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QUANTUM ELECTRODYNAMICS
IN THE INFINITE MOMENTUM FRAME*

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ABSTRACT

We examine the formal foundations of quantum electrodynamics in the infinite momentum frame. We interpret the infinite momentum limit as the change of variables $\tau = 2^{-1/2}(t+z)$, $\mathcal{Z} = 2^{-1/2}(t-z)$, thus avoiding limiting procedures. Starting from the Feynman rules, we derive a τ-ordered perturbation expansion for the S-matrix. We then show how this expansion arises from a canonical formulation of the field theory in the infinite momentum frame. We feel that this approach should lead to convenient approximation schemes for electrodynamics at high energy.

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I. INTRODUCTION

The infinite momentum frame first appeared in connection with current algebra[1] as the limit of a reference frame moving with almost the speed of light. Weinberg[2] asked whether this limit might be more generally useful. He considered the infinite momentum limit of the old-fashioned perturbation diagrams for scalar meson theories and showed that the vacuum structure of these theories simplified in the limit. Later, Susskind[3,4] showed that the infinities which occur among the generators of the Poincaré group when they are boosted to a fast moving reference frame can be scaled or subtracted out consistently. The result is essentially a change of variables. Susskind used the new variables to draw attention to the (two dimensional) Galilean subgroup of the Poincaré group. He pointed out that the simplified vacuum structure and the non-relativistic kinematics of theories at infinite momentum might offer potential-theoretic intuition in relativistic quantum mechanics.

Bardakci and Halpern[5] further analyzed the structure of theories at infinite momentum. They viewed the infinite momentum limit as a change of variables from the laboratory time and z-coordinates to a new "time" $\tau = 2^{-1/2}(t+z)$ and a new "space" coordinate $\mathcal{Z} = 2^{-1/2}(t-z)$. Chang and Ma[6] considered the Feynman diagrams for a ϕ^3 theory and quantum electrodynamics from this point of view and were able to demonstrate the advantages of their approach in several illustrative calculations.

In this paper, we examine the formal foundations of quantum electrodynamics in the infinite momentum frame. We interpret the infinite momentum limit as the change of variables $\tau = 2^{-1/2}(t+z)$, $\mathcal{Z} = 2^{-1/2}(t-z)$, thus avoiding limiting procedures. We derive a τ-ordered perturbation series and show how such a series arises from a canonical formulation of the field theory. We feel that this approach should lead to convenient approximation schemes for electrodynamics at high energy. In particular, we hope to discuss, in a future paper, the recent extensive results of Cheng and Wu[7] on high energy processes in electrodynamics.

We divide this paper into several sections. In Section II we introduce the change of variables which defines the infinite momentum frame and review briefly the structure of the Poincaré group in the new variables. In Section III we deduce, beginning with the Feynman rules, the rules for the construction of scattering amplitudes from τ-ordered diagrams. The results are similar to Weinberg's results concerning the infinite momentum limit of scalar meson theories, but the appearance of spin results in new terms in the infinite momentum hamiltonian.[8] In Section IV we look at the field theoretic basis for the infinite momentum scattering theory rules. We begin with the usual lagrangian and develop the theory along the lines of the canonical formalism usually used in an ordinary reference frame. In the infinite momentum frame, several new features arise. This is because the planes "time" = constant play a preferred role in the canonical formalism, and in the infinite momentum frame these planes are light-like rather than space-like surfaces. We find, however, that it is possible to postulate equal "time" commutation relations which give a formally consistent theory, reproduce the free field theories if the interaction is turned off, and give a formal S-matrix expansion which agrees with the rules found in Section III.

II. CHOICE OF VARIABLES

We shall regard the "infinite momentum frame" as the reference frame obtained by choosing new space-time coordinates (τ, x, y, $\mathcal{Z}$) related to the usual coordinates (t, x, y, z) by

$$\tau = \frac{1}{\sqrt{2}}\,(t + z)$$

$$\mathcal{Z} = \frac{1}{\sqrt{2}}\,(t - z) \quad . \tag{2.1}$$

Thus the τ- and $\mathcal{Z}$-axes of the new frame lie on the light cone, as shown in Fig. 1. The infinite momentum frame is not a Lorentz reference frame, but is, in a certain sense, the limit of a Lorentz reference frame moving in the -z direction with nearly the speed of light.[9]

It will be convenient to use the usual covariant tensor notation for quantities in the new coordinate system. Let $\hat{x}^\mu = (\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3) = (t, x, y, z)$ be the coordinates of a space-time point in the ordinary coordinate system, $x^\mu = (x^0, x^1, x^2, x^3) = (\tau, x, y, \mathcal{Z})$ be the new coordinates of the same point. Then

$$x^\mu = C^\mu{}_\nu\, \hat{x}^\nu \quad , \tag{2.2}$$

where

$$C^\mu{}_\nu = \begin{pmatrix} 2^{-\frac{1}{2}} & 0 & 0 & 2^{-\frac{1}{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-\frac{1}{2}} & 0 & 0 & -2^{-\frac{1}{2}} \end{pmatrix} \quad . \tag{2.3}$$

In general, we shall use hatted symbols for vectors and tensors in the ordinary coordinate system, unhatted symbols for vectors and tensors in the new coordinate system. In particular, we shall use $g_{\mu\nu}$ for the metric tensor in the new coordinate system:

$$g_{\mu\nu} = (C^{-1})^{\alpha}{}_{\mu}\, \hat{g}_{\alpha\beta}\, (C^{-1})^{\beta}{}_{\nu} \tag{2.4}$$

We take for the ordinary metric tensor $\hat{g}_{00} = 1$, $\hat{g}_{11} = \hat{g}_{22} = \hat{g}_{33} = -1$. Then

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{2.5}$$

We use $g_{\mu\nu}$ to lower indices, so that $a_0 = a^3$, $a_3 = a^0$; this may seem confusing, but it has important consequences. For instance, the wave operator $\partial_\mu \partial^\mu = 2\partial_0 \partial_3 - \partial_1 \partial_1 - \partial_2 \partial_2$ is only first order in $\partial_0 = \partial/\partial\tau$.

Let us consider the generators of the Poincaré group in the new notation. Our conventions for the Poincaré algebra in the ordinary notation are

$$\left[\hat{P}^{\mu}, \hat{P}^{\nu}\right] = 0 \qquad \left[\hat{M}_{\mu\nu}, \hat{P}_{\rho}\right] = i(\hat{g}_{\nu\rho}\hat{P}_{\mu} - \hat{g}_{\mu\rho}\hat{P}_{\nu})$$
$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = i(\hat{g}_{\mu\sigma}\hat{M}_{\nu\rho} + \hat{g}_{\nu\rho}\hat{M}_{\mu\sigma} - \hat{g}_{\mu\rho} M_{\nu\sigma} - \hat{g}_{\nu\sigma}\hat{M}_{\mu\rho}) . \tag{2.6}$$

The generators of rotations and boosts are, respectively, $\hat{M}_{ij} = \epsilon_{ijk} J_k$ and $\hat{M}_{i0} = K_i$. Using the matrix $C^{\mu}{}_{\nu}$ to transform from the usual notation to the new notation, we obtain

$$P^{\mu} = (P^0, P^1, P^2, P^3) = (\eta, P^1, P^2, H) \tag{2.7}$$

and

$$M_{\mu\nu} = \begin{pmatrix} 0 & -S_1 & -S_2 & K_3 \\ S_1 & 0 & J_3 & B_1 \\ S_2 & -J_3 & 0 & B_2 \\ -K_3 & -B_1 & -B_2 & 0 \end{pmatrix} , \tag{2.8}$$

where

$$\begin{aligned} \eta &= \frac{1}{\sqrt{2}} (\hat{P}^0 + \hat{P}^3) \\ H &= \frac{1}{\sqrt{2}} (\hat{P}^0 - \hat{P}^3) \\ B_1 &= \frac{1}{\sqrt{2}} (K_1 + J_2) \\ B_2 &= \frac{1}{\sqrt{2}} (K_2 - J_1) \\ S_1 &= \frac{1}{\sqrt{2}} (K_1 - J_2) \\ S_2 &= \frac{1}{\sqrt{2}} (K_2 + J_1) \ . \end{aligned} \tag{2.9}$$

The commutation relations among these generators are, of course, given by (2.6) without the hats. The commutation relations among the operators $H, P^1, P^2, \eta, J_3, B_1, B_2$ are particularly interesting. They are the same as the commutation relations among the symmetry operators of non-relativistic quantum mechanics in two dimensions with

$$H \longrightarrow \text{hamiltonian} \ ,$$

$$\vec{P}_T \longrightarrow \text{momentum} \ ,$$

$\eta \longrightarrow$ mass ,

$J_3 \longrightarrow$ angular momentum ,

B_1 and $B_2 \longrightarrow$ generators of (Galilean) boosts in the x and y directions, respectively.

Indeed, we have

$$
\begin{aligned}
&[H,\vec{P}_T] = [H,\eta] = [\vec{P}_T,\eta] = [J_3,H] = [J_3,\eta] = [\vec{B},\eta] = 0 \\
&[J_3,P^k] = i\epsilon_{k\ell}P^\ell \qquad [J_3,B^k] = i\epsilon_{k\ell}B^\ell \\
&[B_k,H] = -i\,P^k \qquad [B_k,P^\ell] = -i\,\delta_{ij}\,\eta \quad ,
\end{aligned}
\tag{2.10}
$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. The commutation relations (2.10) are the result of an isomorphism between the subgroup of the Poincaré group generated by P^μ, J_3, and $\vec{B}$ and the Galilean symmetry group of non-relativistic quantum mechanics in two dimensions. This isomorphism results in a non-relativistic structure for quantum mechanics in the infinite momentum frame.[10] As one example of this isomorphism, we note that the mass shell condition, $m^2 = P^\mu P_\mu = 2\eta H - \vec{p}_T^{\,2}$, for a free particle implies that the free particle hamiltonian takes the non-relativistic form

$$
H = \frac{\vec{p}_T^{\,2}}{2\eta} + V_0 \quad , \tag{2.11}
$$

where $V_0 = m^2/2\eta$ is a constant potential.

It is easy to verify that the subgroup of the Poincaré group generated by P^1, P^2, η, J_3, B_1, B_2 leaves the planes τ = constant invariant. Thus these operators might be called "kinematical" symmetry operators.

Consider now the operators S_1 and S_2 in connection with our non-relativistic analogy. We find that S_1 and S_2 commute with each other and with the hamiltonian H. Thus they play the role of the "dynamical" symmetry operators sometimes encountered in non-relativistic quantum mechanics.[11] The operators S_1, S_2 form a vector $\vec{S}$ under rotations: $[J_3, S_k] = i\,\epsilon_{k\ell}\, S_\ell$. The commutation relations of $\vec{S}$ with η, $\vec{P}_T$, and $\vec{B}$ are

$$\begin{aligned} &[S_k, \eta] = -i\, P^k \qquad [S_k, P_\ell] = -i\,\delta_{k\ell}\, H \\ &[S_k, B_\ell] = -i\epsilon_{k\ell}\, J_3 + i\,\delta_{k\ell}\, K_3 \quad . \end{aligned} \tag{2.12}$$

Finally, we find from the commutation relations that the operator K_3 serves merely to rescale the operators we have considered so far:

$$\begin{aligned} e^{i\omega K_3}\, \eta\, e^{-i\omega K_3} &= e^{\omega} \eta \\ e^{i\omega K_3}\, \vec{P}_T\, e^{-i\omega K_3} &= \vec{P}_T \\ e^{i\omega K_3}\, H\, e^{-i\omega K_3} &= e^{-\omega}\, H \\ e^{i\omega K_3}\, J_3\, e^{-i\omega K_3} &= J_3' \\ e^{i\omega K_3}\, \vec{B}\, e^{-i\omega K_3} &= e^{\omega}\, \vec{B} \\ e^{i\omega K_3}\, \vec{S}\, e^{-i\omega K_3} &= e^{-\omega}\, \vec{S} \quad . \end{aligned} \tag{2.13}$$

The fact that the operators P^μ, $M_{\mu\nu}$ in the infinite momentum frame transform under z-boosts according to simple scaling laws suggests that the infinite momentum frame may be particularly adapted for high energy approximations.

III. SCATTERING THEORY

In this section, we regard the theory of quantum electrodynamics as being defined by the usual perturbation expansion of the S-matrix in Feynman diagrams. We rewrite the theory in the infinite momentum frame by systematically decomposing each covariant Feynman diagram into a sum of non-covariant τ-ordered diagrams. We consider the Feynman expansion as a formal expansion; thus we shall not be concerned in this paper with the convergence of the perturbation series, or convergence and regularization of the integrals.

A. Propagators

If we wanted to derive t-ordered diagrams from the Feynman diagrams we would begin by writing the Feynman electron propagator in the form

$$S_F(x) = \Theta(t)\, S^{(+)}(x) + \Theta(-t)\, S^{(-)}(x) \quad . \tag{3.1}$$

We will try to do the same thing using $\Theta(\tau)$ instead of $\Theta(t)$.

We start by considering the Klein-Gordon propagator

$$\Delta_F(x) \equiv (2\pi)^{-4} \int d^4p\, e^{-ip_\mu x^\mu} \left[p^\nu p_\nu - m^2 + i\epsilon\right]^{-1} \tag{3.2}$$

$$= (2\pi)^{-4} \int d^2\vec{p}_T \int d\eta\, e^{-i(\eta\mathfrak{z} - \vec{p}_T \cdot \vec{x}_T)} \tag{3.3}$$

$$\times \int dH\, e^{-iH\tau} \left[2\eta H - \vec{p}_T^{\,2} - m^2 + i\epsilon\right]^{-1}$$

We can do the H-integral by contour integration. If $\tau > 0$ we close the contour in the lower half H-plane. The integrand has one pole at $H = (\vec{p}_T^{\,2} + m^2 - i\epsilon)/2\eta$, which is in the lower (upper) half plane if η is positive (negative). Thus we get

$$\Delta_F(x) = \frac{-i}{2(2\pi)^3}\int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \exp\left[-i\left(\frac{\vec{p}_T^{\,2}+m^2}{2\eta}\,\tau + \eta z - \vec{p}_T\cdot\vec{x}_T\right)\right] .$$

Similarly, if $\tau < 0$ we get

$$\Delta_F(x) = \frac{+i}{2(2\pi)^3}\int d^2\vec{p}_T \int_{-\infty}^0 \frac{d\eta}{\eta} \exp\left[-i\left(\frac{\vec{p}_T^{\,2}+m^2}{2\eta}\,\tau + \eta z - \vec{p}_T\cdot\vec{x}_T\right)\right] .$$

Thus (with the change of variable $\vec{p}_T \to -\vec{p}_T$ and $\eta \to -\eta$ for $\tau < 0$) we have the required decomposition for $\Delta_F(x)$ [12]:

$$\Delta_F(x) = \frac{-i}{2(2\pi)^3}\int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \left\{ \Theta(\tau)\, e^{-ip_\mu x^\mu} + \Theta(-\tau)\, e^{+ip_\mu x^\mu} \right\} , \tag{3.4}$$

where

$$p_0 = H(\eta, \vec{p}_T) = \frac{\vec{p}_T^{\,2}+m^2}{2\eta} \tag{3.5}$$

is the free particle hamiltonian. Notice that

$$d^2\vec{p}_T \frac{d\eta}{\eta} = d^3\hat{\vec{p}}/\hat{p}^0$$

is the invariant differential surface element on the mass shell.

We can use the deomposition (3.4) of $\Delta_F(x)$ to derive a decomposition for the electron propagator,

$$S_F(x) \equiv (i\partial_\mu \gamma^\mu + m)\, \Delta_F(x) . \tag{3.6}$$

(In keeping with our convention, the γ^μ are the γ-matrices in the new notation. We shall use $\hat{\gamma}^\mu$ for the γ-matrices in the ordinary notation; thus $\gamma^0 = 2^{-\frac{1}{2}}(\hat{\gamma}^0+\hat{\gamma}^3)$ etc. Table I in Section IV contains some useful identities for the new γ-matrices.)

When we differentiate $\Delta_F(x)$ in (3.4) we get a term proportional to $\Theta(\tau)$, a term proportional to $\Theta(-\tau)$, and a third term proportional to $\delta(\tau) = \partial_0\Theta(\tau)$. As we will see, this third term results in an extra term in the infinite momentum frame hamiltonian. Doing the differentiation we get

$$
\begin{aligned}
S_F(x) = \frac{-i}{2(2\pi)^3} \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \Big\{ & \Theta(\tau) \left[\not{p} + m\right] e^{-ip_\mu x^\mu} \\
& + \Theta(-\tau)\left[-\not{p} + m\right] e^{+ip_\mu x^\mu} \Big\} \\
+ \frac{1}{2(2\pi)^3} \, \delta(\tau)\gamma^0 \int d^2\vec{p}_T & \int_{-\infty}^{\infty} \frac{d\eta}{\eta} \, e^{-i(\eta z - \vec{p}_T \cdot \vec{x}_T)}
\end{aligned}
\tag{3.7}
$$

We will also need a decomposition for the photon propagator. We start with

$$
D_F(x)^{\mu\nu} = (2\pi)^{-4} \int d^4p \, e^{-ip_\mu x^\mu} \, \frac{-g^{\mu\nu}}{p_\mu p^\mu + i\epsilon} \quad . \tag{3.8}
$$

As we will see, a great simplification in the theory will result if we choose the gauge $A^0 = 0$, which might be called the infinite momentum gauge. To write the propagator in this gauge we define the polarization vectors

$$
\begin{aligned}
e_1(p)^\mu &\equiv \frac{1}{\eta} \, (0, \eta, 0, p^1) \\
e_2(p)^\mu &\equiv \frac{1}{\eta} \, (0, 0, \eta, p^2) \quad .
\end{aligned}
\tag{3.9}
$$

These polarization vectors satisfy the orthogonality conditions $e_\lambda{}^\mu e_{\rho\mu} = -\delta_{\lambda\rho}$, $e_\lambda(p)^\mu p_\mu = 0$. By direct calculation, we find

$$
\begin{aligned}
-g^{\mu\nu} = \textstyle\sum_{\lambda=1}^{2} \; & e_\lambda(p)^\mu e_\lambda(p)^\nu - \frac{1}{\eta}\, \delta^\mu{}_3 \, p^\nu - \frac{1}{\eta}\, p^\mu \delta^\nu{}_3 \\
& + \frac{1}{\eta^2}\,(2\eta H - \vec{p}_T{}^2)\,\delta^\mu{}_3 \delta^\nu{}_3 \quad ,
\end{aligned}
\tag{3.10}
$$

Let us make the replacement (3.10) in our integral for $D_F(x)^{\mu\nu}$. We note that the gauge terms $\eta^{-1}\delta^\mu{}_3 p^\nu$ and $\eta^{-1} p^\mu \delta^\nu{}_3$ will not contribute to any physical process because of current conservation. Thus we may drop these terms without changing the theory. This leaves us with

$$D_F(x)^{\mu\nu} = (2\pi)^{-4} \int d^4p \; e^{-ip_\mu x^\mu} \; \frac{\Sigma_\lambda \, e_\lambda(p)^\mu \, e_\lambda(p)^\nu}{p_\mu p^\mu + i\epsilon}$$

$$+ (2\pi)^{-4} \delta^\mu{}_3 \, \delta^\nu{}_3 \int d^4p \; e^{-ip_\mu x^\mu} \; \frac{1}{\eta^2} \; \frac{p_\mu p^\mu}{p_\mu p^\mu + i\epsilon} \quad . \tag{3.11}$$

We can do the H-integration in the first term by contour integration, just as we did for $\Delta_F(x)$. The result is

$$\text{first term} = \frac{-i}{2(2\pi)^3} \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \; (\Sigma_\lambda e_\lambda(p)^\mu \, e_\lambda(p)^\nu)$$

$$\left\{ \Theta(\tau) \, e^{-ip_\mu x^\mu} + \Theta(-\tau) \, e^{+ip_\mu x^\mu} \right\} \quad .$$

In the second term $p_\mu p^\mu / (p_\mu p^\mu + i\epsilon) \to 1$ as $\epsilon \to 0^+$ so that the H-integral is

$$\int_{-\infty}^{\infty} dH \, e^{-iH\tau} = 2\pi \, \delta(\tau) \quad .$$

Thus the second term is

$$\delta(\tau)(2\pi)^{-3} \delta^\mu{}_3 \, \delta^\nu{}_3 \int d^2\vec{p}_T \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2} \; e^{-i(\eta \not{x} - \vec{p}_T \cdot \vec{x}_T)} \quad .$$

This term will result in an extra term in the hamiltonian which is analogous to the Coulomb force term which appears in quantum electrodynamics in the Coulomb gauge.

In sum, then, our photon propagator takes the form

$$D_F^{\mu\nu}(x) = \frac{-i}{2(2\pi)^3}\int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \left(\Sigma_\lambda e_\lambda (p)^\mu e_\lambda (p)^\nu\right)$$

$$\left\{\Theta(\tau)\, e^{-ip_\mu x^\mu} + \Theta(-\tau)\, e^{+ip_\mu x^\mu}\right\} \tag{3.12}$$

$$+ \frac{1}{(2\pi)^3}\, \delta(\tau)\, \delta^\mu_3\, \delta^\nu_3 \int d^2\vec{p}_T \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2}\, e^{-i(\eta\mathfrak{z} - \vec{p}_T\cdot\vec{x}_T)} ,$$

where

$$p_0 = H = \vec{p}_T^{\,2}/2\eta \;. \tag{3.13}$$

B. Diagrams

We start with the usual Feynman rules in coordinate space. For definiteness, let us consider a particular diagram, say the one shown in Figure 2a. We fix our conventions by writing out the contribution of this diagram to the S-matrix:

$$M = (-ie)^3 \int d^4x_1\, d^4x_2\, d^4x_3 \left| \overline{\psi_1(x_1)_c}\, \gamma_\mu\, \psi_3(x_1)_c \right|$$

$$\times \left[\overline{\psi_4(x_2)}\, \gamma_\nu\, i S_F(x_2 - x_3)\, \gamma_\sigma\, \psi_2(x_3)\right] i D_F(x_2 - x_1)^{\mu\nu}\, \epsilon^\sigma(x_3)^* \;. \tag{3.14}$$

The electron wave functions used here are

$$\psi(x) = \left(2(2\pi)^3\right)^{-\frac{1}{2}} e^{-ip_\mu x^\mu}\, u(p, s) \quad , \tag{3.15}$$

where p and s are the momentum and spin of the electron and the spinors $u(p, s)$ are normalized to $\bar{u}u = 2m$. For positrons we use the charge conjugate wave functions

$$\psi(x)_c = \left(2(2\pi)^3\right)^{-\frac{1}{2}} e^{+ip_\mu x^\mu}\, u(p, s)_c \, , \tag{3.16}$$

where p and s are the physical momentum and spin of the positron. The photon wave function is

$$e^{\mu}(x) = \left(2(2\pi)^3\right)^{-\frac{1}{2}} e^{-ip_{\mu}x^{\mu}} e_{\lambda}(p)^{\mu} , \tag{3.17}$$

where $e_{\lambda}(p)$ is one of our infinite momentum gauge polarization vectors. Finally, it may be useful to note that although the γ-matrices appearing explicitly in Eq. (3.14) are, as always, the "new" γ-matrices, the old $\hat{\gamma}^{o}$ still plays a role in $\overline{\psi} = \psi^{\dagger}\hat{\gamma}^{o}$.

We begin the program of deriving the rules for τ-ordered diagrams by inserting the momentum expansions (3.7) and (3.12) for the propagators into (3.14). Let us, for the moment, ignore the contributions to S_F and $D_F^{\mu\nu}$ proportional to $\delta(\tau)$. Then each of the 3! possible τ-orderings of the vertices determines a τ-ordered diagram; let us consider, say, the ordering $\tau_1 < \tau_2 < \tau_3$. For this diagram we draw the picture in Figure 2b. The corresponding contribution to the S-matrix is obtained by inserting $\Theta(\tau_3 - \tau_2)\,\Theta(\tau_2 - \tau_1)$ into (3.14). Thus only one of the $\Theta(\tau)$ or $\Theta(-\tau)$ terms survives from each propagator. We can do the $\vec{x}_T$- and $\mathscr{y}$- integrations to give $(2\pi)^3\delta^2(\vec{p}_{T\,in} - \vec{p}_{T\,out})\,\delta(\eta_{in} - \eta_{out})$ at each vertex. The τ-integrals in this example are

$$\int d\tau_1\, d\tau_2\, d\tau_3\, \Theta(\tau_3 - \tau_2)\,\Theta(\tau_2 - \tau_1)\, \exp(-i[(H_1 - H_3 - H_6)\tau_1 + (H_6 - H_4 - H_7)\tau_2 + (H_7 + H_2 - H_5)\tau_3]). \tag{3.18}$$

With the change of variables

$$\begin{aligned} T_0 &= \tau_1 & \tau_1 &= T_0 \\ T_1 &= \tau_2 - \tau_1 & \tau_2 &= T_0 + T_1 \\ T_2 &= \tau_3 - \tau_2 & \tau_3 &= T_0 + T_1 + T_2 \ , \end{aligned}$$

the τ - integrals become

$$\int dT_0\, dT_1\, dT_2\, \Theta(T_1)\Theta(T_2)\exp(-i[(\mathcal{H}_i-\mathcal{H}_f)T_0 + (\mathcal{H}_1-\mathcal{H}_f)T_1 + (\mathcal{H}_2-\mathcal{H}_f)T_2]), \tag{3.19}$$

where $\mathcal{H}_i = H_1 + H_2$ is the total "energy" of the initial state, $\mathcal{H}_1 = H_3 + H_6 + H_2$ is the total "energy" of the first intermediate state, $\mathcal{H}_2 = H_3 + H_4 + H_7 + H_2$ is the total "energy" of the second intermediate state, and $\mathcal{H}_f = H_3 + H_4 + H_5$ is the total "energy" of the final state. The integrals can now be done using

$$\begin{aligned} \int_{-\infty}^{\infty} dT\, e^{-i\mathcal{H}T} &= 2\pi\,\delta(T) \\ \int_{0}^{\infty} dT\, e^{i\mathcal{H}T} &= \frac{i}{\mathcal{H} + i\epsilon} \quad . \end{aligned} \tag{3.20}$$

Thus we get an overall factor of $(2\pi)\delta(\mathcal{H}_f - \mathcal{H}_i)$ and a factor of $i(\mathcal{H}_f - \mathcal{H} + i\epsilon)^{-1}$ for each intermediate state. With a little thought, one can convince himself that this results is completely general.

We now have to consider the effect of the $\delta(\tau)$ terms in the propagators, which we have so far omitted. To the contributions to the S-matrix from a particular Feynman diagram so far obtained, we should add the contributions obtained by replacing the $\tau \neq 0$ parts of $S_F(x)$ and $D_F(x)^{\mu\nu}$ with the $\delta(\tau)$ part in any of the internal lines. We will use the pictures in Figure 3 for the $\delta(\tau)$ parts of $S_F(x_2 - x_1)$ and $D_F(x_2 - x_1)^{\mu\nu}$. Diagrams containing one or more of these $\delta(\tau)$ internal lines are then treated as before except that we consider structures such as those shown in Figure 4 as single vertices when we do the τ - ordering. Thus we get $(2\pi)^3\delta^2(\vec{p}_{T\,in} - \vec{p}_{T\,out})\delta(\eta_{in} - \eta_{out})$ at each end of a $\delta(\tau)$ internal line, an overall $(2\pi)\delta(\mathcal{H}_f - \mathcal{H}_i)$, and a factor $i(\mathcal{H}_f - \mathcal{H} + i\epsilon)^{-1}$ for each intermediate state between two different "times".

At this point, let us notice that diagrams in which two or more $\delta(\tau)$ parts of propagators are linked together give a zero contribution to the S-matrix. Indeed, consider a diagram containing a part like that shown in Figure 4c. The corresponding contribution to the S-matrix contains $\gamma^o\gamma_\mu\gamma^o$ times $D_F^{\mu\nu}$ or e^μ. Because of our choice of gauge, only $\mu = 1,2,3$ occurs; but, since $\gamma^o\gamma^o = g^{oo} = 0$, we have $\gamma^o\gamma_1\gamma^o = -\gamma^o\gamma^o\gamma_1 = 0$, $\gamma^o\gamma_2\gamma^o = -\gamma^o\gamma^o\gamma_2 = 0$, and $\gamma^o\gamma_3\gamma^o = \gamma^o\gamma^o\gamma^o = 0$. Hence $\gamma^o\gamma_\mu\gamma^o e^\mu = \gamma^o\gamma_\mu\gamma^o D_F^{\mu\nu} = 0$. Now consider a diagram in which the structure shown in Figure 4d occurs. The corresponding contribution to the S-matrix contains a factor $\delta^\mu_3\,\delta^\nu_3\,(\cdots\gamma_\nu\,\gamma^o\cdots) = \delta^\mu_3\,(\cdots\gamma_3\gamma^o\cdots) = \delta^\mu_3\,(\cdots\gamma^o\gamma^o\cdots) = 0$.

We are now in a position to summarize the rules for τ-ordered diagrams. With our choice of gauge there are three types of interactions as shown in Figure 5. These interactions are to be τ-ordered in all possible ways. We then associate the following factors with the parts of the diagram:[13]

i) wave functions $u(p,s)$, $\overline{u(p,s)}$, $\overline{u_c(p,s)}$, $u_c(p,s)$, and $e_\lambda(p)$ for the external lines;

ii) $(\not p + m) = \sum_s u(p,s)\,\overline{u(p,s)}$ for electron propagators;

$(-\not p + m) = -\Sigma_s\, u_c(p,s)\,\overline{u_c(p,s)}$ for positron propagators;

$\Sigma_\lambda\, e_\lambda(p)^\mu\, e_\lambda(p)^\nu$ for photon propagators;

iii) $(4\pi)^{-3/2}\, e\, \gamma_\mu\, \delta(\eta_{out} - \eta_{in})\, \delta^2(\vec{p}_{T\,out} - \vec{p}_{T\,in})$ for each vertex as shown in Figure 5a;

$$\frac{e^2}{4(2\pi)^3}\,\delta^\mu_3\,\delta^\nu_3\,\frac{1}{\eta_0^2}\,\delta(\eta_{out} - \eta_{in})\,\delta^2(\vec{p}_{T\,out} - \vec{p}_{T\,in})\cdots\gamma_\mu\cdots\gamma_\nu\cdots$$

for each vertex as shown in Figure 5b, where η_0 is the total η transferred across the vertex;

$$\frac{e^2}{8(2\pi)^3} \gamma_\nu \gamma^o \gamma_\mu \frac{1}{\eta_0} \delta(\eta_{out} - \eta_{in}) \delta^2(\vec{p}_{T\,out} - \vec{p}_{T\,in})$$ for each vertex as shown in Figure 5c;

iv) an overall factor of $-2\pi i \, \delta(\mathscr{H}_f - \mathscr{H}_i)$, and a factor of $(\mathscr{H}_f - \mathscr{H} + i\epsilon)^{-1}$ for each intermediate state;

v) the usual overall sign from the Wick reduction, determined by the structure of the original Feynman diagram;

vi) an integration $\int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta}$ for each internal line.

Note that since each line carries positive η and η is conserved in each interaction, vacuum diagrams like those shown in Figure 6 cannot occur.

In the next section we shall develop the canonical field theory for quantum electrodynamics in the infinite momentum frame. As we will see, the hamiltonian we will obtain reproduces the scattering theory we have developed here.

IV. CANONICAL FIELD THEORY

A. Equations of Motion

We base our field theory on the usual lagrangian density[14]

$$\mathscr{L}(x) = \bar{\Psi}\left\{(\tfrac{1}{2}\overleftrightarrow{\partial}_{\mu} - e A_{\mu})\gamma^{\mu} - m\right\}\Psi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} , \tag{4.1}$$

where the electromagnetic field tensor $F^{\mu\nu}$ is related to the potential A^{μ} by $F^{\mu\nu} = \partial^{\nu} A^{\mu} - \partial^{\mu} A^{\nu}$. Variation of the fields Ψ, $\bar{\Psi}$, and A^{μ} give the Dirac equation and Maxwell's equations:

$$\left\{(i\,\partial_{\mu} - e A_{\mu})\gamma^{\mu} - m\right\}\Psi = 0 \tag{4.2}$$

$$\partial_{\lambda} F^{\mu\lambda} = e\bar{\Psi}\gamma^{\mu}\Psi \equiv J^{\mu}. \tag{4.3}$$

It will be convenient to work in the infinite momentum gauge, $A^{o}(x) = 0$. In this gauge the field tensor is related to the potential by

$$F^{o\mu} = -\partial^{o} A^{\mu} = -\partial_3 A^{\mu} \qquad (\mu = 1, 2, 3) \quad . \tag{4.4}$$

In order to completely specify the gauge, we must choose boundary conditions for $A^{\mu}(x)$. For reasons of symmetry, we will require that $A^{\mu}(x^o, x^1, x^2, +\infty) = -A^{\mu}(x^o, x^1, x^2, -\infty)$. With these boundary conditions, the solution of (4.4) is

$$A^{\mu}(x) = -\tfrac{1}{2}\int d\xi\ \epsilon(x^3 - \xi)\ F^{o\mu}(x^o, x^1, x^2, \xi) , \tag{4.5}$$

where

$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \quad .$$

It is perhaps not obvious that the gauge conditions we have imposed are consistent with Maxwell's equations. Thus it is reassuring to note that the definition (4.5) of $A^{\mu}(x)$ works for the classical electromagnetic field. If the field $F^{\mu\nu}(x)$ is produced by a current which, say, is non-zero only in a bounded space-time region, then the components $F^{0\mu}(x)$ go to zero like $(x^3)^{-2}$ as $|x^3| \to \infty$. Thus the integral (4.5) is well defined. Using the homogeneous Maxwell's equations, $\partial^{\mu}F^{\nu\lambda} + \partial^{\nu}F^{\lambda\mu} + \partial^{\lambda}F^{\mu\nu} = 0$, one can easily show that the potential A^{μ} defined by (4.5) satisfies $\partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu} = F^{\mu\nu}$ for all indices μ, ν.

We have eliminated one component of $A^{\mu}(x)$ by our choice of gauge. Only two of the remaining three components can be independent dynamical variables, since the three components of $A^{\mu}(x)$ are related at any "time" x^0 by the differential equation

$$\partial_3(\partial_1 A^1 + \partial_2 A^2 + \partial_3 A^3) = -\partial_{\mu}F^{0\mu} = -J^0 \quad . \tag{4.6}$$

It will be convenient to regard A^1 and A^2 as the independent components. Then A^3 satisfies

$$\partial_3\partial_3 A^3 = -\partial_3\partial_j A^j - J^0 \quad .$$

(We adopt the convention that Latin indices are to be summed from 1 to 2.) The solution of this equation which equals A^3 as defined by (4.5) is

$$A^3(x) = -\tfrac{1}{2}\int d\xi\, |x^3 - \xi| \left\{ \partial_3\partial_j A^j(x^0, x^1, x^2, \xi) + J^0(x^0, x^1, x^2, \xi) \right\} . \tag{4.7}$$

To see that this equation reproduces our definition of A^3 in terms of F^{03}, write it as [15]

$$
\begin{aligned}
A^3(x) &= -\tfrac{1}{2}\int d\xi \; |x^3-\xi| \; \partial_3 F^{03}(x^0,x^1,x^2,\xi) \\
&= -\tfrac{1}{2}\int d\xi \left(\frac{\partial}{\partial x^3} |x^3-\xi|\right) F^{03}(x^0,x^1,x^2,\xi) \\
&= -\tfrac{1}{2}\int d\xi \; \epsilon(x^3-\xi)\, F^{03}(x^0,x^1,x^2,\xi) \;.
\end{aligned}
\tag{4.8}
$$

Thus only two components, $A^1(x)$ and $A^2(x)$, of $A^\mu(x)$ are dynamical variables. $A^0(x)$ is identically zero, and $A^3(x)$ is determined at any "time" x^0 by $A^1(x)$, $A^2(x)$, and $\Psi(x)$ at that x^0 by means of Eq. (4.7). This reduction in the number of independent components of A^μ is a familiar feature of quantum electrodynamics in any reference frame.

In the infinite momentum frame, we find that the number of independent components of the electron field $\Psi(x)$ is also reduced from four to two. In order to show this we pause briefly to examine the properties of the infinite momentum γ-matrices, $\gamma^\mu = C^\mu_{\ \nu}\, \hat{\gamma}^\nu$. The "ordinary" γ-matrices $\hat{\gamma}^\mu$ are chosen to satisfy $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\hat{g}^{\mu\nu}$ and $\hat{\gamma}^{\mu\dagger} = \gamma_\mu$. Thus the infinite momentum γ-matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\gamma^{\mu\dagger} = \gamma_\mu$. From this it follows easily that $P_+ \equiv \tfrac{1}{2}\gamma^3\gamma^0$ and $P_- \equiv \tfrac{1}{2}\gamma^0\gamma^3$ are hermitian projection operators with $P_+P_- = 0$ and $P_+ + P_- = 1$. These facts, as well as some others that we will need later are listed for convenient reference in Table 1.

It will be helpful to have a specific representation of the γ-matrices in mind. We will consistently use

$$
\hat{\gamma}^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad \hat{\gamma}^\alpha = \begin{pmatrix} 0 & -\sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix} \quad (\alpha = 1,2,3) \;, \tag{4.9}
$$

where σ^1, σ^2, σ^3 are the usual 2×2 Pauli matrices. With this choice for the γ^μ, we find that

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.10}$$

By applying the projection matrices $P_\pm$ to the electron field $\Psi(x)$ we obtain two two-component fields which we call $\Psi_+(x)$ and $\Psi_-(x)$:

$$\Psi_+ \equiv P_+ \Psi = \begin{pmatrix} \Psi_1 \\ 0 \\ 0 \\ \Psi_4 \end{pmatrix} \qquad \Psi_- \equiv P_- \Psi = \begin{pmatrix} 0 \\ \Psi_2 \\ \Psi_3 \\ 0 \end{pmatrix} . \tag{4.11}$$

With this preparation completed, we are ready to examine the dynamics of the electron field $\Psi(x)$. If we multiply the Dirac equation by γ^o and recall that $\gamma^o \gamma^o = 0$, we obtain

$$(i\partial_3 - eA_3)\, \gamma^o \gamma^3 \Psi = \gamma^o [-(i\partial_j - eA_j)\, \gamma^j + m] \Psi .$$

Using our γ-matrix identities, this becomes

$$(i\partial_3 - eA_3)\, \Psi_- = \tfrac{1}{2} [(i\partial_j - eA_j)\, \gamma^j + m] \gamma^o \Psi_+ .$$

This differential equation is considerably simplified because of our choice of gauge, $A_3 = A^o = 0$. Thus

$$\partial_3 \Psi_- = -\frac{i}{2} [(i\partial_j - eA_j)\, \gamma^j + m] \gamma^o \Psi_+ . \tag{4.12}$$

For reasons of symmetry, we write the solution of Eq. (4.12) as

$$\Psi_-(x) = -\frac{i}{4}\int d\xi\ \epsilon(x^3-\xi)\Big\{\big[i\partial_j - eA_j(x^o,\vec{x}_T,\xi)\big]\gamma^j + m\Big\}\gamma^o\Psi_+(x^o,\vec{x}_T,\xi)\ . \tag{4.13}$$

Thus the two components of $\Psi_-(x)$ are dependent variables in the infinite momentum frame. They are determined at any "time" x^o by the independent fields $\Psi_+(x)$ and $A^j(x)$ at the same x^o. We recall that the dependent variable $A^3(x)$ is determined at any x^o by A^j and J^o at that x^o. It is reassuring to note that the dependence of $J^o(x)$ on the independent fields Ψ_+, A^j is very simple:

$$J^o = \overline{\Psi}\gamma^o\Psi = \Psi^\dagger\hat{\gamma}^o\gamma^o\Psi = \sqrt{2}\ \Psi_+^\dagger\Psi_+\ . \tag{4.14}$$

What are the equations of motion for our independent fields $A^j(x)$ and $\Psi_+(x)$? For $A^j(x)$ we have the Maxwell's equations

$$\partial_\nu(\partial^\nu A^j - \partial^j A^\nu) = J^j\ ,$$

or

$$\begin{aligned} 2\partial_0\partial_3 A^j &= J^j + \partial^j\partial_\nu A^\nu - \partial_i\partial^i A^j \\ &= J^j + \partial^j\partial_3 A^3 + \partial^j\partial_i A^i - \partial_i\partial^i A^j \\ &= J^j + \partial^j\partial_3 A^3 + \partial_i F^{ij}\ . \end{aligned} \tag{4.15}$$

Using the definition (4.5) of A^j in terms of F^{oj}, we have

$$\partial_0 A^j(x) = \tfrac{1}{2}\int d\xi\ \epsilon(x^3-\xi)\,\partial_0\partial_3 A^j(x^o,\vec{x}_T,\xi)\ . \tag{4.16}$$

Substituting into (4.17) from (4.16), we obtain

$$\partial_0 A^j(x) = \tfrac{1}{4}\,\partial^i \int d\xi\; \epsilon(x^3-\xi)\,\partial_3 A^3(x^o,\vec{x}_T,\xi)$$

$$+\tfrac{1}{4}\int d\xi\; \epsilon(x^3-\xi)\left\{J^j(x^o,\vec{x}_T,\xi) + \partial_i F^{ij}(x_0,\vec{x}_T,\xi)\right\} .$$

Since the integral in the first term is just $2A^3(x)$ because of Eq. (4.5), we have, finally,

$$\partial_0 A^j(x) = \tfrac{1}{2}\,\partial^j A^3(x) + \tfrac{1}{4}\int d\xi\; \epsilon(x^3-\xi)\left\{J^j(x^o,\vec{x}_T,\xi) + \partial_i F^{ij}(x_0,\vec{x}_T,\xi)\right\} . \tag{4.17}$$

We can obtain the equation of motion for $\Psi_+(x)$ by multiplying the Dirac equation by γ^3. After making use of some of our γ-matrix identities, we obtain

$$\partial_0 \Psi_+(x) = -ie A^3(x)\,\Psi_+(x) - \frac{i}{2}\left[(i\partial_j - eA_j(x))\gamma^j + m\right]\gamma^3\,\Psi_-(x) . \tag{4.18}$$

B. Momentum and Angular Momentum

The invariance of the langrangian under the Poincaré group provides us, using Noether's theorem, with a conserved momentum tensor $T_\mu^{\ \lambda}(x)$ and a conserved angular momentum tensor $J_{\mu\nu}^{\ \ \lambda}(x)$:

$$T_\mu^{\ \lambda} = \overline{\Psi}\,\frac{i}{2}\overleftrightarrow{\partial}_\mu\,\gamma^\lambda\,\Psi + (\partial_\mu A_\alpha)\,F^{\lambda\alpha} - g_\mu^{\ \lambda}\,\mathscr{L} , \tag{4.19}$$

$$J_{\mu\nu}^{\ \ \lambda} = x_\mu T_\nu^{\ \lambda} - x_\nu T_\mu^{\ \lambda} + S_{\mu\nu}^{\ \ \lambda} , \tag{4.20}$$

where

$$S_{\mu\nu}^{\ \ \lambda} = \frac{i}{8}\,\overline{\Psi}\,(\gamma^\lambda[\gamma_\mu,\gamma_\nu] + [\gamma_\mu,\gamma_\nu]\gamma^\lambda)\,\Psi + F^\lambda_{\ \mu}A_\nu - F^\lambda_{\ \nu}A_\mu . \tag{4.21}$$

If the fields satisfy the equations of motion, then $T_\mu^{\ \lambda}$ and $J_{\mu\nu}^{\ \ \lambda}$ are conserved:

$$\partial_\lambda T_\mu^{\ \lambda} = 0 \qquad\qquad \partial_\lambda J_{\mu\nu}^{\ \ \lambda} = 0 . \tag{4.22}$$

Thus the total momentum,

$$P_\mu \equiv \int d^2\vec{x}_T dx^3 \, T_\mu{}^o \,, \tag{4.23}$$

and the total angular momentum,

$$M_{\mu\nu} = \int d^2\vec{x}_T dx^3 \, J_{\mu\nu}{}^o \quad , \tag{4.24}$$

are constants of the motion. In our quantum theory, P_μ and $M_{\mu\nu}$ are the generators of the Poincare group.[16]

We recall from our discussion of the Poincaré group in Section II that the operators $P_1, P_2, P_3, M_{12}, M_{13}$, and M_{23} are "kinematical" symmetry operators in that the subgroups of the Poincare group which they generate leaves the planes τ = constant invariant. Thus we might expect that they take a particularly simple form. Indeed, we find that

$$T_\alpha{}^o = \sqrt{2}\,\Psi_+^\dagger \frac{i}{2} \overleftrightarrow{\partial}_\alpha \Psi_+ - (\partial_\alpha A^i)(\partial_3 A_i) \qquad (\alpha = 1,2,3) \tag{4.25}$$

$$J_{12}{}^o = x_1 T_2{}^o - x_2 T_1{}^o + \sqrt{2}\,\Psi_+^\dagger \frac{i}{2} \gamma_1 \gamma_2 \Psi_+ + A^1(\partial_3 A^2) - A^2(\partial_3 A^1) \tag{4.26}$$

$$J_{13}{}^o = x_1 T_3{}^o - x_3 T_1{}^o \tag{4.27}$$

$$J_{23}{}^o = x_2 T_3{}^o - x_3 T_2{}^o \;. \tag{4.28}$$

Note that these operators involve only the independent fields Ψ_+ and A^i, and thus do not depend on the coupling constant e.

The most important operator in the theory is, of course, the hamiltonian $H = P_0$. From the definition (4.19) we have

$$T_0^{\,o} = \overline{\Psi}\,\frac{i}{2}\,\overleftrightarrow{\partial_0}\,\gamma^o\,\Psi + (\partial_0 A_\lambda)\,F^{o\lambda} - \overline{\Psi}\left\{(\frac{i}{2}\,\partial_\mu - eA_\mu)\,\gamma^\mu - m\right\}\Psi + \frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu}\,.$$

The first two terms cancel the terms in the lagrangian containing ∂_0, and we are left with

$$\begin{aligned} T_0^{\,o} &= -\overline{\Psi}\left(\frac{i}{2}\sum_{\nu=1}^{3}\overleftrightarrow{\partial_\nu}\,\gamma^\nu - m\right)\Psi + eA_\mu\,\overline{\Psi}\,\gamma^\mu\Psi \\ &\quad + \frac{1}{2}\,F^{12}F_{12} - \frac{1}{2}\,(\partial_3 A^3)(\partial_3 A^3) - (\partial_j A^3)(\partial_3 A^j)\,. \end{aligned} \tag{4.29}$$

C. <u>Momentum Space Expansions of the Fields; Commutation Relations</u>

Let $\Psi_+(\vec{p}_T, \eta; \tau)$ be the Fourier transform, at the "time" τ, of $\Psi_+(x)$, so that

$$\Psi_+(\tau, \vec{x}_T, \xi) = (2\pi)^{-3/2}\int d^2\vec{p}_T\,d\eta\;e^{-i(\eta\xi - \vec{p}_T\cdot\vec{x}_T)}\;\Psi_+(\vec{p}_T, \eta; \tau)\;. \tag{4.30}$$

It will be useful to define operators $b(\vec{p}_T, \eta; s; \tau)$ and $d(\vec{p}_T, \eta; s; \tau)$, where s takes the values $\pm\frac{1}{2}$, by

$$\begin{aligned} 2^{-\frac{1}{4}}\,\eta^{-\frac{1}{2}}\,b(\vec{p}_T, \eta; +\tfrac{1}{2}; \tau) &= \Psi_{+1}(\vec{p}_T, \eta; \tau) \quad \text{for } \eta > 0\,, \\ 2^{-\frac{1}{4}}\,\eta^{-\frac{1}{2}}\,b(\vec{p}_T, \eta; -\tfrac{1}{2}; \tau) &= \Psi_{+4}(\vec{p}_T, \eta; \tau) \quad \text{for } \eta > 0\,, \\ 2^{-\frac{1}{4}}\,\eta^{-\frac{1}{2}}\,d^\dagger(\vec{p}_T, \eta; +\tfrac{1}{2}; \tau) &= \Psi_{+4}(-\vec{p}_T, -\eta; \tau) \quad \text{for } \eta > 0\,, \\ 2^{-\frac{1}{4}}\,\eta^{-\frac{1}{2}}\,d^\dagger(\vec{p}_T, \eta; -\tfrac{1}{2}; \tau) &= \Psi_{+1}(-\vec{p}_T, -\eta; \tau) \quad \text{for } \eta > 0\,. \end{aligned} \tag{4.31}$$

Then our Fourier expansion of $\Psi_+(x)$ takes the form

$$\begin{aligned} \Psi_+(\tau, \vec{x}_T, \xi) &= 2^{-1/4}(2\pi)^{-3/2}\int d^2\vec{p}_T\int_0^\infty d\eta\;\eta^{-1/2}\sum_{s=\pm 1/2} \\ &\times\left\{w(s)\,e^{-i(\eta\xi - \vec{p}_T\cdot\vec{x}_T)}\,b(p; s; \tau) + w(-s)\,e^{+i(\eta\xi - \vec{p}_T\cdot\vec{x}_T)}\,d^\dagger(p; s; \tau)\right\}, \end{aligned} \tag{4.32}$$

where the spinors w(s) are

$$w(+\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad w(-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{4.33}$$

Let us see what the electron parts of the momentum operators P_1, P_2, P_3 look like in momentum space. Taking the operators P_α from (4.23) and (4.25), and doing a little algebra, we get

$$\begin{aligned} P_\alpha(\text{electron}) &= \int d^2\vec{x}_T \, d\mathscr{y} \sqrt{2}\, \Psi_+^\dagger(\tau,\vec{x}_T,\mathscr{y}) \, \frac{i}{2} \overleftrightarrow{\partial}_\alpha \Psi_+(\tau,\vec{x}_T,\mathscr{y}) \\ &= \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \sum_{s=\pm\frac{1}{2}} p_\alpha \left\{ b^\dagger(p;s;\tau) b(p;s;\tau) - d(p;s;\tau) d^\dagger(p;s;\tau) \right\} \\ &\qquad\qquad (\alpha = 1,2,3) \ . \end{aligned} \tag{4.34}$$

Up until now we have not mentioned the commutation relations of our independent fields. The form of (4.34) makes a very clear suggestion as to what commutation relations to choose. We are led to interpret $b(p;s;\tau)$ and $d(p;s;\tau)$ as destruction operators for electrons and positrons, respectively. (The minus sign in (4.34) can then be disposed of by normal ordering.) We thus postulate the covariant anticommutation relations

$$\left\{ b(p;s;\tau), b^\dagger(p';s';\tau) \right\} = \left\{ d(p;s;\tau), d^\dagger(p';s';\tau) \right\} = \delta_{ss'} \, \eta \, \delta(\eta - \eta') \, \delta^2(\vec{p}_T - \vec{p}_T{}') \ , \tag{4.35}$$

with all other anticommutators vanishing. Transforming back to coordinate space, we obtain the following equal-τ anticommutation relations:

$$\left\{\Psi_+(\tau,\vec{x}_T,\mathscr{z}),\ \Psi_+^\dagger(\tau,\vec{x}_T',\mathscr{z}')\right\} = \frac{1}{\sqrt{2}}P_+\delta(\mathscr{z}-\mathscr{z}')\,\delta^2(\vec{x}_T-\vec{x}_T')$$

$$\left\{\Psi_+(\tau,\vec{x}_T,\mathscr{z}),\ \Psi_+(\tau,\vec{x}_T',\mathscr{z}')\right\} = \left\{\Psi_+^\dagger(\tau,\vec{x}_T,\mathscr{z}),\Psi_+^\dagger(\tau,\vec{x}_T',\mathscr{z}')\right\} = 0\ . \tag{4.36}$$

We will use the same procedure to find commutation rules for the field $A^j(x)$. Since $A^j(x)$ is to be hermitian field, we write its Fourier expansion as

$$A^j(\tau,\vec{x}_T,\mathscr{z}) = \left[2(2\pi)^3\right]^{-\frac{1}{2}}\int d^2\vec{p}_T\int_0^\infty\frac{d\eta}{\eta}\sum_{\lambda=1}^{2}\delta_{\lambda j}$$

$$\times\left\{e^{-i(\eta\mathscr{z}-\vec{p}_T\cdot\vec{x}_T)}a(\vec{p}_T,\eta;\lambda;\tau)+e^{+i(\eta\mathscr{z}-\vec{p}_T\cdot\vec{x}_T)}a^\dagger(\vec{p}_T,\eta;\lambda;\tau)\right\}. \tag{4.37}$$

In terms of the operators $a(p;\lambda;\tau)$, the photon part of the momentum P_α is

$$P_{\alpha(\text{photon})} = -\int d^2\vec{x}_T d\mathscr{z}\,(\partial_\alpha A^i(\tau,\vec{x}_T,\mathscr{z}))(\partial_3 A_i(\tau,\vec{x}_T,\mathscr{z}))$$

$$=\int d^2\vec{p}_T\int_0^\infty\frac{d\eta}{\eta}\,p_\alpha\sum_{\lambda=1}^{2}\tfrac{1}{2}\left\{a(p;\lambda;\tau)\,a^\dagger(p;\lambda;\tau)+a^\dagger(p;\lambda;\tau)\,a(p;\lambda;\tau)\right\} \tag{4.38}$$

$$(\alpha=1,2,3)\ .$$

The interpretation of (4.38) is clear if we let the operators $a(p;\lambda;\tau)$ be destruction operators for photons and normal order the expression for P_α. Thus we are led to postulate the covariant commutation relations

$$\left[a(p;\lambda;\tau),\ a^\dagger(p';\lambda';\tau)\right] = \delta_{\lambda\lambda'}\,\eta\,\delta(\eta-\eta')\delta^2(\vec{p}_T-\vec{p}_T')$$

$$\left[a(p;\lambda;\tau),\ a(p',\lambda',\tau)\right] = 0\ . \tag{4.39}$$

Transforming back to coordinate space, we obtain easily the equal-τ commutation relations

$$\left[\partial_3 A^i(\tau,\vec{x}_T,\mathscr{y}),\ A^j(\tau,\vec{x}'_T,\mathscr{y}')\right] = -\frac{i}{2}\,\delta_{ij}\,\delta(\mathscr{y}-\mathscr{y}')\,\delta^2(\vec{x}_T-\vec{x}'_T)\ . \tag{4.40}$$

Utilizing the relation (4.5) between A^i and $\partial_3 A^i = -F^{oi}$, we obtain

$$\left[A^i(\tau,\vec{x}_T,\mathscr{y}),\ A^j(\tau,\vec{x}'_T,\mathscr{y}')\right] = -\frac{i}{4}\,\delta_{ij}\,\epsilon(\mathscr{y}-\mathscr{y}')\,\delta^2(\vec{x}_T-\vec{x}'_T)\ . \tag{4.41}$$

We also assume, of course, that the photon creation and destruction operators commute (at equal τ) with the fermion creation and destruction operators. Thus

$$\left[A^i(\tau,\vec{x}_T,\mathscr{y}),\ \Psi_+(\tau,\vec{x}'_T,\mathscr{y}')\right] = 0\ . \tag{4.42}$$

Our field theory in the infinite momentum frame is based on the equal-τ commutation relations (4.36), (4.41), and (4.42). We would expect, a priori, that dynamical effects could propagate from one point to another in a plane τ = constant along a line $\vec{x}_T$ = constant (i.e. along a light cone). Thus we might expect that the commutation relations would depend on the coupling constant e. The commutation relations among the independent fields of the theory are in fact independent of e. However, the electrodynamic interaction does affect in the equal-τ commutation relations among the components of the complete fields $A^\mu(x)$ and $\Psi(x)$, since the charge e appears in the definition of the "auxiliary" components, A^3 and Ψ_-, of the fields. We find, for instance, that

$$\left[A^3(\tau,\vec{x}_T,\mathscr{y}),\ \Psi_+(\tau,\vec{x}'_T,\mathscr{y}')\right] = \frac{e}{2}\,|\mathscr{y}-\mathscr{y}'|\,\delta^2(\vec{x}_T-\vec{x}'_T)\Psi_+(\tau,\vec{x}'_T,\mathscr{y}')\ .$$

We can gain further confidence in the equal-τ commutation relations by using them to show that the operators P_μ and $M_{\mu\nu}$ actually generate translations and Lorentz transformations when commuted with the independent fields of the theory. The verification for the "kinematical" operators is particularly simple because these operators involve only the independent fields. One finds

$$\begin{aligned}
i\left[P_j, A^i(x)\right] &= \partial_j A^i(x) \qquad\qquad i\left[P_j, \Psi_+(x)\right] = \partial_j \Psi_+(x) \\
i\left[\eta, A^i(x)\right] &= \partial_3 A^i(x) \qquad\qquad i\left[\eta, \Psi_+(x)\right] = \partial_3 \Psi_+(x) \\
i\left[J_3, A^i(x)\right] &= (x_1\partial_2 - x_2\partial_1)\, A^i(x) - \epsilon_{ij}\, A^j(x) \\
i\left[J_3, \Psi_+(x)\right] &= (x_1\partial_2 - x_2\partial_1)\, \Psi_+(x) + \tfrac{1}{2}\, \gamma_1\gamma_2\Psi_+(x) \\
i\left[B_j, A^i(x)\right] &= (x_3\partial_j - x_j\partial_3)\, A^i(x) \\
i\left[B_j, \Psi_+(x)\right] &= (x_3\partial_j - x_j\partial_3)\, \Psi_+(x)\ .
\end{aligned} \tag{4.43}$$

It is considerably more tedious to show that the operators H, S_1, S_2, and K_3 have the proper commutation relations with the fields. We present in Appendix A some details of the calculation which verifies the crucial assertion

$$\begin{aligned}
i\left[H, A^i(x)\right] &= \partial_0 A^i(x) \\
i\left[H, \Psi_+(x)\right] &= \partial_0 \Psi_+(x)\quad .
\end{aligned} \tag{4.44}$$

Similar but lengthier algebra gives

$$
\begin{aligned}
i\left[K_3, A^i(x)\right] &= (x_0\partial_3 - x_3\partial_0)\, A^i(x) \\
i\left[K_3, \Psi_+(x)\right] &= (x_0\partial_3 - x_3\partial_0)\, \Psi_+(x) \\
i\left[S_i, A^j(x)\right] &= (x_i\partial_0 - x_0\partial_i)\, A^j(x) - g^j{}_i A_0(x) + \partial^j \Lambda_i(x) \\
i\left[S_i, \Psi_+(x)\right] &= (x_i\partial_0 - x_0\partial_i)\, \Psi_+(x) + \tfrac{1}{2}\gamma_i\gamma_0\, \Psi(x) - ie\, \Lambda_i(x)\, \Psi_+(x) \quad ,
\end{aligned}
\tag{4.45}
$$

where $\Lambda_i(x) = \frac{1}{2}\int d\xi\; \epsilon(x^3 - \xi)\, A_i(x^o, \vec{x}_T, \xi)$ is that function which preserves the gauge during the Lorentz transformation.[17]

D. Free Fields

Let us see how the methods of the preceeding sections work if the interaction is turned off. Consider first the electron field $\Psi(x)$. With no interaction, each component of $\Psi(x)$ satisfies the Klein-Gordon equation

$$
(2\partial_0\partial_3 + \partial_i\partial^i + m^2)\, \Psi(x) = 0 \quad . \tag{4.46}
$$

Using this in the Fourier expansion (4.32) of $\Psi_+(x)$, we find that the operators $b(p;s;\tau)$, $d^\dagger(p;s;\tau)$ satisfy the differential equations

$$
\begin{aligned}
(-2i\eta \frac{\partial}{\partial\tau} + \vec{p}_T^{\,2} + m^2)\, b(p;s;\tau) &= 0 \\
(+2i\eta \frac{\partial}{\partial\tau} + \vec{p}_T^{\,2} + m^2)\, d^\dagger(p;s;\tau) &= 0 \quad .
\end{aligned}
$$

Solving these equations, we get

$$
\begin{aligned}
b(p;s;\tau) &= e^{-ip_0\tau}\, b(p;s;0) \\
d^\dagger(p;s;\tau) &= e^{+ip_0\tau}\, d^\dagger(p;s;0) \quad ,
\end{aligned}
\tag{4.47}
$$

where $p_0 = (\vec{p}_T^{\,2} + m^2)/2\eta$ is the free particle hamiltonian. Thus the Fourier expansion for $\Psi_+(x)$ takes the form

$$\Psi_+(x) = \left(2(2\pi)^3\right)^{-\frac{1}{2}} \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \sum_{s=\pm\frac{1}{2}} \left\{ 2^{\frac{1}{4}} \eta^{\frac{1}{2}} w(s) e^{-ip_\mu x^\mu} b(p;s;0) + 2^{\frac{1}{4}} \eta^{\frac{1}{2}} w(-s) e^{+ip_\mu x^\mu} d^\dagger(p;s;0) \right\} . \tag{4.48}$$

The auxiliary field $\Psi_-(x)$ is given in terms of $\Psi_+(x)$ by equation (4.13),

$$\Psi_-(x) = -\frac{i}{4} \int d\xi \; \epsilon(x^3 - \xi)(i\partial_j \gamma^j + m)\gamma^o \Psi_+(x^o, \vec{x}_T, \xi) . \tag{4.49}$$

Substituting the Fourier expansion of $\Psi_+(x)$ into this equation and doing the ξ-integration we obtain

$$\begin{aligned}\Psi_-(x) = \left(2(2\pi)^3\right)^{-\frac{1}{2}} \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \sum_{s=\pm\frac{1}{2}} &\left\{ 2^{-\frac{3}{4}} \eta^{-\frac{1}{2}} (p_j \gamma^j + m) \gamma^o w(s) e^{-ip_\mu x^\mu} b(p;s;0) \right. \\ &\left. - 2^{-\frac{3}{4}} \eta^{-\frac{1}{2}} (-p_j \gamma^j + m) \gamma^o w(-s)\, e^{+ip_\mu x^\mu} d^\dagger(p;s;0) \right\} .\end{aligned} \tag{4.50}$$

We have now only to add $\Psi_+(x)$ and $\Psi_-(x)$ to obtain the complete field $\Psi(x)$:

$$\Psi(x) = \left(2(2\pi)^3\right)^{-\frac{1}{2}} \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \sum_{s=\pm\frac{1}{2}} \left\{ u(p,s)\, e^{-ip_\mu x^\mu} b(p;s;0) + v(p,s)\, e^{+ip_\mu x^\mu} d^\dagger(p;s;0) \right\} , \tag{4.51}$$

where

$$u(p,s) = 2^{\frac{1}{4}}\eta^{\frac{1}{2}}\left(1 + \frac{p_j\gamma^j + m}{2\eta}\gamma^o\right) w(s)$$

$$v(p,s) = 2^{\frac{1}{4}}\eta^{\frac{1}{2}}\left(1 + \frac{p_j\gamma^j - m}{2\eta}\gamma^o\right) w(-s) \quad . \tag{4.52}$$

Recalling the definition of the spinors w(s) from Eq. (4.33), we can calculate u(p, s) and v(p, s). We find

$$u(p,+\tfrac{1}{2}) = 2^{-\frac{1}{4}}\eta^{-\frac{1}{2}}\begin{pmatrix}\sqrt{2}\ \eta\\ p^1 + ip^2\\ m\\ 0\end{pmatrix} \qquad u(p,-\tfrac{1}{2}) = 2^{-\frac{1}{4}}\eta^{-\frac{1}{2}}\begin{pmatrix}0\\ m\\ -p^1 + ip^2\\ \sqrt{2}\ \eta\end{pmatrix}$$

$$v(p,+\tfrac{1}{2}) = 2^{-\frac{1}{4}}\eta^{-\frac{1}{2}}\begin{pmatrix}0\\ -m\\ -p^1 + ip^2\\ \sqrt{2}\ \eta\end{pmatrix} \qquad v(p,-\tfrac{1}{2}) = 2^{-\frac{1}{4}}\eta^{-\frac{1}{2}}\begin{pmatrix}\sqrt{2}\ \eta\\ p^1 + ip^2\\ -m\\ 0\end{pmatrix} . \tag{4.53}$$

If the field $\Psi(x)$ which we have obtained in the infinite momentum frame is to be equal to the usual free Dirac field, then the spinors u(p, s) should be solutions of the Dirac equation normalized to $\bar{u}(p,s)u(p,s') = 2m\,\delta_{ss'}$, and the spinors v(p, s) should be related to u(p, s) by charge conjugation. Indeed, a quick check shows that this is the case.

The destruction operator $b(p;s;\tau)$ destroys an electron with momentum p described by the Dirac spinor u(p, s). Using the explicit form of u(p, s), we can clarify the physical meaning of the spin index s. In a short calculation presented in Appendix B, we find that s is the helicity of the electron as measured in a Lorentz reference frame moving with (almost) the speed of light in the $-z$ direction.

We can also check to see that, with the interaction turned off, our field $A^\mu(x)$ is just the usual free photon field (in the appropriate gauge). The calculation is completely analogous to the calculation for $\Psi(x)$, so we just state the result. With $e = 0$, we find

$$A^\mu(x) = \left(2(2\pi)^3\right)^{-\frac{1}{2}} \int d^2\vec{p}_T \int_0^\infty \frac{d\eta}{\eta} \sum_{\lambda=1}^{2} e_\lambda(p)^\mu \times \left\{ e^{-ip_\mu x^\mu} a(p;\lambda;0) + e^{+ip_\mu x^\mu} a^\dagger(p;\lambda;0) \right\} , \tag{4.54}$$

where the $e_\lambda(p)^\mu$ are just the infinite momentum gauge polarization vectors defined in Eq. (3.9). Using the explicit representation of the polarization vectors, we can clarify the physical meaning of the index λ. An easy calculation shows that the creation operators $2^{-\frac{1}{2}} (a^\dagger(p;1;0) \pm i a^\dagger(p;2;0))$ create photons with helicity ± 1.

E. Scattering Theory

We have seen that infinite momentum quantum electrodynamics is the same as ordinary quantum electrodynamics in the trivial case $e = 0$. The two theories can be compared for $e \neq 0$, at least formally, by constructing the S-matrix in old fashioned perturbation theory in the infinite momentum frame and comparing it with the S-matrix given by the τ-ordered diagrams of Section III.

The perturbation expansion of the S-matrix takes a familiar form once we have divided the hamiltonian into a free part and an interaction part. To make this division, we start with the hamiltonian density $T^o_0(x)$:

$$
\begin{aligned}
T^o_0 = & -\bar{\Psi}\left(\left[\tfrac{i}{2}\overleftrightarrow{\partial}_j - eA_j\right]\gamma^j - m\right)\Psi - \bar{\Psi}\,\tfrac{i}{2}\overleftrightarrow{\partial}_3\gamma^3\Psi + eA^3\bar{\Psi}\gamma^o\Psi + \tfrac{1}{2}F^{12}F_{12} \\
& - \tfrac{1}{2}(\partial_3 A^3)(\partial_3 A^3) - (\partial_j A^3)(\partial_3 A^j) \quad .
\end{aligned}
\tag{4.55}
$$

The integrated hamiltonian can be somewhat simplified if we realize that the first term is equal to -2 times the second term after an integration by parts in the transverse variables x^1, x^2. To see this, write -2 times the second term as

$$
\bar{\Psi}\, i\overleftrightarrow{\partial}_3\gamma^3\Psi = \Psi^\dagger\hat{\gamma}^o\gamma^3 i\overleftrightarrow{\partial}_3\Psi = \sqrt{2}\,\Psi^\dagger_-\, i\overleftrightarrow{\partial}_3\Psi_- \; .
$$

Using Eq. (4.12) for $\partial_3\Psi_-$, this is

$$
\begin{aligned}
& -\frac{1}{\sqrt{2}}\Psi^\dagger_-\gamma^o\left[(i\overrightarrow{\partial}_j - eA_j)\gamma^j - m\right]\Psi_+ \\
& -\frac{1}{\sqrt{2}}\Psi^\dagger_+\gamma^3\left[(-i\overleftarrow{\partial}_j - eA_j)\gamma^j - m\right]\Psi_- \quad .
\end{aligned}
$$

With an integration by parts in the transverse variables, we can replace $i\overrightarrow{\partial}_j$ and $-i\overleftarrow{\partial}_j$ by $\frac{i}{2}\overleftrightarrow{\partial}_j$ and obtain

$$
-\frac{1}{\sqrt{2}}\Psi^\dagger(P_-\gamma^o P_+ + P_+\gamma^3 P_-)\left[(\tfrac{i}{2}\overleftrightarrow{\partial}_j - eA_j)\gamma^j - m\right]\Psi \; .
$$

But $P_-\gamma^o P_+ + P_+\gamma^3 P_- = \gamma^o + \gamma^3 = \sqrt{2}\,\hat{\gamma}^o$, so this is just

$$-\overline{\Psi}\left[\left(\frac{i}{2}\overleftrightarrow{\partial_j} - eA_j\right)\gamma^j - m\right]\Psi \quad .$$

Thus the hamiltonian density can be rewritten as

$$T_0^o = \overline{\Psi}\,\frac{i}{2}\overleftrightarrow{\partial_3}\,\gamma^3\Psi + eA^3\overline{\Psi}\gamma^o\Psi + \frac{1}{2}F^{12}F_{12} - \frac{1}{2}(\partial_3A^3)(\partial_3A^3) - (\partial_jA^3)(\partial_3A^j) \quad . \tag{4.56}$$

At this point we realize that part of the interaction is buried in the dependence of Ψ_- and A^3 on e. In order to bring out this dependence we write Ψ_- as the sum of a "free" part ψ_- and an "interaction part" Υ, where

$$\psi_-(x) = -\frac{i}{4}\int d\xi\;\epsilon(x^3-\xi)\left\{i\partial_j\gamma^j + m\right\}\gamma^o\,\Psi_+(x^o,\vec{x}_T,\xi) \tag{4.57}$$

$$\Upsilon(x) = \frac{ie}{4}\int d\xi\;\epsilon(x^3-\xi)\;A_j(x^o,\vec{x}_T,\xi)\,\gamma^j\gamma^o\,\Psi_+(x^o,\vec{x}_T,\xi) \quad . \tag{4.58}$$

We also define $\psi_+ = \Psi_+$ and $\psi = \psi_+ + \psi_-$. Similarly, we write $A^3 = \mathscr{A}^3 + \phi$, where

$$\mathscr{A}^3(x) = -\frac{1}{2}\int d\xi\;|\,x^3-\xi\,|\;\partial_3\partial_jA^j(x^o,\vec{x}_T,\xi) \tag{4.59}$$

$$\phi(x) = -\frac{1}{2}\int d\xi\;|\,x^3-\xi\,|\;J^o(x^o,\vec{x}_T,\xi) \quad , \tag{4.60}$$

and we put $\mathscr{A}^j = A^j$, $\mathscr{A}^o = 0$. Let us insert $\Psi = \psi + \Upsilon$ and $A^\mu = \mathscr{A}^\mu + \delta_3^\mu\,\phi$ into our hamiltonian density (4.56) and simplify the result.

From the first term in T_0^o we get four terms

$$\begin{aligned}\overline{\Psi}\,\frac{i}{2}\overleftrightarrow{\partial_3}\gamma^3\Psi &= \sqrt{2}\,\Psi_-^\dagger\,\frac{i}{2}\overleftrightarrow{\partial_3}\Psi_- = \sqrt{2}\,\psi_-^\dagger\,\frac{i}{2}\overleftrightarrow{\partial_3}\,\psi_- \\ &\quad + \sqrt{2}\,\Upsilon^\dagger\,\frac{i}{2}\overleftrightarrow{\partial_3}\,\Upsilon + \sqrt{2}\,\psi_-^\dagger\,\frac{i}{2}\overleftrightarrow{\partial_3}\,\Upsilon + \sqrt{2}\,\Upsilon^\dagger\,\frac{i}{2}\overleftrightarrow{\partial_3}\,\psi_- \quad .\end{aligned} \tag{4.61}$$

The first two terms can be left as they stand. The integrated form of the third term can be integrated by parts so that $\frac{i}{2}\overleftrightarrow{\partial}_3$ is replaced by $i\overrightarrow{\partial}_3$. This integration by parts can be justified simply by using the definitions (4.57) and (4.58) to write

$$-\int d\mathscr{y}\ \partial_3\psi_-^\dagger(\mathscr{y})\,\Upsilon(\mathscr{y}) = -\tfrac{1}{2}\int d\mathscr{y}d\xi\ \partial_3\psi_-^\dagger(\mathscr{y})\,\epsilon(\mathscr{y}-\xi)\,\partial_3\Upsilon(\xi)$$

$$= +\tfrac{1}{2}\int d\mathscr{y}d\xi\ \partial_3\psi_-^\dagger(\mathscr{y})\,\epsilon(\xi-\mathscr{y})\,\partial_3\Upsilon(\xi)$$

$$= +\int d\xi\ \psi_-^\dagger(\xi)\,\partial_3\Upsilon(\xi)\ .$$

Similarly, we can replace $\frac{i}{2}\overleftrightarrow{\partial}_3$ by $-\frac{i}{2}\overleftarrow{\partial}_3$ in the fourth term. Then, making use of the definition (4.58) of Υ, we obtain for the sum of the third and fourth terms of (4.61)

$$\frac{e}{\sqrt{2}}\,\psi^\dagger(P_-\gamma^o P_+ + P_+\gamma^3 P_-)\mathscr{A}_j\gamma^j\psi = e\mathscr{A}_j\bar{\psi}\,\gamma^j\psi \quad . \tag{4.62}$$

Turning now to the second term in T_0^o, we write simply

$$eA^3\bar{\Psi}\,\gamma^o\Psi = eA^3\bar{\Psi}_+\gamma^o\Psi_+ = e\mathscr{A}^3\bar{\psi}\,\gamma^o\psi + e\phi\bar{\psi}\,\gamma^o\psi \quad . \tag{4.63}$$

The third term in T_0^o can be left unchanged since it involves only $A^j = \mathscr{A}^j$. The fourth term requires some work. With an integration by parts we can make the replacement[18]

$$-\tfrac{1}{2}\,(\partial_3A^3)(\partial_3A^3) \rightarrow +\tfrac{1}{2}\,A^3\,\partial_3\partial_3A^3 \quad .$$

Writing $A^3 = \mathscr{A}^3 + \phi$, we obtain the sum

$$\tfrac{1}{2}\mathscr{A}^3\,\partial_3\partial_3\mathscr{A}^3 + \tfrac{1}{2}\,\phi\,\partial_3\partial_3\phi + \tfrac{1}{2}\,\phi\,\partial_3\partial_3\mathscr{A}^3 + \tfrac{1}{2}\mathscr{A}^3\,\partial_3\partial_3\,\phi \quad . \tag{4.64}$$

We write the first and second terms simply as

$$\tfrac{1}{2}\mathscr{A}^3\partial_3\partial_3\mathscr{A}^3 = -\tfrac{1}{2}\mathscr{A}^3\partial_3\partial_j\mathscr{A}^j \tag{4.65}$$

and

$$\tfrac{1}{2}\phi\,\partial_3\partial_3\phi = -\tfrac{1}{2}\phi\,J^o = -\frac{e}{2}\phi\,\overline{\psi}\,\gamma^o\psi \quad . \tag{4.66}$$

We see, with use of the definitions (4.59) and (4.60), that the integrated forms of the third and fourth terms in (4.64) are equal. Indeed,

$$\int d\mathscr{y}\,(\partial_3\partial_3\phi(\mathscr{y}))\mathscr{A}^3(\mathscr{y}) = \tfrac{1}{2}\int d\mathscr{y}d\xi\,(\partial_3\partial_3\phi(\mathscr{y}))\,|\,\mathscr{y}-\xi\,|\,(\partial_3\partial_\xi\mathscr{A}^3(\xi)) = \int d\xi\,\phi(\xi)(\partial_3\partial_\xi\mathscr{A}^3(\xi)) \quad .$$

Thus we can write for the sum of the last two terms in (4.64)

$$\tfrac{1}{2}\phi\,\partial_3\partial_\xi\mathscr{A}^3 + \tfrac{1}{2}\mathscr{A}^3\partial_3\partial_3\phi \rightarrow \phi\,\partial_3\partial_3\mathscr{A}^3 = -\phi\,\partial_3\partial_j\mathscr{A}^j \quad . \tag{4.67}$$

Finally, we consider the fifth term of T_0^o, which we write, using an integration by parts of the variables x^1, x^2, as

$$-(\partial_j A^3)(\partial_3 A^j) \rightarrow A^3\partial_3\partial_j A^j = \phi\,\partial_3\partial_j\mathscr{A}^j + \mathscr{A}^3\partial_3\partial_j\mathscr{A}^j \quad . \tag{4.68}$$

The integrated hamiltonian is now in the form we wanted. Adding up the pieces, we have

$$H = H_0 + V \quad , \tag{4.69}$$

where

$$H_0 = \int d^2\vec{x}_T d\mathscr{y} \left\{ \sqrt{2}\, \psi_-^\dagger \frac{i}{2} \overleftrightarrow{\partial}_3 \psi_- + \frac{1}{2} F^{12} F_{12} + \frac{1}{2} \mathscr{A}^3 \partial_3 \partial_j \mathscr{A}^j \right\} \tag{4.70}$$

$$V = \int d^2\vec{x}_T d\mathscr{y} \left\{ e\mathscr{A}_\mu \overline{\psi} \gamma^\mu \psi + \sqrt{2}\, \Upsilon^\dagger \frac{i}{2} \overleftrightarrow{\partial}_3 \Upsilon + \frac{1}{2} e\, \phi\, \overline{\psi} \gamma^0 \psi \right\} . \tag{4.71}$$

If we work in the Schroedinger picture, we can evaluate all Heisenberg operators at "time" $\tau = 0$. We note that the Fourier expansions of the fields ψ (x) and $\mathscr{A}^\mu$(x) at $\tau = 0$ in terms of creation and destruction operators are the same as the expansions (4.51) and (4.54) for free fields. Thus the free hamiltonian H_0 generates the free motion of the quanta created by $a^\dagger(p;\lambda;0)$, $b^\dagger(p;s;0)$, $d^\dagger(p;s;0)$. The remaining part of the hamiltonian, V, gives rise to the scattering of these quanta.

We can formally calculate the scattering matrix with the aid of the "old fashioned" perturbation theory expansion

$$S_{fi} = -2\pi\, i\, \delta(\mathscr{H}_f - \mathscr{H}_i) \left\{ V + V(\mathscr{H} - H_0 + i\epsilon)^{-1} V + \cdots \right\} . \tag{4.72}$$

In a field theory in an ordinary Lorentz frame, this formula leads to a set of rules for calculating scattering matrix elements using time ordered diagrams. In the present case, we are led in the same way[19] to rules for τ-ordered diagrams.

These rules are the same as the rules developed directly from the covariant Feynman rules in Section III. This can be seen by calculating a few matrix elements of the interaction hamiltonian V. One finds that the interaction term

$$V_1 = \int d^2\vec{x}_T d\mathscr{y}\; e\mathscr{A}_\mu \overline{\psi} \gamma^\mu \psi \tag{4.73}$$

gives the "ordinary" vertices of Figure 5a. The second term in V, when written out in full using the definition of Υ, is

$$V_2 = -\frac{ie^2}{4}\int d\vec{x}_T d\mathscr{z}\, d\xi\, \epsilon(\mathscr{z}-\xi)\,\overline{\psi}(0,\vec{x}_T,\mathscr{z})\,\gamma^{\mu}\mathscr{A}_{\mu}(0,\vec{x}_T,\mathscr{z})\,\gamma^{0}\gamma^{\nu}\mathscr{A}_{\nu}(0,\vec{x}_T,\xi)\,\psi(0,\vec{x}_T,\xi)\ . \tag{4.74}$$

Using

$$\int d\mathscr{z}\; e^{i\eta\mathscr{z}}\,\epsilon(\mathscr{z}) = \frac{2i}{\eta}\quad , \tag{4.75}$$

one finds that the interaction V_2 gives the vertices of Figure 5c.

The third term in V, written out in full, is

$$V_3 = -\frac{e^2}{4}\int d^2\vec{x}_T\, d\mathscr{z}\, d\xi\, |\,\mathscr{z}-\xi\,|\;\overline{\psi}(0,\vec{x}_T,\mathscr{z})\,\gamma^{0}\psi(0,\vec{x}_T,\mathscr{z})\,\overline{\psi}(0,\vec{x}_T,\xi)\,\gamma^{0}\psi(0,\vec{x}_T,\xi)\ . \tag{4.76}$$

Using

$$\int d\mathscr{z}\; e^{i\eta\mathscr{z}}\,|\,\mathscr{z}\,| = -\frac{2}{\eta^2}\quad , \tag{4.77}$$

it is easily shown that the interaction V_3 gives the "Coulomb" vertices of Figure 5b.

Thus when we formally calculate the S-matrix from canonical field theory developed in the infinite momentum frame, we get the same results as when we directly transform the S-matrix for ordinary quantum electrodynamics to the infinite momentum frame.

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APPENDIX A

In this appendix we will show that the canonical hamiltonian presented in section 4B generates the correct equations of motion for the independent field operators $A^i(y)$. We begin with expression (4.56) for the hamiltonian,

$$H = \int d\vec{x} \left\{ i\, 2^{-1/2}\, \Psi_-^\dagger \overleftrightarrow{\partial}_3 \Psi_- + A^3 J^o - \frac{1}{2} \partial_3 A^3 \partial_3 A^3 + \frac{1}{2} F^{12} F_{12} - \partial_j A^3 \partial_3 A^j \right\} . \tag{A1}$$

In order to compute $\left[H, A^i(y)\right]$ we need to first compute two rather complicated equal-τ commutators which we list here,

$$\left[\Psi_-(x), A^i(y)\right]\Big|_{x^o=y^o} = -\frac{e}{16} \int_{-\infty}^{\infty} d\xi\; \epsilon(x^3-\xi)\; \epsilon(\xi - y^3) \delta^2(\vec{x}_T - \vec{y}_T)\, \gamma^i \gamma^o\, \Psi_+(y^o, \vec{y}_T, \xi)$$

$$\left[A^3(x), A^i(y)\right]\Big|_{x^o=y^o} = \frac{1}{4i} \left| x^3 - y^3 \right| \partial^i \delta^2(\vec{x}_T - \vec{y}_T) \quad . \tag{A2}$$

These relations follow from the definitions of the auxiliary fields, $\Psi_-(y)$ and $A^3(y)$, and the basic equal-τ commutators of the independent fields.

With these preliminaries done, we can compute

$$\begin{aligned}
\left[H, A^i(y)\right] = 2^{1/2} \int d\vec{x} \left[\Psi_-^\dagger(x)\, \frac{i}{2} \overleftrightarrow{\partial}_3\, \Psi_-(x), A^i(y)\right]_{x^o=y^o} \\
+ \int d\vec{x}\; J^o(x) \left[A^3(x), A^i(y)\right]_{x^o=y^o} \\
- \frac{1}{2} \int d\vec{x} \left[\partial_3 A^3(x)\; \partial_3 A^3(x), A^i(y)\right]_{x^o=y^o} \\
+ \frac{1}{2} \int d\vec{x} \left[F^{12}(x)\; F_{12}(x), A^i(y)\right]_{x^o=y^o} \\
- \int d\vec{x} \left[\partial_j A^3(x)\; \partial_3 A^j(x), A^i(y)\right]_{x^o=y^o} \quad .
\end{aligned} \tag{A3}$$

For convenience we label these five terms (I.), (II.), (III.), (IV.), (V.), and compute each in its turn:

$$(\mathrm{I.}) = 2^{1/2}\int d\vec{x}\left[\Psi_-^\dagger(x)\,\frac{i}{2}\overleftrightarrow{\partial}_3\Psi_-(x),A^i(y)\right]_{x^o=y^o} \tag{A4}$$

$$= i\,2^{-1/2}\int d\vec{x}\left\{\Psi_-^\dagger(x)\left[\partial_3\Psi_-(x),A^i(y)\right]_{x^o=y^o}+\left[\Psi_-^\dagger(x),A^i(y)\right]_{x^o=y^o}\partial_3\Psi_-(x)\right.$$

$$\left.-\partial_3\Psi_-^\dagger(x)\left[\Psi_-(x),A^i(y)\right]_{x^o=y^o}-\left[\partial_3\Psi_-^\dagger(x),A^i(y)\right]_{x^o=y^o}\Psi_-(x)\right\}$$

$$= i\,2^{-1/2}\int d\vec{x}\left\{-\frac{e}{8}\,\epsilon(x^3-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\Psi_-^\dagger(x)\,\gamma^i\gamma^o\,\Psi_+(x)\right.$$

$$-\frac{e}{16}\int d\xi\;\epsilon(x^3-\xi)\,\epsilon(\xi-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\Psi_+^\dagger(y^o,\vec{x}_T,\xi)\;\gamma^3\gamma^i\partial_3\Psi_-(x)$$

$$+\frac{e}{16}\int d\xi\;\epsilon(x^3-\xi)\,\epsilon(\xi-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\partial_3\Psi_-^\dagger(x)\;\gamma^i\gamma^o\,\Psi_+(y^o,\vec{x}_T,\xi)$$

$$\left.+\frac{e}{8}\,\epsilon(x^3-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\Psi_+^\dagger(x)\;\gamma^3\gamma^i\,\Psi_-(x)\right\}$$

$$= ie\,2^{-7/2}\int dx^3\;\epsilon(x^3-y^3)\left\{\Psi_-^\dagger(y^o,\vec{y}_T,x^3)\gamma^o\gamma^i\Psi_+(y^o,\vec{y}_T,x^3)+\Psi_+^\dagger(y^o,\vec{y}_T,x^3)\gamma^3\gamma^i\,\Psi_-(y^o,\vec{y}_T,x^3)\right\}$$

$$-ie\,2^{-9/2}\int dx^3\int d\xi\,\epsilon(x^3-\xi)\,\epsilon(\xi-y^3)\left\{\Psi_+^\dagger(y^o,\vec{y}_T,\xi)\gamma^3\gamma^i\partial_3\Psi_-(y^o,\vec{y}_T,x^3)+\partial_3\Psi_-^\dagger(y^o,\vec{y}_T,x^3)\gamma^o\gamma^i\Psi_+(y^o,\vec{y}_T,\xi\right.$$

$$= \frac{1}{4i}\int d\xi\;\epsilon(y^3-\xi)\,J^i(y^o,\vec{y}_T,\xi)\quad . \tag{A5}$$

We have observed in this calculation that

$$\Psi_-(y) = \frac{1}{2}\int d\xi\;\epsilon(y^3-\xi)\;\partial_3\Psi_-(y^o,\vec{y}_T,\xi)$$

and

$$J^i(y) = 2^{-1/2}\, e\left\{\Psi_-^\dagger(y)\,\gamma^o\gamma^i\,\Psi_+(y) + \Psi_+^\dagger(y)\,\gamma^3\gamma^i\,\Psi_-(y)\right\} .$$

Continuing,

$$(\text{II.}) = \int d\vec{x}\; J^o(x)\left[A^3(x),\; A^i(y)\right]_{x^o=y^o} \tag{A6}$$

$$= \frac{1}{4i}\int d\vec{x}\; J^o(x)\,|x^3-y^3|\,\partial^i\delta^2(\vec{x}_T-\vec{y}_T)$$

$$= -\frac{1}{4i}\int dx^3\,|x^3-y^3|\,\partial^i J^o(y^o,\vec{y}_T,x^3) \;. \tag{A7}$$

Next,

$$(\text{III.}) = -\frac{1}{2}\int d\vec{x}\left[\partial_3 A^3(x)\;\partial_3 A^3(x),\; A^i(y)\right]_{x^o=y^o} \tag{A8}$$

$$= -\int d\vec{x}\;\partial_3 A^3(x)\left[\partial_3 A^3(x),\; A^i(y)\right]_{x^o=y^o}$$

$$= \frac{1}{4i}\int d\vec{x}\;\partial_3 A^3(x)\;\partial_i\,\delta^2(\vec{x}_T-\vec{y}_T)\,\epsilon(x^3-y^3)$$

$$= -\frac{1}{4i}\,\partial^i\int dx^3\;\epsilon(y^3-x^3)\;\partial_3 A^3(y^o,\vec{y}_\perp,x^3)$$

$$= -\frac{1}{2i}\,\partial^i A^3(y) \;. \tag{A9}$$

We have applied here the definition (4.5) of $A^3(x)$.

The fourth term becomes

$$(\text{IV.}) = \frac{1}{2}\int d\vec{x}\left[F^{12}(x)\;F_{12}(x),\; A^i(y)\right]_{x^o=y^o} \tag{A10}$$

$$= \int d\vec{x}\;F^{12}(x)\left[\partial^2 A^1(x) - \partial^1 A^2(x),\; A^i(y)\right]_{x^o=y^o}$$

$$= \frac{1}{4i}\int d\vec{x}\; F^{12}(x)\left\{\delta_{1i}\,\partial^2\delta^2(\vec{x}_T-\vec{y}_T)\;\epsilon(x^3-y^3) - \delta_{2i}\,\partial^1\delta^2(\vec{x}_T-\vec{y}_T)\;\epsilon(x^3-y^3)\right\}$$

$$= \frac{1}{4i}\left\{\delta_{1i}\int dx^3\;\epsilon(x^3-y^3)\,\partial_2 F^{12}(y^o,\vec{y}_T,x^3) + \delta_{2i}\int dx^3\;\epsilon(x^3-y^3)\,\partial_1 F^{21}(y^o,\vec{y}_T,x^3)\right\}$$

$$= -\frac{1}{4i}\int dx^3\;\epsilon(y^3-x^3)\,\partial_j F^{ij}(y^o,\vec{y}_T,x^3)\quad . \tag{A11}$$

Finally,

$$(V.) = -\int d\vec{x}\left[\partial_j A^3(x)\;\partial_3 A^j(x),\; A^i(y)\right]_{x^o=y^o} \tag{A12}$$

$$= -\int d\vec{x}\left\{\partial_j A^3(x)\;\partial_3\left[A^j(x),\; A^i(y)\right]_{x^o=y^o} + \partial_j\left[A^3(x),\; A^i(y)\right]_{x^o=y^o}\;\partial_3 A^j(x)\right\}$$

$$= -\frac{1}{4i}\int d\vec{x}\left\{\partial_j A^3(x)\,\delta_{ij}\;\delta^2(\vec{x}_T-\vec{y}_T)\,\partial_3\;\epsilon(x^3-y^3) + \partial_j\partial^i\,\delta^2(\vec{x}_T-\vec{y}_T)\,|x^3-y^3|\,\partial_3 A^j(x)\right\}$$

$$= -\frac{1}{4i}\int d\vec{x}\left\{2\partial_i A^3(x)\delta^3(\vec{x}-\vec{y}) + |x^3-y^3|\,\partial^i\,\partial_3\,\partial_j A^j(x)\delta^2(\vec{x}_T-\vec{y}_T)\right\}$$

$$= -\frac{1}{2i}\,\partial_i A^3(y) - \frac{1}{4i}\,\partial^i\int dx^3\,|x^3-y^3|\,\partial_3\,\partial_j A^j(y^o,\vec{y}_T,x^3)\quad . \tag{A13}$$

Collecting these five terms, we have the result

$$\left[H, A^i(y)\right] = \frac{1}{4i}\int dx^3\;\epsilon(y^3-x^3)\left\{J^i(y^o,\vec{y}_T,x^3) + \partial_j F^{ji}(y^o,\vec{y}_T,x^3)\right\}$$

$$- \frac{1}{4i}\,\partial^i\int dx^3\,|x^3-y^3|\left\{\partial_3\,\partial_j A^j(y^o,\vec{y}_T,x^3) + J^o(y^o,\vec{y}_T,x^3)\right\}\quad . \tag{A14}$$

Recalling the relation (4.7) for $A^3(x)$, we have, more simply,

$$\left[H, A^i(y)\right] = \frac{1}{4i}\int dx^3\;\epsilon(y^3-x^3)\left\{J^i(y^o,\vec{y}_T,x^3) + \partial_j F^{ji}(y^o,\vec{y}_T,x^3)\right\} + \frac{1}{2i}\,\partial^i A^3(y)\quad . \tag{A15}$$

Referring to (4.17), we see that we have indeed verified our claim,

$$\left[H, A^{i}(y)\right] = \frac{1}{i}\,\partial_0 A^{i}(y) \quad . \tag{A16}$$

The verification of the Heisenberg relation

$$\left[H, \Psi_+(y)\right] = \frac{1}{i}\,\partial_0 \Psi_+(y) \tag{A17}$$

is also tedious but straight-forward.

APPENDIX B

We will discuss here the physical meaning of the spinors appearing in the expansion (4.51) of the free Dirac field. We will show that these spinors are eigenstates of helicity referred to a Lorentz reference frame moving with (almost) the speed of light in the -z direction. To do this, we consider, using the representation (4.9) of the γ-matrices, the helicity operator in the laboratory frame

$$h_0(p) = \frac{1}{2}\frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|} = \frac{1/2}{|\vec{p}|}\begin{pmatrix} p^3 & p_- & 0 & 0 \\ p_+ & -p^3 & 0 & 0 \\ 0 & 0 & p^3 & p_- \\ 0 & 0 & p_+ & -p^3 \end{pmatrix}, \tag{B1}$$

where[20] $p_\pm = p^1 \pm ip^2$. Then the operator which measures helicity from a reference frame moving in the z-direction with a velocity $v_z = -\tanh(\omega)$ is[21]

$$h_\omega(p) = \exp\left(-\frac{\omega}{2}\gamma^o\gamma^3\right) h_0(q) \exp\left(\frac{\omega}{2}\gamma^o\gamma^3\right), \tag{B2}$$

where

$$q^\mu = \Lambda(\omega)^\mu{}_\nu\, p^\nu \quad ,$$

$$\Lambda(\omega)^\mu{}_\nu = \begin{pmatrix} \cosh\omega & 0 & 0 & \sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix}.$$

We compute

$$h_\omega(p) = \frac{1/2}{|\vec{q}|}\begin{pmatrix} q^3 & e^{-\omega}q_- & 0 & 0 \\ e^{\omega}q_+ & -q^3 & 0 & 0 \\ 0 & 0 & q^3 & e^{\omega}q_- \\ 0 & 0 & e^{-\omega}q_+ & -q^3 \end{pmatrix}. \tag{B3}$$

Now let $\omega \rightarrow \infty$. Then

$$|\vec{q}|,\ q^3 \rightarrow \frac{1}{2} e^{\omega} (p^o + p^3) = 2^{-1/2} e^{\omega} \eta ,$$

and

$$h_{\omega}(p) \rightarrow h_{\infty}(p) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \sqrt{2}\, p_+/\eta & -1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{2}\, p_-/\eta \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{B4}$$

We can now verify that the spinors listed in (4.53) are eigenstates of $h_{\infty}(p)$:

$$\begin{aligned} h_{\infty}(p)\ u(p, \pm 1/2) &= \pm \frac{1}{2}\ u(p, \pm 1/2) \\ h_{\infty}(p)\ v(p, \pm 1/2) &= \mp \frac{1}{2}\ v(p, \pm 1/2) . \end{aligned} \tag{B5}$$

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8. These results extend and systematize the results of Chang and Ma, Ref. 5. See also, Susskind, Ref. 4 and D. Flory, Yeshiva preprint (April 1969), unpublished.
9. Cf. Susskind, Refs. 3 and 4.
10. Cf. Susskind, Refs. 3, 4, and Bardakci and Halpern, Ref. 5.
11. Cf. L. I. Schiff, Quantum Mechanics (McGraw Hill Inc., New York, 1968); p. 234ff, 3rd Edition.

12. Here and elsewhere, we encounter a singularity at $\eta = 0$. In this paper it will not be necessary to specify the precise nature of these singularities.
13. We have done the momentum integrations over the $\delta(\tau)$ lines and rearranged the factors of π, i, etc.
14. We use the notation $a \overleftrightarrow{\partial}_\mu b$ for $a(\partial_\mu b) - (\partial_\mu a)b$.
15. For classical fields, the integral (4.8) converges because $\partial_3 F^{03}$ goes to zero like $(x^3)^{-3}$ as $x^3 \to \infty$. Furthermore, no surface term arises in the integration by parts since F^{03} falls off like $(x^3)^{-2}$ as $x^3 \to \infty$. Note, however, that it is not permissible to integrate by parts in Eq. (4.7).

16. Of course, this remains to be verified using the commutation relations of the fields, which we discuss in part C.

17. Cf. J. Bjorken and S. Drell, Relativistic Quantum Fields, (McGraw Hill Inc., New York, 1965); p. 88ff.

18. We may find some reassurance about this in the fact that, in classical electrodynamics, the surface term $A^3 \partial_3 A^3$ vanishes like $\mathfrak{z}^{-2}$ as $\mathfrak{z} \to \infty$.

19. Of course, we encounter most of the usual problems too. Cf. W. Heitler, The Quantum Theory of Radiation, (Oxford University Press, New York, 1966); p. 276ff, 3rd Edition.

20. In this appendix all quantities are referred to the ordinary coordinate system. We omit the hats.

21. Cf. J. Bjorken and S. Drell, Relativistic Quantum Mechanics, (McGraw Hill Inc., New York, 1964); p. 18ff.

TABLE I

γ-Matrix Identities

$$\left\{\gamma^{\mu}, \gamma^{\nu}\right\} = 2\, g^{\mu\nu} \qquad \gamma^{\mu\dagger} = \gamma_{\mu}$$

$$P_+ \equiv \tfrac{1}{2}\gamma^3\gamma^0 \qquad P_- \equiv \tfrac{1}{2}\gamma^0\gamma^3$$

$$P_{\pm}{}^{\dagger} = (P_{\pm})^2 = P_{\pm}$$

$$P_+ + P_- = 1 \qquad P_+P_- = P_-P_+ = 0$$

$$\gamma^3 P_+ = P_-\gamma^3 = 0 \qquad \gamma^3 P_- = P_+\gamma^3 = \gamma^3$$

$$\gamma^0 P_- = P_+\gamma^0 = 0 \qquad \gamma^0 P_+ = P_-\gamma^0 = \gamma^0$$

$$\hat{\gamma}^0 = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3) = \frac{1}{\sqrt{2}}(P_-\gamma^0 P_+ + P_+\gamma^3 P_-)$$

$$\hat{\gamma}^0\gamma^0 = \sqrt{2}\, P_+ \qquad \hat{\gamma}^0\gamma^3 = \sqrt{2}\, P_-$$

FIGURE CAPTIONS

1. The coordinate axes of the infinite momentum frame.
2. Typical Feynman diagram in coordinate space (a), and in momentum space after τ-ordering (b).
3. Pictures used for the $\delta(\tau)$ terms in the electron propagator (a) and the photon propagator (b).
4. Structures considered as single vertices. Structures like (c) and (d) give zero.
5. Vertices in the infinite momentum frame.
6. Typical diagrams that vanish because of η-conservation.

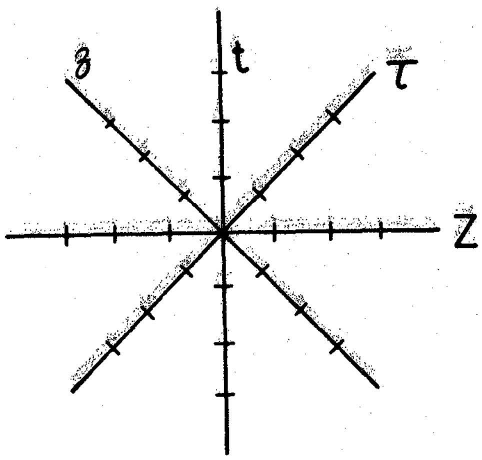

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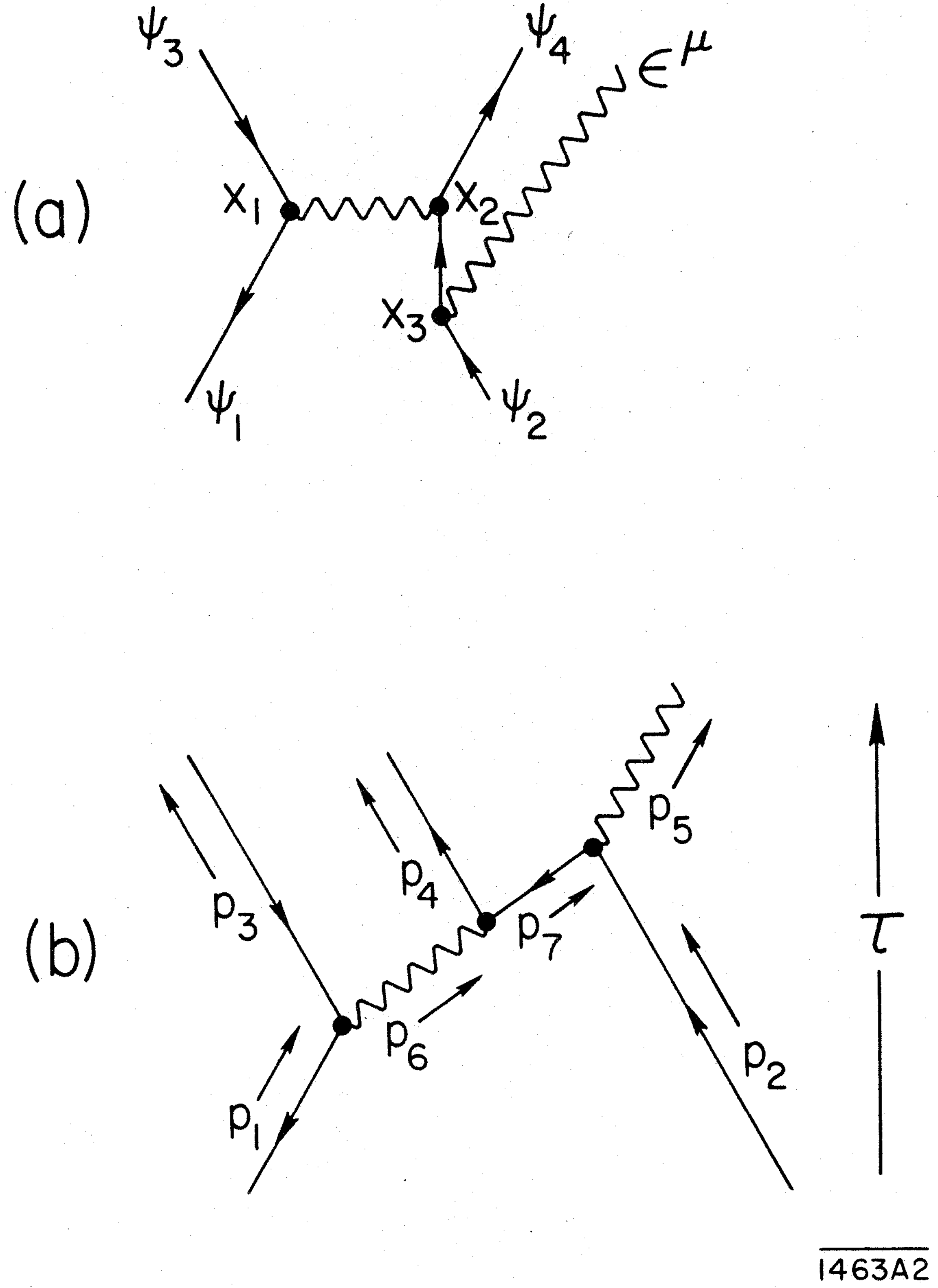
Fig. 1



(a)
ψ_3
ψ_4
ϵ^μ
x_1
x_2
x_3
ψ_1
ψ_2
(b)
p_3
p_4
p_5
p_7
p_6
p_1
p_2
τ
1463A2

Fig. 2

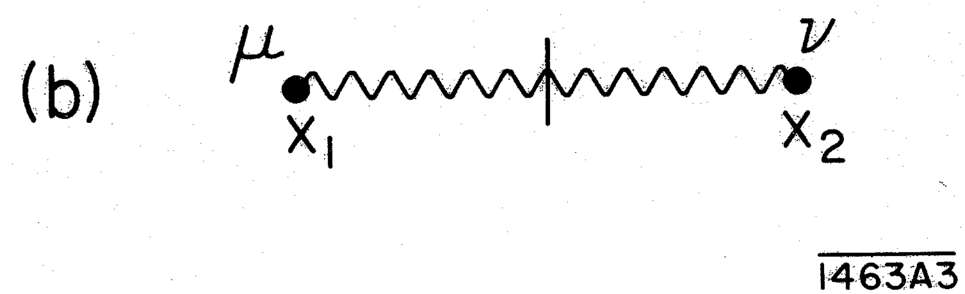


Fig. 3

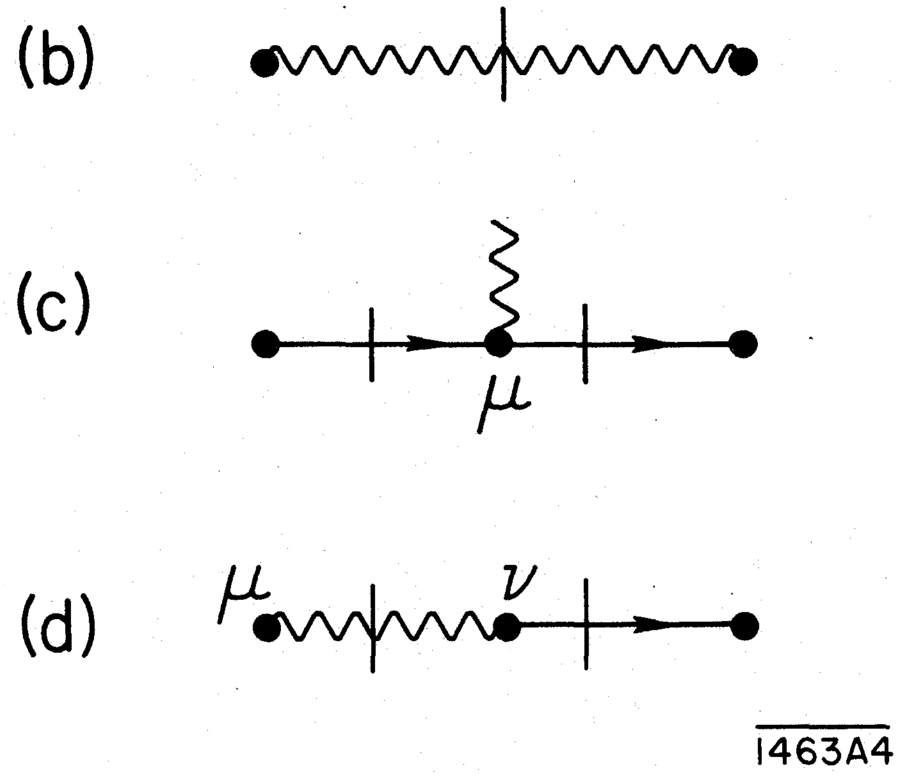


Fig. 4

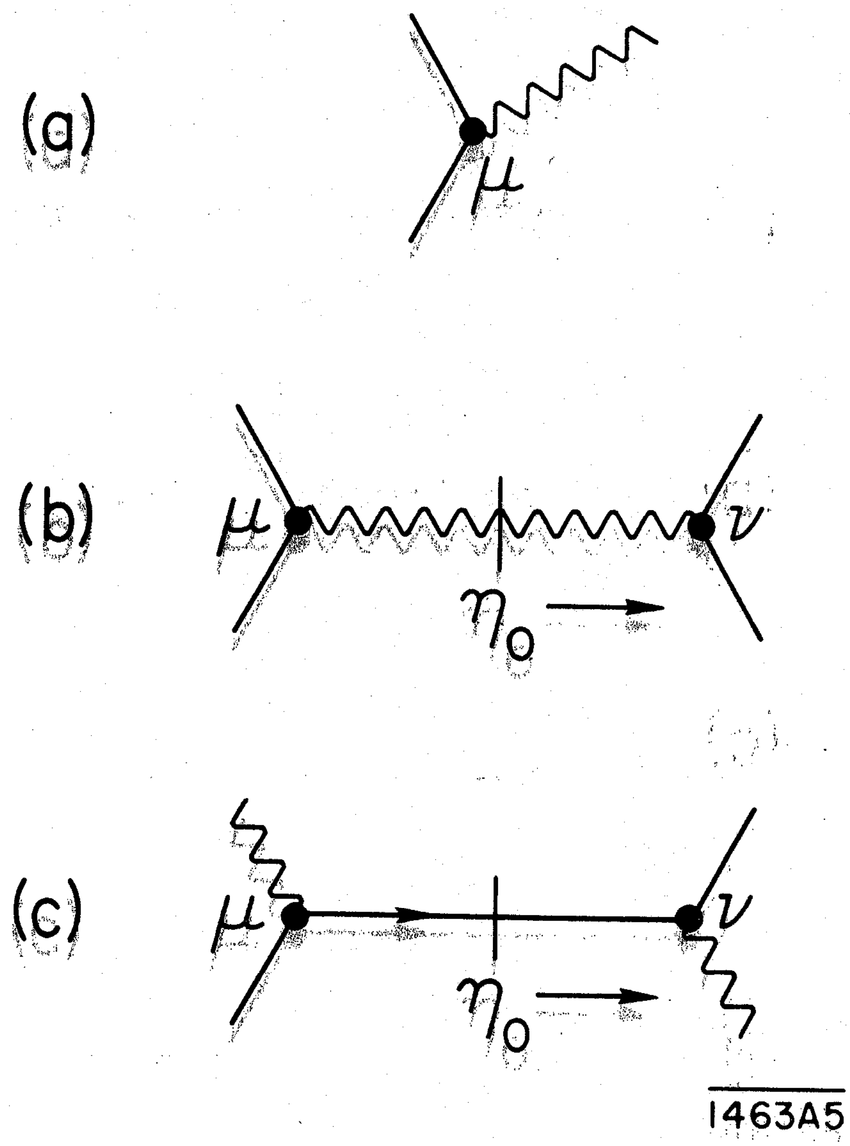


Fig. 5

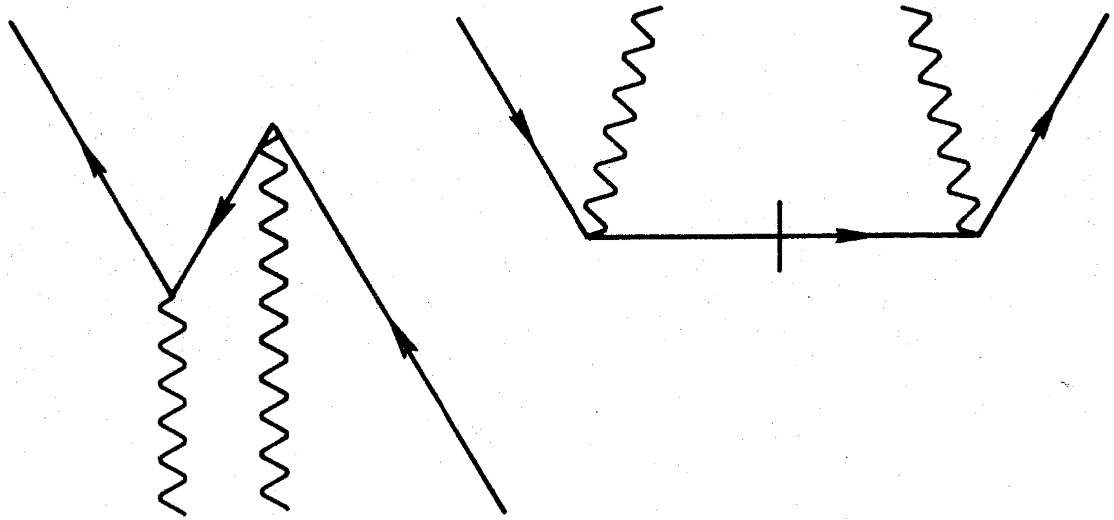


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Fig. 6