidiagonalized form are:

$$a^{1}_{1-1} = a^{1}_{-11} = \rho^{1}_{11} + \rho^{1}_{1-1}$$

$$a^{1}_{00} = \rho^{1}_{00} + 2\rho^{1}_{11} \tag{58}$$

The antidiagonalize ρ^{2} by the same rotation requires

$$\frac{\mathrm{Im}\ \rho^{2}_{10}}{\mathrm{Im}\ \rho^{2}_{1-1}} = \frac{1}{2}\ \frac{\rho^{o}_{1-1}}{\mathrm{Re}\ \rho^{o}_{10}} \tag{59}$$

The elements of the antidiagonalized matrix are:

$$\mathrm{Im}\ a^{2}_{1-1} = -\ \mathrm{Im}\ a^{2}_{-11} = \mathrm{Im}\ \rho^{2}_{10}\sqrt{\frac{1 - 3\rho^{o}_{11} - 3\rho^{o}_{1-1}}{-\rho^{o}_{1-1}}} \tag{60}$$

$$a^{2}_{00} = 0$$

In order to diagonalize at the same time ρ^3 requires the two measurable matrix elements to be zero:

$$\mathrm{Im}\ \rho^3_{1-1} = \mathrm{Im}\ \rho^3_{10} = 0 \tag{61}$$

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TABLE I

The form of the density matrices $\rho^0, \rho^1, \rho^2, \rho^3$, making use of hermiticity and parity conservation. The underlined density matrix elements are measurable from the decay angular distributions. The lower half of the matrices is obtained by hermitian conjugation.

$$\rho^0 = \begin{pmatrix} \underline{\frac{1}{2}(1-\rho^0_{00})} & \underline{\mathrm{Re}\,\rho^0_{10}} + i\,\mathrm{Im}\,\rho^0_{10} & \underline{\mathrm{Re}\,\rho^0_{1-1}} \\ & \underline{\rho^0_{00}} & -(\underline{\mathrm{Re}\,\rho^0_{10}} - i\,\mathrm{Im}\,\rho^0_{10}) \\ & & \underline{\frac{1}{2}(1-\rho^0_{00})} \end{pmatrix}$$

$$\rho^1 = \begin{pmatrix} \underline{\rho^1_{11}} & \underline{\mathrm{Re}\,\rho^1_{10}} + i\,\mathrm{Im}\,\rho^1_{10} & \underline{\mathrm{Re}\,\rho^1_{1-1}} \\ & \underline{\rho^1_{00}} & -(\underline{\mathrm{Re}\,\rho^1_{10}} - i\,\mathrm{Im}\,\rho^1_{10}) \\ & & \underline{\rho^1_{11}} \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} \rho^2_{11} & \mathrm{Re}\,\rho^2_{10} + i\,\underline{\mathrm{Im}\,\rho^2_{10}} & \underline{i\,\mathrm{Im}\,\rho^2_{1-1}} \\ & 0 & (\mathrm{Re}\,\rho^2_{10} - i\,\underline{\mathrm{Im}\,\rho^2_{10}}) \\ & & -\rho^2_{11} \end{pmatrix}$$

$$\rho^3 = \begin{pmatrix} \rho^3_{11} & \mathrm{Re}\,\rho^0_{10} + i\,\underline{\mathrm{Im}\,\rho^3_{10}} & \underline{i\,\mathrm{Im}\,\rho^3_{1-1}} \\ & 0 & \mathrm{Re}\,\rho^3_{10} - i\,\underline{\mathrm{Im}\,\rho^3_{10}} \\ & & -\rho^3_{11} \end{pmatrix}$$

TABLE II

Restrictions on the density matrix elements

1. $0 \le \rho^{0}_{00} \le 1$

2. $\left|\rho^{0}_{1-1}\right| \le \frac{1}{2}(1 - \rho_{00})$

3. $(\mathrm{Re}\,\rho^{0}_{10})^2 \le \frac{1}{4}\rho^{0}_{00}\,(2 - \rho^{0}_{00} - \mathrm{Re}\,\rho^{0}_{1-1})$

4. $\left|\mathrm{Im}\,\rho^{3}_{1-1}\right| \le \frac{1}{2}(1 - \rho^{0}_{00})$

5. $\left|\mathrm{Im}\,\rho^{3}_{10}\right| \le \sqrt{\frac{1}{2}\rho^{0}_{00}\,(1 - \rho^{0}_{00})}$

6. $\left|\rho^{1}_{00}\right| \le \rho^{0}_{00}$

7. $\left|\rho^{1}_{11}\right| \le \frac{1}{2}(1 - \rho^{0}_{00})$

8. $\left|\rho^{1}_{1-1}\right| \le \frac{1}{2}(1 - \rho^{0}_{00})$

9. $\left|\mathrm{Re}\,\rho^{1}_{10}\right| \le \sqrt{\frac{1}{2}\rho^{0}_{00}\,(1 - \rho^{0}_{00})}$

10. $\left|\mathrm{Im}\,\rho^{2}_{1-1}\right| \le \frac{1}{2}(1 - \rho^{0}_{00})$

11. $\left|\mathrm{Im}\,\rho^{2}_{10}\right| \le \sqrt{\frac{1}{2}\,\rho^{0}_{00}\,(1 - \rho^{0}_{00})}$

12. $(\mathrm{Re}\,\rho^{0}_{10} \pm \mathrm{Re}\,\rho^{1}_{10})^2$

$$\le \frac{1}{8}\left[\left\{\frac{1}{2} + \left(\frac{1}{2}\rho^{0}_{00} \mp \rho^{1}_{11}\right) - \left(\rho^{0}_{1-1} \pm \rho^{1}_{1-1}\right)\right\}^2 - \left\{\frac{3}{2}\,\rho^{0}_{00} \pm \rho^{1}_{00} \mp \rho^{1}_{11} + \rho^{0}_{1-1} \pm \rho^{1}_{1-1} - \frac{1}{2}\right\}^2\right]$$

It would be too tedious to write down inequality 13.

TABLE III

Separation of natural and unnatural parity exchanges in the t-channel. The expressions with the upper (lower) sign give the measurable natural (unnatural) parity contributions to the density matrix.

$$\rho_{00}^{o\binom{N}{U}} = \rho_{00}^{o} \mp \rho_{00}^{1} \qquad \rho_{00}^{1\binom{N}{U}} = \rho_{00}^{1} \mp \rho_{00}^{o}$$

$$\mathrm{Re}\,\rho_{10}^{o\binom{N}{U}} = \mathrm{Re}\,\rho_{10}^{o} \mp \mathrm{Re}\,\rho_{10}^{1} \qquad \mathrm{Re}\,\rho_{10}^{1\binom{N}{U}} = \mathrm{Re}\,\rho_{10}^{1} \mp \mathrm{Re}\,\rho_{10}^{o}$$

$$\rho_{1-1}^{o\binom{N}{U}} = \rho_{1-1}^{o} \pm \rho_{11}^{1} \qquad \rho_{11}^{1\binom{N}{U}} = \rho_{11}^{1} \pm \rho_{1-1}^{o}$$

$$\rho_{11}^{o\binom{N}{U}} = \frac{1}{2}(1 - \rho_{00}^{o}) \pm \rho_{1-1}^{1} \qquad \rho_{1-1}^{1\binom{N}{U}} = \rho_{1-1}^{1} \pm \frac{1}{2}(1 - \rho_{00}^{o})$$