

## STATISTICAL MODEL FOR ELECTRON-POSITRON

## ANNIHILATION INTO HADRONS\*

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## ABSTRACT

A statistical model for the production of multibody hadronic states by  $e^+e^-$  annihilation is discussed. We associate the secondary hadron momentum distributions for colliding beam processes with the exponentially falling transverse-momentum distributions in hadron-hadron collisions. The consequence of this picture is that at high energies hadron multiplicity rises linearly with center-of-mass energy unlike the  $\log s$  behavior for the multiplicity of secondaries in hadron-hadron collisions. If the total annihilation cross section is assumed to have a power fall-off  $\sim s^{-m}$ , the  $n$ -pion cross sections follow a Poisson distribution with the most probable multiplicity  $n_\pi = \sqrt{s}/\langle E_\pi \rangle + 3 - m$  and  $\langle E_\pi \rangle \sim 375$  MeV. An alternate statistical model based on jets is also briefly discussed. The storage rings now being constructed or envisaged should easily distinguish between the various possibilities.

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## I. INTRODUCTION

In the next few years, electron-positron storage rings will be developed capable of producing hadron systems of total mass  $\sqrt{s}$  up to  $\sim 6$  GeV or higher. Aside from predictions for the energy dependence of the total annihilation cross section into hadrons,<sup>1,2</sup> there has been little discussion concerning the composition, multiplicity, and other properties expected for the multibody hadron final states.<sup>3</sup> It is not so clear what to expect, even qualitatively. The process  $e^+ + e^- \rightarrow$  hadrons at high energy differs from almost all other hadron processes inasmuch as (within the one-photon-exchange approximation) the hadrons are produced via the decay of an arbitrarily heavy virtual photon. One picture of such a decay would be that the virtual photon decays into an intermediate state consisting of a virtual pair of "bare" constituent partons<sup>4</sup> (such as a bare quark-antiquark pair) which subsequently decay in some way into hadrons—mainly pions. If this were the case, one could anticipate anisotropy and the existence of an axis in the distribution of hadron products; in other words the hadrons "remember" the direction along which the bare constituents were emitted. Under these circumstances, the transverse momenta  $p_{\perp}$  of the secondaries relative to the axis for a given event would be no more than a few hundred MeV, while the longitudinal momenta could be much larger. In other words the momentum distributions and the (slow) increase of multiplicity with energy would be much like the situation in ordinary hadron-hadron collisions.

The observation of such "jets" in colliding-beam processes would be most spectacular. It is not our intention here to study such a possibility further. Instead, we consider the case in which there are no high hadron momenta in the final state. Because all directions are equivalent in the center-of-mass frame, we associate the secondary hadron momentum distributions for colliding beam

processes with the transverse-momentum distributions in hadron-hadron collisions. These fall off exponentially and are characterized by a mean momentum of a few hundred MeV. The most immediate consequence of this picture is that the hadron multiplicity rises linearly with center-of-mass energy, quite unlike the case in hadron-hadron collisions. For example, given this picture we predict that at an energy of  $\sim 1.5$  GeV/lepton ( $\sqrt{s} = 3$  GeV), states containing 8 to 10 pions will be most prevalent; at  $\sim 3$  GeV/lepton typical hadron states are expected to contain on the order of 15-20 pions; at 6 GeV/lepton the number is 30-40 pions.

## II. THE STATISTICAL MODEL

A striking phenomenological feature of high energy hadron collisions is the fact that the distribution of transverse momenta of the secondaries is quite well represented by an exponential law<sup>5</sup>:

$$N(p_{\perp}) dp_{\perp} \propto p_{\perp} e^{-\frac{p_{\perp}}{b}} dp_{\perp} \quad (1)$$

with  $b = 1/2 \langle p_{\perp} \rangle \approx 150-200$  MeV/c depending upon the mass of the secondary. For pions,  $\langle p_{\perp} \rangle \approx 300$  MeV/c. The relation (1) has been checked over a range of transverse momentum from  $\sim 10$  MeV/c to 1.5 GeV/c. The approximate constancy of  $\langle p_{\perp} \rangle$  with energy has been checked for incident nucleon energies extending from a few GeV to cosmic-ray energies of  $\sim 10^4-10^5$  GeV. The longitudinal momentum distribution is much broader; as a consequence of the slow increase of multiplicity with energy, the energy per secondary increases as the center-of-mass energy increases.

In going over to electron-positron collisions we take the same exponential form (1), with  $p_{\perp}$  replaced by  $|p|$  and  $1/2 dp_{\perp}^2 = p_{\perp} dp_{\perp}$  replaced by the invariant

phase-space  $d^3 p/E$

$$N(p) d^3 p \propto e^{-\frac{|p|}{b'}} \frac{d^3 p}{E} \quad (2)$$

If we choose  $b'$  such that  $\langle p_1 \rangle$  remains unchanged from its value in hadron-hadron processes, we find, for pions

$$2b = \langle p_1 \rangle$$

$$\approx \frac{\pi}{2} b' \left[ \frac{\int_0^\infty dx x^3 (x^2 + m_\pi^2/b'^2)^{-1/2} e^{-x}}{2 \int_0^\infty dx x^2 (x^2 + m_\pi^2/b'^2)^{-1/2} e^{-x}} \right] \quad (3)$$

This factor in brackets, for reasonable  $b'$ , is nearly unity and gives  $b' \approx 1.2b \approx 175$  MeV. An immediate result is that if, as we expect to be the case, pions dominate the secondaries, the mean energy of a pion is

$$\langle E_\pi \rangle = 2b' \left[ \frac{\int_0^\infty dx x^2 e^{-x}}{2 \int_0^\infty dx e^{-x} x^2 (x^2 + m_\pi^2/b'^2)^{-1/2}} \right] \quad (4)$$

$$\approx 375 \text{ MeV}$$

By equipartition this leads to a crude estimate of multiplicity

$$n_{\pi} \simeq \frac{\sqrt{s}}{\langle E_{\pi} \rangle} \simeq \frac{\sqrt{s}}{375 \text{ MeV}} \quad (5)$$

In the following, we try to improve and refine this estimate.

### III. ANNIHILATION INTO $n$ PIONS

The differential cross section for annihilation into  $n$  pions may be written as

$$d\sigma^n = \frac{1}{2 \cdot 4} \left( \frac{16\pi^2 \alpha^2}{q^4} \right) \frac{\text{Tr} \left[ \not{p}_+ \gamma_{\mu} \not{p}_- \gamma_{\nu} \right]}{4E_+ E_-} dJ^{\mu\nu} \quad (6)$$

with the hadron matrix element  $dJ^{\mu\nu}$  given by

$$dJ^{\mu\nu} = \sum_n \langle 0 | j^{\mu}(0) | n \rangle \langle n | j^{\nu}(0) | 0 \rangle (2\pi)^4 \delta^4(P_n - q) \prod_{i=1}^n \frac{d^3 p_i}{2E_i (2\pi)^3} \quad (7)$$

The statistical assumption that we adopt is that in the center-of-mass frame

$$dJ^{\mu\nu} = a_n (q^{\mu} q^{\nu} - g^{\mu\nu} q^2) e^{-\sum_i a |p_i|} (2\pi)^4 \delta^4(P_n - q) \prod_{i=1}^n \frac{d^3 p_i}{2E_i (2\pi)^3} \quad (8)$$

with  $a_n$  a slowly varying function of  $q^2$  and  $a = 2/\langle E_{\pi} \rangle$ . We then find for  $d\sigma^n$ , the expression

$$d\sigma^n = \frac{8\pi^2 \alpha^2}{q^2} a_n \left\{ \prod_{i=1}^n e^{-a|p_i|} \frac{d^3 p_i}{2E_i (2\pi)^3} \right\} (2\pi)^4 \delta^4(P_n - q) \quad (9)$$

We discuss in the appendix the relationship of this form with the single-pion momentum distribution (2). Clearly the essentially uncorrelated distribution (9) should only be expected to have validity, if at all, for large  $n$ . We shall only apply the results consequent from (9) for  $n \geq 4$ . All correlation effects, isospin requirements, Bose-statistics, and effects associated with higher mass particles ( $K$ ,  $N$ ,  $\bar{N}$ ,  $Y$ ,  $\bar{Y}$ ) have been neglected. In a statistical model such as this, we

expect the  $K/\pi$  ratio  $\sim 10^{-1}$  and the  $p/\pi$  ratio  $\sim 10^{-2}$ , of the order of the ratio in hadron-hadron collisions. We also expect the 2-body and quasi-two-body final states to be a very small fraction of the total yield. For example, the cross section for  $e^+ + e^- \rightarrow p + \bar{p}$  can be estimated from the dipole fit to the electromagnetic form factor to decrease as  $\sim s^{-5}$ .

Returning to the cross section (9), we carry out the integrals over the pion phase-space. The pion mass makes only a minor modification to the kinematics, and we neglect it here; then all the phase-space integrals can be performed. Introducing a Fourier-transform on the  $\delta^4(P_n - q)$  we get

$$\begin{aligned}
 \sigma^n &= \frac{8\pi^2 \alpha^2}{q^2} a_n \int d^4x e^{-iq \cdot x} \left[ \int \frac{d^3p}{2p(2\pi)^3} e^{ipt - ip_i \cdot x} e^{-ap} \right]^n \\
 &= \frac{8\pi^2 \alpha^2}{s} a_n \int d^3x dt e^{-i t \sqrt{s}} \left\{ \frac{1}{4\pi^2 [(a-it)^2 + x^2]} \right\}^n \\
 &= \frac{8\pi^2 \alpha^2}{s} a_n \int_{-\infty}^{\infty} dt \frac{e^{-i t \sqrt{s}}}{(4\pi^2)^n} \frac{2\pi^2 (2n-4)!}{(n-1)! (n-2)!} \frac{1}{[2(a-it)]^{2n-3}} \\
 &= \frac{\pi \alpha^2}{s} a_n \frac{s^{n-2} e^{-a \sqrt{s}}}{(16\pi^2)^{n-2} (n-1)! (n-2)!} \\
 &\equiv c_n s^{n-3} e^{-a \sqrt{s}}
 \end{aligned} \tag{10}$$

Therefore, as a function of  $s$  the cross section for producing  $n$  pions follows a distribution which is sharply peaked about the value

$$a \sqrt{s} \approx 2n-6 \tag{11}$$

(provided  $a_n$  varies slowly with  $s$ ). Thus

$$n \approx 3 + \frac{\sqrt{s}}{\langle E_\pi \rangle} \approx 3 + \frac{\sqrt{s}}{375 \text{ MeV}} \tag{12}$$

which for large  $n$  is in agreement with our original estimate (5). We also see that as long as  $a_n$  indeed varies slowly with  $s$ , it is sufficient to set it equal to its value at the  $s$  for which  $\sigma^n$  attains its maximum, Eq. (11).

Various dynamical models have predicted the asymptotic behavior of the total annihilation cross section. The most optimistic quark-model estimate gives<sup>1</sup>

$$\sigma_{\text{tot}}(s) \sim s^{-1} \quad s \rightarrow \infty \quad (13)$$

while the gauge-field algebra gives the prediction<sup>2</sup>

$$\sigma_{\text{tot}}(s) < s^{-2} (\log s)^{-1} \quad s \rightarrow \infty \quad (14)$$

and probably

$$\sigma_{\text{tot}}(s) \sim s^{-3} \quad (15)$$

It is interesting to combine such behaviors with the statistical-model forms.

If, for large  $s$ ,

$$s^m \sigma_{\text{tot}}(s) \rightarrow \text{constant} \quad (16)$$

then the  $c_n$  in (10) can be estimated. If we set

$$s^m \sigma_{\text{tot}}(s) = \text{constant} = c \quad (17)$$

we get, from (10),

$$\sum_n c_n s^{n+m-3} = s^m \sigma_{\text{tot}} e^{a\sqrt{s}} = c e^{a\sqrt{s}} \approx c (e^{a\sqrt{s}} + e^{-a\sqrt{s}}) = 2c \sum_{k=0}^{\infty} \frac{(a^2 s)^k}{(2k)!} \quad (18)$$

Equating coefficients we have<sup>(6)</sup>

$$c_n = 2c \frac{a^{2n+2m-6}}{(2n+2m-6)!} \quad (19)$$

and finally, for the cross section to produce  $n$  pions, with the aid of (10)

$$\sigma_n = \frac{2c}{s^m} (a \sqrt{s})^{2n+2m-6} \frac{e^{-a\sqrt{s}}}{(2n+2m-6)!} \quad (20)$$

Notice the distribution of multiplicity  $n_\pi$  for a given  $s$  is a Poisson distribution with  $\Delta n_\pi \sim \frac{1}{\sqrt{n_\pi}}$ .

#### IV. CONCLUSIONS

The two models we have discussed, "jet" and "statistical" are most likely extreme limiting cases, with the truth somewhere in between. The most immediate experimental distinction between them, apart from the qualitative "visual" difference,<sup>7</sup> is the energy dependence of the mean multiplicity, which we compare in Fig. 1 for p-p and  $\pi$ -nucleon collisions<sup>8</sup> (supposed to roughly represent the case of the "jet" model) and for  $e^+ - e^-$  collisions in the statistical model, according to Eq. (12) predicting  $n_\pi \approx 3 \sqrt{s}/\text{GeV}$ . The storage rings now being constructed or envisaged should easily distinguish between the two extreme cases.

For the statistical model, our conclusions are summarized in Eq. (20). From that equation, it can be deduced that with  $\sigma_{\text{tot}} \sim s^{-m}$  ( $s \rightarrow \infty$ ), then at energy  $\sqrt{s}$ , the most probable multiplicity is

$$n_\pi = \frac{\sqrt{s}}{\langle E_\pi \rangle} + 3 - m \quad (21)$$

with  $\langle E_\pi \rangle \approx 375$  MeV. The multiplicity at this  $\sqrt{s}$  falls off rapidly for larger and smaller  $n_\pi$ , with width  $\Delta n_\pi \sim \sqrt{n_\pi}$ . Likewise the cross section for fixed  $n_\pi$  falls off rapidly for energies  $\sqrt{s}$  larger and smaller than the optimum given above by (11), again with  $\Delta(\sqrt{s}) \sim \sqrt{\langle E_\pi \rangle \sqrt{s}} \approx \sqrt{\frac{s}{n_\pi}}$ . In Figs. 2a and 2b we plot the partial cross sections for  $m = 1$  and 2.

Most of the anticipatory interest in colliding beam experiments has resided in 2-body channels (study of vertex functions) and the possibilities of relatively clean



resonance spectroscopy. Yet it is quite probable that the bulk of the events will not fall into these categories. In a new phenomenon such as this, where even qualitative properties are a matter of speculation, it would be a surprise if careful study of the more common complicated events did not reveal important and fundamental facts regarding hadron dynamics. It is with this in mind that this study was carried out.

#### V. ACKNOWLEDGEMENTS

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APPENDIX

SINGLE PARTICLE DISTRIBUTIONS

Let us consider a simple factorizable representation for the distribution of transverse momentum of secondaries in hadron-hadron collisions,

$$dN = d^2 p_1 \dots d^2 p_n f(p_1) \dots f(p_n) \delta^2(\Sigma p_i) \quad (\text{A. 1})$$

where  $f(p)$  is a peaked function of the magnitude of the transverse momentum.

The resulting single particle distribution is

$$\frac{dN}{d^2 p_1} = f(p_1) \int \frac{d^2 x}{(2\pi)^2} e^{i \underline{p}_1 \cdot \underline{x}} \left[ \int d^2 k e^{i \underline{k} \cdot \underline{x}} f(k) \right]^{n-1} \quad (\text{A. 2})$$

The Fourier transform

$$\tilde{f}(x) \equiv \int d^2 k e^{i \underline{k} \cdot \underline{x}} f(k) \quad (\text{A. 3})$$

is maximum at  $x = 0$ ; for small  $x$

$$\begin{aligned} \tilde{f}(x) &\cong \int d^2 k f(k) \left[ 1 + i \underline{k} \cdot \underline{x} - \frac{(\underline{k} \cdot \underline{x})^2}{2} + \dots \right] \\ &\cong \tilde{f}(0) \left[ 1 - \frac{1}{4} x^2 \langle k^2 \rangle \right] \end{aligned} \quad (\text{A. 4})$$

The important range of integration in (A.2) is small  $x$ , so that when  $\tilde{f}(x)$  is raised to a large power it can be approximated by a Gaussian:

$$\frac{dN}{d^2 p_1} \propto f(p_1) \int d^2 x e^{i \underline{p}_1 \cdot \underline{x}} e^{-\left(\frac{n-1}{4}\right) x^2 \langle k^2 \rangle} \propto f(p_1) e^{-\frac{p_1^2}{(n-1) \langle k^2 \rangle}} \quad (\text{A. 5})$$

Suppose

$$f(p) = e^{-p/b}, \quad 2b = \langle k \rangle = .3 \text{ BeV} \quad (\text{A. 6})$$

then

$$\langle k^2 \rangle = 6b^2 \cong .135 \text{ BeV}^2 \quad (\text{A. 7})$$

and

$$\frac{dN}{dp_1^2} \propto e^{-6.5 p_1} e^{-\frac{7.5}{(n-1)} p_1^2} \quad (\text{A. 8})$$

with  $p_1$  in (BeV). For a typical multiplicity  $n \sim 5$ , the distribution remains exponential until the Gaussian takes over at  $p_1 \sim 4$  BeV. Thus  $f(p_1)$  can be identified experimentally with the single particle distribution.

For the annihilation process into  $n$  particles we made the ansatz

$$dN = \frac{d^3 p_1}{p_1} \dots \frac{d^3 p_n}{p_n} f(p_1) \dots f(p_n) \delta^4(q - \sum p_i) \quad (\text{A. 9})$$

Let us consider the single particle distribution for the annihilation processes where the incident energy is averaged over an interval which includes the major contribution to the cross section  $\sigma_n$ . Then, ignoring mass corrections

$$\int dq_0 \frac{dN}{d^3 p_1} \propto \frac{f(p_1)}{p_1} \int d^3 x e^{ip_1 \cdot x} \left[ \int \frac{d^3 k}{k} f(k) e^{ik \cdot x} \right]^{n-1} \propto \frac{f(p_1)}{p_1} \exp \left[ -\frac{p_1^2}{(n-1)k^2} \right] \quad (\text{A. 10})$$

Again the Gaussian is ineffective and  $f(p_1)/p_1$  can be experimentally identified with a single particle distribution. It might be noted that if  $f(p)$  is a pure exponential, then in Eq. (A.9),

$$f(p_1) \dots f(p_n) = e^{-a(p_1 + \dots + p_n)} = e^{-aq_0}, \quad (\text{A. 11})$$

Thus  $dN/d^3 p$  is independent of  $a$  (and  $\langle p \rangle$ ) and is only determined by phase-space. For  $\sqrt{s}$  near the maximum of  $\sigma_n(s)$  it follows that the single particle distribution from pure phase-space must reduce back to the single-particle distribution  $\exp[-ap]$ .

## FOOTNOTES AND REFERENCES

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6. Notice that the coefficients  $a_n$  in (10) for large  $n$  can now be written as an area factor  $[A]^{n-2}$  times a slowly-varying function of  $n$  and  $s$ .
7. For example, the underlying axes of the jets would imply a non-zero quadrupole moment for the angular distribution of the hadrons. Quantitatively, one could check for non-degenerate eigenvalues of the tensor

$$T_{\alpha\beta} = \frac{\sum_i \left[ \frac{3}{2} p_{\alpha}^i p_{\beta}^i - \frac{1}{2} \delta_{\alpha\beta} p_i^2 \right]}{\sum_i p_i^2}$$

(averaged over events) where  $p_i$  is the momentum of the  $i^{\text{th}}$  hadron.

8. The data have been summarized by O. Czyzewski, the 14th International Conference on High Energy Physics, (Vienna, 1968); p. 367.

#### FIGURE CAPTIONS

1. Charged hadron multiplicity given by the statistical model for  $e^+ e^-$  annihilation compared to the observed multiplicity for p-nucleon and  $\pi$ -nucleon scattering. The experimental points are from the data compilation of Ref. 8.
2. The annihilation cross sections into  $n \geq 4$  pions and total cross section according to the statistical model assuming for high energies

(a)  $\sigma_{\text{tot}} \sim s^{-1}$ ,

(b)  $\sigma_{\text{tot}} \sim s^{-2}$ .

The curves are given as functions of  $x = \sqrt{s}/\langle E_{\pi} \rangle$  and also energy/lepton for a typical value  $\langle E_{\pi} \rangle = 375$  MeV.

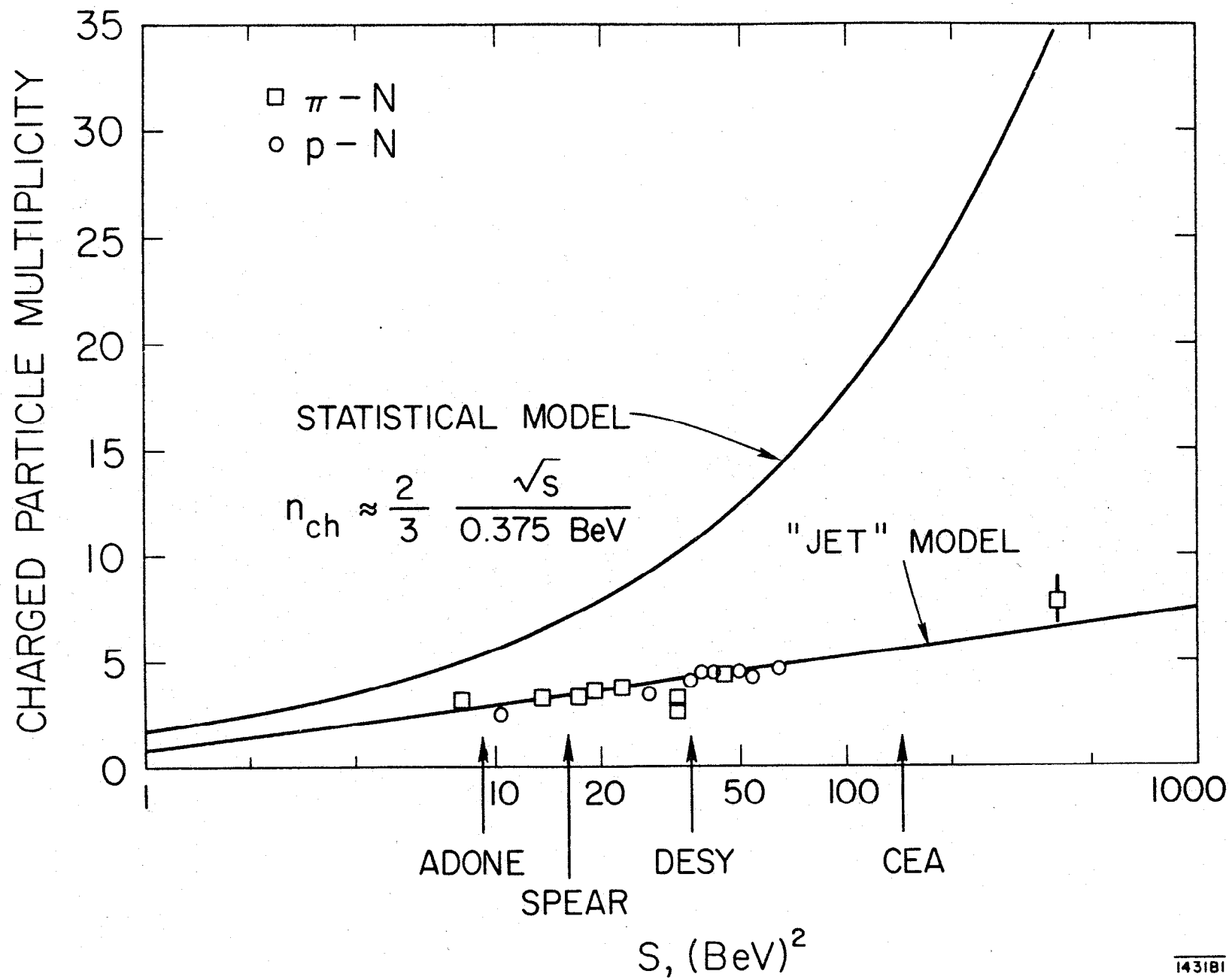


Fig. 1

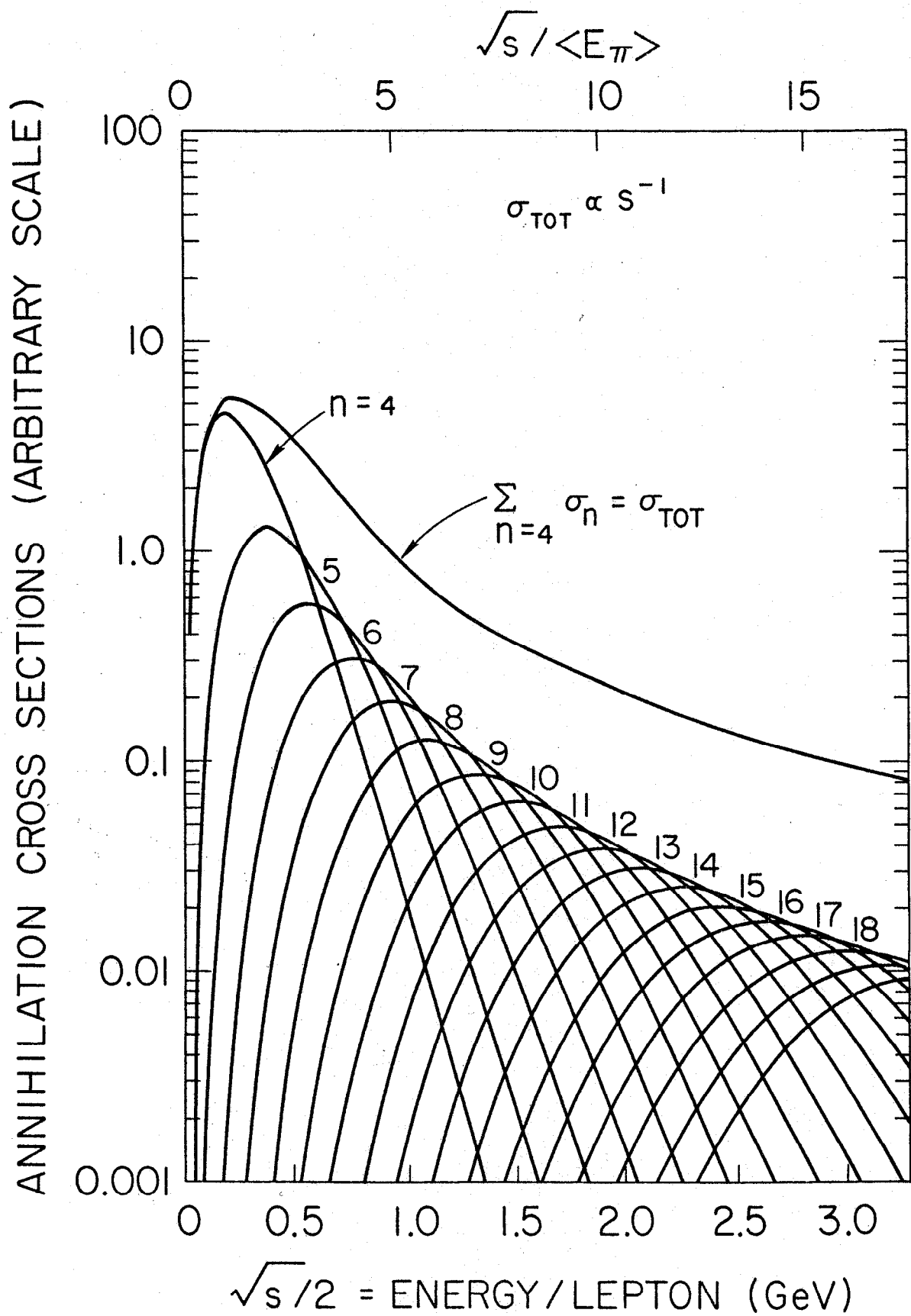


Fig. 2a

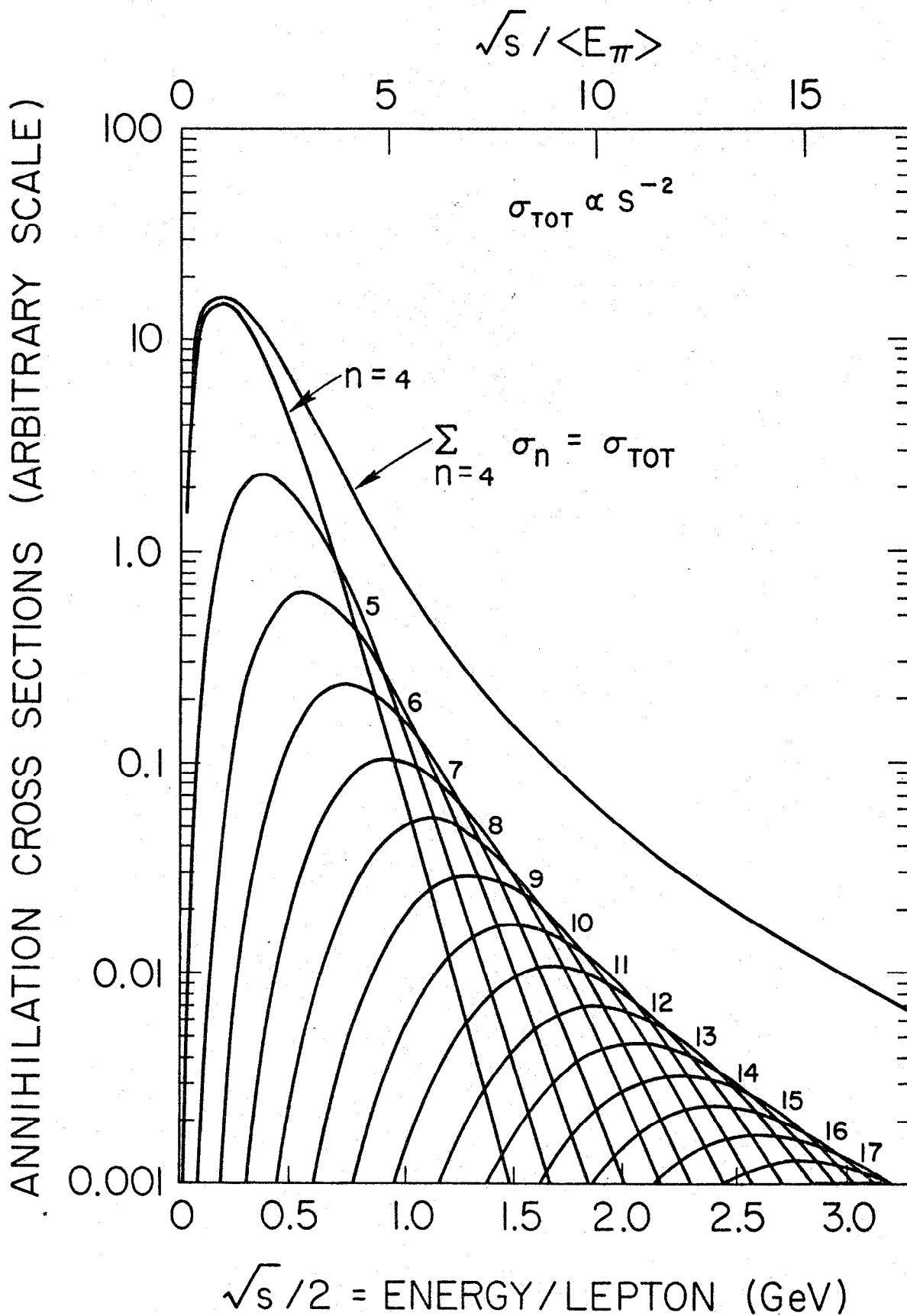


Fig. 2b