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AN INVESTIGATION OF ASYMPTOTIC BEHAVIOR IN FIELD THEORY — THE ELASTIC FORM FACTOR*

Thomas Appelquist and Joel R. Primack†
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We examine the question whether local relativistic quantum field theory can account for the observed asymptotic behavior of the electromagnetic form factor of the proton. Particular attention is devoted to the theory of pions and nucleons interacting via isospin-symmetric pseudoscalar coupling, with minimal coupling to the electromagnetic field. New techniques are developed for calculating and summing all asymptotically dominant contributions to vertex functions and to propagators from all orders of perturbation theory. The results indicate that summing Feynman graphs in local quantum field theory is unlikely to provide an explanation of the observed rapid decrease of the form factor at large momentum transfers.

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I. INTRODUCTION

A question of great interest in high energy physics is whether quantum field theory will ever have any calculational value in the realm of strong interactions. It seems unlikely that a simple field theory can be used in the region where processes are dominated by resonances. However, in the asymptotic region where the momenta involved are much larger than any of the masses and where cross sections vary smoothly with the momenta, one might hope that a field-theoretic approach would have some value. In this region, the small distance structure of hadrons, which seems to be much simpler than the long range resonance structure, is evidently being probed.

Some examples of this smooth asymptotic behavior are the large s, small t behavior of scattering amplitudes predicted by Regge theory; the high-energy, large-momentum-transfer behavior of two-body hadronic processes[1]; the behavior of the inelastic electron-proton scattering structure functions W_1 and W_2[2]; and the rapidly decreasing behavior of the electromagnetic form factor of the nucleon for large spacelike q^2. The simplest of these things to analyze is the form factor since one can then deal with at most three-point functions and the kinematics are quite easy to handle. In this paper, we will examine the possibility of using field theory to explain the asymptotic behavior of the nucleon electromagnetic form factor.

During the last decade, a series of beautiful experiments on elastic electron-proton scattering, beginning with the work of Hofstadter and collaborators and extending to the recent SLAC results,[3] has allowed the determination of the electromagnetic form factor of the proton up to momentum transfers of $-q^2 \approx 25\ (\mathrm{GeV/c})^2$. These experiments have shown that the magnetic Sachs form factor G_M^P decreases asymptotically at least as fast as $\sim 1/q^4$. If the scaling law

$G_E^P = G_M^P/\kappa_p$, which has been verified experimentally for $-q^2 \lesssim 3\ (\mathrm{GeV/c})^2$, continues to hold at least approximately for large $-q^2$, then we can conclude that the proton's Dirac form factor[4] also decreases the same way for large $-q^2$, that is $F_1^P \sim 1/q^4$. The rapid decrease of the electromagnetic form factors is rich in physical implications, and it begs for deep theoretical understanding.

This understanding has been sought in a number of different directions.[5] It will be helpful to summarize here those that have some bearing on our field theoretic approach and we will do this in Section II, under the following rubrics: (a) resonance fits, (b) composite models, (c) bootstrap models, (d) statistical theories, and (e) related analyses of field theory.

Let us state more specifically what we mean by a field theoretic approach. The only solutions which now exist to realistic field theories are perturbative solutions to renormalizable theories. The renormalizable field theory which first comes to mind to describe the electromagnetic form factor of the nucleon is the isospin-symmetric pion-nucleon theory with the usual pseudoscalar γ_5 coupling[6]

$$\mathscr{H}_I = ig_0 \bar{\psi} \gamma_5 \vec{\tau} \psi \cdot \vec{\pi} + \lambda_0 (\vec{\pi}^2)^2 , \tag{I.1}$$

with minimal electromagnetic coupling. The pion self-coupling term is included in (I.1), as usual, to compensate for the logarithmic divergence of π-π scattering graphs. It is this field theory that we shall analyze in detail in this paper. Our analysis will also lead us to several general conclusions about the use of field theory in the asymptotic region.

Our approach will be to attempt to derive the large q^2 behavior of the form factor by summing the complete leading asymptotic term from each order of perturbation theory. The hope is that the sum will converge — or that it can be continued to a meaningful expression — for asymptotic values of q^2. We develop a simple iterative method for determining the leading asymptotic behavior of vertex and self-energy graphs and for summing arbitrary sets of these graphs over all orders. The scheme is simplified by the use of Ward's identity to calculate the self-energy graphs at each stage, thereby avoiding complications associated with overlapping divergences.

In Section III, we present the results of calculating the leading logarithmic contributions for all vertex and self-energy graphs in first (lowest) order perturbation theory and all vertex graphs in second order perturbation theory. (In this section and throughout the rest of the paper, most of the explicit calculations are relegated to appendices.)

The results of Section III suggest two things. First of all, the asymptotically leading logarithmic contributions in each order come from the ultraviolet region of integration independently of which legs of the vertex function are taken off mass shell. This is a result of the p-wave γ_5 coupling, which suppresses the infrared region. In fact, it can be shown[7] that there are no infrared divergences in this theory in the limit $m_\pi \rightarrow 0$. These results will be contrasted in the discussion section with those from theories which emphasize the infrared region.

The other thing suggested by these low order results is that the leading contribution to the vertex from any graph which has a leading contribution in its order of perturbation theory can be determined iteratively in terms of its vertex and self-energy subgraphs. We emphasize that our techniques have no relation to those of Tiktopolous,[8] who was investigating a theory where the leading contributions come from the infrared region of integration.

The machinery for calculating the leading lograithmic contributions of ladder graphs in any order of perturbation theory is set up in Section IV. The sum of leading logarithms in the ladder graphs for the isoscalar electromagnetic form factor is found to have the very interesting asymptotic form

$$F_1^S(q^2) \sim \exp\left[\frac{-3g^2}{32\pi^2}\log(-q^2/m^2)\right] \approx (m^2/-q^2)^{1.8} \qquad (I.2)$$

using the value $(g^2/4\pi) \approx 15$ for the renormalized strong interaction coupling constant. However, the importance of this result is made very doubtful by the fact that the isovector form factor sums to a growing exponential, and also that when we include all graphs which give a leading contribution in each order, which is done in Section V, a much less interesting form results. The proton's Dirac form factor then sums to

$$F_1^P(q^2) \sim \left(1 - \frac{10g^2}{32\pi^2} \log \frac{-q^2}{m^2}\right)^{3/10} . \tag{I.3}$$

Similar very unphysical results of summing the leading logarithmic contributions in each order appear also in the propagators and in the γ-N and π-N vertex functions with either one, two, or three legs asymptotic. These results for the completely (three legs) asymptotic π-N vertex and the asymptotic propagators were first obtained by Landau and collaborators[9] by solving, to leading logarithmic accuracy, a set of approximate integral equations, and by Bogoliubov and collaborators[10] using the renormalization group. These methods are reviewed in Section II.E.

The leading logarithmic contributions to the neutron's Dirac form factor cancel in each order of perturbation theory. This is due to the fact that these terms, which go like $g^{2n} \log^n(-q^2/m^2)$, come from the ultraviolet region of integration and that since there is no elementary neutron-γ coupling, the sum of all neutron-γ vertex graphs in each order contains no leading logarithmic divergence.

The above results show that if physically sensible results are to emerge from summing over all orders of perturbation theory, nonleading logs in each order must be included. This possibility is investigated in Section VI. It is found that the unphysical behavior remains when next-to-leading logs $\left(g^{2n} \log^{n-1}(-q^2/m^2)\right)$ are summed, indicating that a sum of all logs in all orders will also be plagued with these problems. We conclude by discussing other renormalizable field theories and by making several general observations about the use of perturbative sums in field theory for investigating asymptotic behavior.

In Appendix A, we give the parametric integral notation used in doing the calculations in this paper. Appendix B contains some second order calculations. The general proof of the iteration scheme for ladder graphs is given in Appendix C. We treat the n^{th} order ladder graph explicitly and then point out how to treat an arbitrary graph.

II. PREVIOUS APPROACHES

We will not attempt to review all previous theoretical work on the asymptotic behavior of form factors — for that, we refer the reader to the several excellent review articles.[5] Rather, we will discuss several diverse approaches for purposes of comparison or contrast with our field theoretical approach.

A. Resonance Fits

Resonance approximations generally start from the dispersion relation for the form factor in a narrow resonance approximation. Evidently a cancellation must occur among the resonances in order to be consistent with $F(q^2)$ decreasing asymptotically faster than $1/q^2$. Such a cancellation is difficult or impossible to arrange, using known resonances. Thus resonance fits have not shed much light on the reasons for the rapid decrease of the form factor. It would indeed have been surprising were the dispersion expression for the form factor in the asymptotic region to be dominated by one or a few resonances near threshold. Perhaps the recent attempt by Jengo and Remiddi[11] and others to take into account some of the cut contribution in a resonance approximation by including the poles associated with the $J = 1$ daughters of all the Regge recurrences of the ρ, a la Veneziano, is a reasonable direction in which to extend resonances models.

B. Composite Models

It has long been supposed that some sort of composite structure is responsible for the rapid decrease of form factors. It is not difficult to understand that a large momentum transfer $-q^2$ should more readily disintegrate such a composite system, the larger $-q^2$ is.

Drell, Finn, and Goldhaber[12] have examined nonrelativistic potential theory models, and have shown that if the function $g(r) = r|\psi(r)|^2$ is well defined for $r = 0$ and is well-behaved at infinity, then asymptotically

$$F(q^2) \sim \frac{g(0)}{q^2} + \frac{g''(0)}{q^4} + \frac{g^{(iv)}(0)}{q^6} + \ldots \tag{II.1}$$

Since $|\psi(0)|^2$ is zero or finite for any but pathological potentials, $g(0) = 0$ and consequently $F(q^2) \sim 1/q^4$. One can obtain faster decrease by arranging to have the higher derivatives of g vanish; indeed, $F(q^2)$ falls off asymptotically faster than any inverse power if all the even derivatives of $g(r)$ vanish at the origin. The potential $V(r) = V_0/r^2$ can lead to exponential decrease for certain values of V_0; and, in general, if $V(r) \sim r^{-2(1+p)}$ as $r \to 0$, then $F(q^2)$ decreases like $\exp\left(-q^{p/(p+1)}\right)$ times inverse powers of q and an oscillating factor. It turns out that $F \sim e^{-\sqrt{q/q_0}}$ fits the large $-q^2$ proton form factor data fairly well,[1] a fact which gives us little theoretical insight but from which we draw the moral that the data cannot yet distinguish between inverse power and fractional exponential asymptotic behavior.

More recently, Ball and Zachariasen[13] constructed a simple relativistic composite model which gives the asymptotic behavior $F \sim 1/q^4$ times one or two powers of $\log(-q^2)$. Their model is represented in Fig. 1; they solve the linear integral equation Fig. 1b to obtain the asymptotic behavior of Γ; this is then substituted into Fig. 1a to determine the asymptotic behavior of the form factor. The origin of the $1/q^4$ dependence can be visualized simply: one power of $1/q^2$ arises from the Feynman propagator of the constituent which absorbs the photon, and the other comes from the exchange of a (scalar) meson between the constituents. Very similar results have also been derived from essentially the same model on the somewhat firmer basis of the Bethe-Salpeter equation in ladder approximation by Ciafaloni and Menotti[14] and by Amati et al.[15] In addition, these authors found that if a composite particle is formed by combining two particles in a P-state instead of an S-state, then asymptotically the form factor acquires another factor of $1/q^2$; similarly, higher-spin composite particles acquire additional factors of $1/q^2$. Ciafaloni and Menotti[14] emphasize that the asymptotic behavior of the form factor in a composite model is also dependent upon the nature of the particle exchanged and its coupling to the constituents; the results mentioned above were obtained assuming scalar exchange with linear scalar coupling in Fig. 1b. Menotti[14] further observes that the spins assumed for the constituents can have a considerable influence on the asymptotic behavior of the form factor in a composite model.

Besides the model dependence of the sort we have just mentioned, there is a somewhat deeper difficulty with composite models. In general, if we construct a dynamical model in which a hadron is regarded as being made up of constituents, whether observable or fictitious, there is little reason to suppose that the number of these constituents is two, three,[16] or any particular finite number for that matter. However, as Stack[17] has observed in a nonrelativistic model and as Amati, Jengo et al.,[15] found relativistically also, the asymptotic behavior of the form factor in a composite model depends strongly on the number of constituents, with extra factors of $1/q^2$ for each additional constituent. This observation has led to the proposal of infinitely composite models with faster-than-polynomial decrease of the form factor. Two species of such models will be discussed here, "bootstrap" models in the next subsection, and "statistical" models in the one after that.

In some sense, the quantum theory of local, interacting fields, which we examine in this paper, is an alternative to an infinitely composite model. Before discussing our work, we review, in Section II.E, the previous field-theoretic treatments of form factors.

C. Bootstrap and Infinitely Composite Models

Stack[17] investigated a particular nonrelativistic model in which the proton is regarded as a bound state of a proton and an infinite number of pions. For potentials of the Coulomb or Yukawa form, Stack was able to show that the form factor decreases exponentially for large momentum transfers: $F(q) \sim e^{-aq}$. He then argued that the precise features of his model did not matter in obtaining exponential fall-off; rather it is the assumption of infinite compositeness which was crucial.

A more sophisticated bootstrap model along somewhat different lines has been developed by Harte.[18] For simplicity, let us discuss the case of a theory in which there is only one species of particle, self-coupled with a ϕ^3 interaction. Harte considers nonlinear equations of the sort represented in Fig. 2, and imposes a "crossing-symmetric bootstrap" condition by requiring that the asymptotic behavior of the hadron vertex function Γ, which is assumed as an input to the equations in Fig. 2, be reproduced on the rhs of this equation. Harte gives arguments that asymptotically the electromagnetic form factor $F(q^2) \sim \Gamma(q^2)$, up to a polynomial factor. The simplest nonrelativistic and relativistic models gave $\Gamma(q^2) \sim (q^2)^{\beta} e^{-a\sqrt{-q^2}}$, but further consideration led to the conclusion that the integral equation implicit in Fig. 2 is actually consistent with more general asymptotic behavior. Specifically, if Γ is assumed to have the asymptotic form

$$\Gamma(q^2, p^2, p'^2) \sim \beta(q^2, p^2, p'^2)\, e^{-a\, g(q^2)\, g(p^2)\, g(p'^2)} \qquad \text{(II.2)}$$

when one or more of the variables q^2, p^2, p'^2 approach infinity, where β is polynomial, then this will be a solution if

$$g(0) = 0, \quad \lim_{q^2 \to \infty} \frac{\log q^2}{g(q^2)} = 0 \, , \qquad \text{(II.3)}$$

and

$$\lim_{q^2 \to \infty} \frac{g(q^2)}{(q^2)^\gamma} = 0 \quad \text{for } \gamma > \frac{1}{4} \, . \qquad \text{(II.4)}$$

It should be understood that although (II.2) is the only form of asymptotic solution which Harte was able to find, he has not been able to prove that this solution is unique.

D. Statistical Theories

An interesting example of a statistical approach is provided by the work of Kastrup[19] and Mack.[20] They constructed a model for meson production in high-momentum-transfer collisions involving hadrons, in analogy with electromagnetic bremsstrahlung. Such a model is suggested by the fact that most nucleon-nucleon collisions are quasi-elastic. The secondaries are mostly pions (about 80%), with the ratio between pions and kaons, the energy of the secondary mesons in the cm frame (usually $\lesssim$ 1 GeV), and the average transverse momentum of the secondaries ($\simeq$ 0.3 GeV/c) all approximately independent of the energy of the primaries.

If it is assumed that recoil is negligible in the emission of the secondary soft particles, and if complications of spin, isospin, and parity are also ignored, then the secondaries are emitted independently and with a Poisson distribution. For electron-proton scattering with the emission of n secondary soft mesons,

$$d\sigma^n(s,t) = \frac{\bar{n}^n}{n!}\; e^{-\bar{n}}\, d\sigma^{tot}(s,t), \quad n=0,1,\ldots, \tag{II.5}$$

where $\bar{n}\,(s,t)$ is the average number of secondaries at given s and t, and

$$d\sigma^{tot} = \sum_{n=0}^{\infty} d\sigma^n \ .$$

Combining (II.5) with the Rosenbluth formula (for n=0), they obtain for $s \gg -t \gg m^2$ the result

$$\bar{n} \sim -\log\left[G_M^P(t)\right] \quad . \tag{II.6}$$

If $\bar{n}$ can now be calculated on the basis of a dynamical hypothesis of some sort, then the asymptotic behavior of G_M^P can be obtained. Mack[20] now assumes a "partially conserved dilation current"[19] in order to derive an expression for $d\sigma^1$; thus attaching a soft meson to all nucleon lines in all Feynman graphs for the form factor, he also uses a sort of generalized Ward identity. After some further approximations, a nonlinear differential equation is obtained which has the asymptotic solution

$$G_M^P(t) \sim \exp\left[-A \log^2(-a\, t)\right] \ , \tag{II.7}$$

which fits the data[3] at least as well as any other parameterization.

Although both the statistical and the dynamical assumptions underlying the model of Kastrup and Mack are doubtless oversimplified, the basic ideas seem rather attractive. Much more probably remains to be learned from statistical or bremsstrahlung models of scattering with large momentum transfer.

E. Field Theory

There are few field theoretic approaches which deal directly with the electromagnetic form factors. However, there have been several treatments of off-shell vertex functions and propagators which attempt to go beyond low-order perturbation theory in the asymptotic region. Since these relate closely to our work, we shall also discuss them here.

It is useful to distinguish between field theories on the basis of the regions in the Feynman integrations which are responsible for the leading asymptotic contributions. We say that the leading logarithms originate in the ultraviolet region of integration if the leading asymptotic logarithms resulting from Feynman integrals which are cut off (or regularized), and not renormalized, contain the ratio of the asymptotic variable $-q^2$ and the ultraviolet cutoff. Otherwise, the asymptotic logarithms will contain the ratio of $-q^2$ and nonasymptotic scalars, for example masses or infrared cutoffs. It is possible to show[7] that the pseudoscalar π-N theory has no infrared divergences as $m_\pi \to 0$, and no asymptotic logarithms arising from the infrared region.

Sudakov,[21] and more recently Cassandro and Cini[22] and Jackiw,[23] have considered a theory in which the leading asymptotic logarithms can arise from the infrared region of integration, namely spinor electrodynamics with a photon of mass $\mu \neq 0$. Assuming that the external momenta satisfy the inequalities

$$|q^2| \gg |p^2, |p'^2| \gg m^2 = (\text{fermion mass})^2 , \qquad (\text{II}.8)$$

Sudakov was able to show that in all orders of perturbation theory, the terms with the greatest number of large logarithmic factors are those in which all the virtual photon lines overlap the point of emission of the external photon q — i.e., ladder and crossed-ladder graphs (see Fig. 3). Furthermore, all ladders and crossed-ladders with the same number of rungs contribute equally. The sum of all leading logs of order e^{2n} was found by Sudakov to be

$$\Gamma_\mu^{(2n)} = \frac{\gamma_\mu}{n!} \left(- \frac{e^2}{8\pi^2} \log \left| \frac{q^2}{p^2} \right| \log \left| \frac{q^2}{p'^2} \right| \right)^n , \quad (e^2/4\pi = \alpha);$$

clearly, this is an exponential series, with the result

$$\Gamma_\mu = \gamma_\mu \exp\left(-\frac{e^2}{8\pi^2}\log\left|\frac{q^2}{p^2}\right|\log\left|\frac{q^2}{p'^2}\right|\right). \tag{II.9}$$

Cassandro and Cini[22] attempted to evaluate the sum of leading logarithmic terms for the physically more interesting kinematics

$$-q^2 \gg p^2 = p'^2 = m^2 \tag{II.10}$$

using the methods developed by Tiktopolous and others[8] for determining the leading asymptotic behavior of Feynman graphs. Jackiw[24] noted a counting error in the calculation of Cassandro and Cini, and by an independent analysis of the asymptotic behavior of the relevant graphs — which are the same as those analyzed by Sudakov — he was able to give arguments which make quite plausible the result

$$\Gamma_\mu = \gamma_\mu \exp\left(-\frac{e^2}{16\pi^2}\log^2\frac{-q^2}{\mu^2}\right) \tag{II.11}$$

for the asymptotic region (II.10). This form of asymptotic behavior seems to fit the data on electromagnetic form factors as well as any. Interestingly enough, (II.11) has the same form as Eq. (II.7), derived by Mack on the basis of what appears at first to be a rather different model. A common feature of both models, however, is their emphasis upon the infrared region of boson momenta. This is expressed in the massive QED model through the infrared dominance of the integrations, and in the Mack model by the bremsstrahlung-like treatment of the inelastic amplitude, which is related to the elastic amplitude through a statistical hypothesis.

In the period 1954-6, Landau and collaborators[9] sought to determine the asymptotic behavior of Feynman propagators and vertex functions in quantum electrodynamics and in pseudoscalar π-N theory (we will confine our attention to the π-N case). They were able to show that the approximate integral equation for the π-N vertex $\Gamma_5(q^2, p^2, p'^2)$ represented in Fig. 4, together with Dyson's equations for the pion and nucleon propagators, corresponds to the sum of the asymptotically leading logarithmic terms from all Feynman graphs for the vertex. It was then possible to solve this system of equations in the asymptotic region where the Lorentz invariant arguments of Γ_5, S, and D are all much larger than the square of the nucleon or pion mass. Essentially the same results were also obtained by Bogoliubov and collaborators[10] using the renormalization group. The expressions obtained are most easily written if we define a quantity

$$Q(k^2) = 1 - \frac{10\, g_0^2}{32\pi^2} \log\left(\frac{-k^2}{\Lambda^2}\right),$$

where g_0 is the unrenormalized strong coupling constant, and Λ^2 an ultraviolet cutoff point (or, in Bogoliubov's formulation, g_0 is an intermediate-renormalized coupling constant, and Λ^2 is the corresponding normalization point). Then, for the symmetric pseudoscalar πN theory,

$$\begin{aligned} \not{k}\, S(k^2) &\sim Q(k^2)^{-3/10} \\ k^2\, D(k^2) &\sim Q(k^2)^{-4/5} \\ \Gamma_5(q^2, p^2, p'^2) &\sim Q(k^2)^{1/5} \end{aligned} \tag{II.12}$$

where

$$|q^2|,\ |p^2|,\ |p'^2| \gg m^2,\ m_\pi^2, \tag{II.13}$$

and according to Landau *et al.*,[9] if $|q^2|$, $|p^2|$ and $|p'^2|$ are of the same order, or two of them are much greater than the third, then k^2 is equal to the greatest of the squares. These formulas clearly possess difficulties for $-k^2$ sufficiently large that $Q(k^2) \leq 0$. Using the Kallen-Lehmann representation to force the correct analytic behavior, as suggested by Redmond,[24, 25] allows removal of the ghost poles and cuts at $Q(k^2) = 0$; but it does not clarify the interpretation of Eq. (II.12) for large $-k^2$. Of course, the restriction (II.13) on the external momenta makes the physical significance of the results (II.12) somewhat uncertain in any case. Such a restriction to asymptotic momenta seems to be a feature of Landau's method and of treatments based on the renormalization group.

It will become clear that the restriction (II.13) is not necessary to sum the leading asymptotic terms over all orders. We shall be able to sum over all orders of renormalized perturbation theory for the case of the asymptotic form factor (two legs on mass shell). The result is (I.3) (similar to (II.12) except with Λ^2 replaced by the nucleon mass m^2 and g_0 replaced by the renormalized coupling constant g).

In all these cases with the γ_5 π-N theory, the leading contributions come from the ultraviolet integration region and in each case the sum of leading logs does not produce reasonable asymptotic behavior nor correct analytic behavior. In the case of massive QED, a kinematic region $\left((\mathrm{II}.8) \text{ or } (\mathrm{II}.10)\right)$ can be found where the leading terms are infrared and a reasonable asymptotic behavior (II.11) apparently results. Note that for the asymptotic propagators and the completely asymptotic vertex functions $\left(-p^2 = -p'^2 = -q^2 \gg \mu^2, m^2\right)$ in QED, the ultraviolet region dominates and the sum of leading terms again has unreasonable asymptotic behavior. We shall return to these points in Section VI.

III. LOW ORDER RESULTS

In this section we will present several first and second order results which should illustrate the general features of our iterative method of summing over all orders. A useful graphical notation will also be introduced.

We begin by considering the proton-γ vertex to first order in g^2, which we denote by $\Lambda_{\mu 1}(p,p')$. The two graphs are shown in Fig. 5 and the resulting amplitude is

$$\Lambda_{\mu 1}(p,p') = g^2 \int \frac{d^4k}{(2\pi)^4} \left\{ \gamma_5 \frac{i}{\not{p}' - \not{k} - m} \gamma_\mu \frac{i}{\not{p} - \not{k} - m} \gamma_5 \frac{i}{k^2 - m_\pi^2} \right. \tag{III.1}$$

$$\left. + 2\gamma_5 \frac{i}{\not{k} - m} \gamma_5 \frac{i}{(p'-k)^2 - m_\pi^2} (p'_\mu + p_\mu - 2k_\mu) \frac{i}{(p-k)^2 - m_\pi^2} \right\} .$$

Once the usual renormalization subtraction at $\not{p} = \not{p}' = m$, $q = 0$ is carried out, the leading contribution as $-q^2 \to \infty$ with the fermion legs on the mass shell can easily be extracted. The calculation of the second term (III.1), corresponding to the graph 5b, is carried out in Appendix A. It is found that apart from the isotopic factor of two, the two graphs give the same leading asymptotic contribution to the renormalized first order vertex function[26]

$$\tilde{\Lambda}_{\mu 1}(\not{p} = \not{p}' = m) = -3\gamma_\mu \, G \log(-q^2/m^2) \tag{III.2}$$

where

$$G = \frac{g^2}{32\pi^2} \approx 0.6 \quad . \tag{III.3}$$

Thus through first order, the Dirac form factor of the proton is

$$F_{1p}(q^2) = 1 - 3G \log(-q^2/m^2) + \dots \quad . \tag{III.4}$$

The calculation of Appendix A shows that the leading contribution (III.2) comes from the ultraviolet region of integration and that its form is independent of which legs are taken asymptotic. By this we mean that if any of its legs are taken asymptotic, then

$$\tilde{\Lambda}_{\mu 1}(p,p') \approx -3\gamma_\mu \, G \log(-k^2/m^2) \tag{III.5}$$

where $|k^2|$ is the largest of $|p^2|$, $|p'^2|$ and $|q^2|$. This form-invariance will be seen to hold in all orders and is a very important aspect of our iterative scheme for summing over all orders.

A similar calculation shows that the two graphs of Fig. 6 which contribute to the neutron factor in first order give leading asymptotic contributions which exactly cancel each other. Since there is no zeroth order "bare" coupling of the photon to the neutron, the neutron form factor remains zero through first order in our approximation of keeping only the leading asymptotic terms. Two other first order amplitudes which are calculated in the same way and which we will need later are the charged pion form factor F_π[27] and the pion-nucleon vertex Γ_5. The relevant graphs are shown in Fig. 7, and the results of extracting the leading asymptotic terms are

$$F_\pi(q^2) = 1 - 8G \log(-q^2/m^2) + \ldots \qquad \text{(III. 6)}$$

and

$$\Gamma_5(q^2, \not{p} = \not{p}' = m) = \gamma_5\left[1 - 2G \log(-q^2/m^2) + \ldots\right] . \qquad \text{(III. 7)}$$

It is found that the pion-γ vertex and the pion-nucleon vertex also have the same asymptotic form independently of which legs are taken asymptotic.

The iterative method which we develop here is based on the fact that, since the dominant contribution comes from the ultraviolet integration region, the asymptotic value of an n-th order vertex (or self-energy) graph can be determined in terms of the asymptotic value of its vertex and self-energy subgraphs. In order to illustrate this by examining the second order proton form factor graphs, we will need the asymptotic form of the first order pion and nucleon self-energy graphs shown in Fig. 8. Both mass and wave function renormalization subtractions must be made and the leading asymptotic contribution comes from the ultraviolet region of integration. In the asymptotic region the complete renormalized nucleon and pion self-energies take the form

$$\tilde{\Sigma}(p) \approx \not{p} \sum_{n=1}^{\infty} \Sigma_n G^n \log^n\left(\frac{-p^2}{m^2}\right) \qquad \text{(III. 8)}$$

and

$$\widetilde{\Pi}(p^2) \approx p^2 \sum_{n=1}^{\infty} \Pi_n G^n \log^2 \left(\frac{-p^2}{m^2}\right) \quad . \tag{III.9}$$

The first order (n=1) results are

$$\Sigma_1 = 3 \tag{III.10}$$

and

$$\Pi_1 = 8 \quad . \tag{III.11}$$

We next consider the second order (i.e., g^4) contributions to the proton-γ vertex and start by examining the uncrossed ladder graph of Fig. 9a where only π_0's are exchanged. The details of the calculation are given in Appendix B and we simply give the result and important points of the calculation here. The mass of the pion may again be set equal to zero and the dominant term comes from the ultraviolet integration region. Both the second order vertex subgraph and the graph as a whole are logarithmically divergent and must be subtracted. Each subtraction gives rise to one logarithm in the asymptotic region; and with the proton legs on the mass shell, the leading term is

$$\gamma_\mu G^2 \frac{1}{2!} \log^2 (-q^2/m^2) \quad .$$

This result along with (III.2) suggests that the ladder graphs for a theory with only π_0's and protons give rise to an exponential series. We shall confirm this to all orders in the next section. It is again found that the above form is independent of which legs are taken asymptotic. Thus, for example, suppose we take q = 0 and $-p^2 = -p'^2 \gg m^2$. Then the leading term is

$$\gamma_\mu G^2 \frac{1}{2!} \log^2 (-p^2/m^2) \quad .$$

Another second order graph is the crossed ladder of Fig. 9b. This graph contains only one overall logarithmic divergence and the single necessary subtraction leads to a single asymptotic logarithm which comes from the ultraviolet region of integration.

The general rule for determining the number of asymptotic logarithms can now be conjectured: the leading asymptotic behavior of a vertex or self-energy graph is $\log^n(-k^2/m^2)$ where n is the number of primitive divergences in the graph and thus the number of subtractions which must be made. ($|k^2|$ is the maximum of $|p^2|$, $|p'^2|$ and $|q^2|$.) This will be verified to all orders in the next sections.

In order to elucidate the general methods of the next section, we will complete the second order calculation of the proton-γ vertex and relate it to the first order vertex and self-energy results. The above rule immediately tells us that the leading $\log^n(-k^2/m^2)$ contributions to the asymptotic vertex in n-th order come only from uncrossed ladders and uncrossed ladders with vertex and self-energy corrections. All such second order proton-γ vertex graphs are listed in Fig. 10 using the graphical notation already introduced. The ladder graph which we have already calculated is contained in Fig. 10a.

The sum of leading terms in each order in the asymptotic proton form factor can be written in the form

$$F_{1p} \approx 1 + \sum_{n=1}^{\infty} p_n \, G^n \log^n(-q^2/m^2) \ , \qquad \text{(III.12)}$$

where, from (III.4), $p_1 = -3$. We have explicitly calculated the contribution of each of the graphs of Fig. 10 to p_2. Besides the ladder graph calculation, we also present in Appendix B the details of the calculation of the graph of Fig. 10d, which includes a fermion self-energy subgraph. The contribution of each of the graphs of Fig. 10 to p_2 is listed under the graph itself. Thus the complete second order coefficient is

$$p_2 = \frac{-21}{2!} \tag{III.13}$$

which verifies the result (I.3) through second order in $G = g^2/32\pi^2$.

We can now explain these results in a way which will make it clear that the leading terms in each order can be obtained iteratively. The discussion which follows will be supplemented below by a more careful treatment.

We consider first the contribution of Fig. 10d which is explicitly calculated in Appendix B. This graph is formed by inserting a second order self-energy subgraph into the second order vertex graph of Fig. 5a. The leading asymptotic contribution of this second order vertex graph can be reduced to the form

$$-G \int_1^{-q^2/m^2} \frac{dx}{x} = -G \log(-q^2/m^2) \quad . \tag{III.14}$$

Now since it is the ultraviolet integration region which dominates, we can determine the contribution of Fig. 10d by inserting the (spacelike) asymptotic value of the second order fermion self-energy into the left-hand side of (III.14). Using (III.10), this gives

$$-3\, G^2 \int_1^{-q^2/m^2} \frac{dx}{x} \log x = -3G^2 \, \frac{\log^2(-q^2/m^2)}{2} \quad , \tag{III.15}$$

which is just the contribution of Fig. 10d to the asymptotic Dirac form factor.

The contribution of each of the graphs of Fig. 10 can be related to the asymptotic contributions of second order graphs in the same way. Thus consider the graph of Fig. 10h. The second order vertex subgraph with one pion leg and one nucleon leg taken asymptotic has the same form as (III.7). It is $-2\gamma_5 G \log(-k^2/m^2)$. Inserting this into the second order graph gives

$$-2\, G \int_1^{-q^2/m^2} (-2G \log x) \frac{dx}{x} = 4G^2 \, \frac{\log^2(-q^2/m^2)}{2} \quad , \tag{III.16}$$

which is the contribution of Fig. 10h to the asymptotic Dirac form factor. We will make extensive use of these techniques for determining asymptotic behavior in treating the general case in the next section.

The calculation of the second order neutron-γ vertex proceeds in the same way. It is found that for this case the leading contributions from each order sum to zero. This cancellation can be understood by the following argument. The leading logarithmic term in the unrenormalized amplitude corresponding to some second order neutron-γ vertex graph g has the form

$$n_2^g \frac{G^2}{2!} \log^2 (\Lambda^2/-q^2) \quad ,$$

where n_2^g is some constant coefficient. After the necessary subtractions have been made, the leading asymptotic behavior of the renormalized amplitude will be

$$n_2^g \frac{G^2}{2!} \log^2 (-q^2/m^2) \quad .$$

Now since there is no bare neutron-γ coupling to be renormalized, we know that the leading logarithmically divergent terms in each order in the unrenormalized amplitudes must sum to zero and hence the leading asymptotic terms in the renormalized vertex also sum to zero.

An analysis similar to that for the nucleon-γ vertex gives the second order asymptotic pion-γ and nucleon-pion vertices. The sum of leading terms in the pion-γ vertex taking only the γ momentum asymptotic and keeping the pion legs on mass shell is

$$F_\pi(q^2) \approx 1 + \sum_{n=1}^{\infty} \pi_n G^n \log^n (-q^2/m^2) \qquad \text{(III.17)}$$

where

$$\pi_1 = -8, \qquad \pi_2 = \frac{-16}{2!} \qquad \text{(III.18)}$$

The sum of leading terms in the nucleon-pion vertex with only the pion leg asymptotic and the nucleons on mass shell is

$$\Gamma_5(q^2, \not{p} = \not{p}' = m) \approx \gamma_5 \left[1 + \sum_{n=1}^{\infty} v_n G^n \log^2(-q^2/m^2)\right] . \qquad \text{(III.19)}$$

The first and second order coefficients are

$$v_1 = -2 \qquad v_2 = \frac{-16}{2!} \qquad \text{(III.20)}$$

The second order asymptotic self-energies, which will be needed to calculate the third order vertex functions, can be calculated in the same way. However, a simpler procedure, which avoids dealing with the overlapping divergences in the self-energy graphs, is to relate the self-energies via the Ward identity to the proton-γ and pion-γ vertices. We will use this technique to all orders in G in Section V.

IV. SUMMATION OF LADDER GRAPHS

The second order results of the previous section have indicated that the exact leading contribution from Feynman graphs of any order can be determined iteratively. In this section, we will develop this method for uncrossed ladder graph contributions to the nucleon and pion electromagnetic form factors.

For simplicity we first consider a theory of protons and neutral pions so that the n-th order graph is given by Fig. 11. The leading term in each order will give a contribution to the nucleon electromagnetic form factors proportional to γ_μ. It has been shown in the previous section that through second order in G, with $-q^2 \gg m^2$ and $\not{p} = \not{p}' = m$, the ladder graphs give

$$\Gamma_\mu(q^2, \not{p} = \not{p}' = m) = \gamma_\mu \left[1 - G \log(-q^2/m^2) + \frac{1}{2!} G^2 \log^2(-q^2/m^2) + \ldots\right] . \qquad \text{(IV.1)}$$

This certainly suggests an exponential series, and in order to confirm this we examine the n-th order ladder graph. We denote the n-th order term in this series by $\Gamma_{\mu,n}(q^2)$. In Appendix C it will be shown that $\Gamma_{\mu,n}$ is given by

$$\Gamma_{\mu,n}(q^2) = \gamma_\mu (-1)^n G^n \frac{\log^n(-q^2/m^2)}{n!} \quad , \qquad \text{(IV.2)}$$

which confirms the exponential character of the series and shows that the sum of leading contributions from the ladder graphs for the π^o-p theory with $-q^2 \gg m^2$ is

$$F_{1(\text{ladder})}(q^2) = \exp\left[-G \log(-q^2/m^2)\right] = (-q^2/m^2)^{-G} \ . \qquad \text{(IV.3)}$$

This is not an unreasonable result since it says that the form factor drops off faster asymptotically as the coupling strength increases.

The above analysis is easily extended to the complete isospin-symmetric theory. The n-th order ladder graph contributions to the proton-γ, neutron-γ and pion-γ vertices are shown in Fig. 12 in terms of the (n-1)-th order ladder graphs. The first order (n=1) graphs in this set of graphical equations are identical to the first order graphs in the complete series. They are given in Figs. 5, 6 and 7. Apart from isotopic spin factors, the leading contribution from any n-th order ladder graph can again be shown to be of the form (IV.2) by an analysis similar to that of Appendix C. The only divergences are the n vertex subgraphs and each subtraction gives rise to one power of the log.

Another type of logarithmic divergence which might seem to enter into the ladder graphs is the "box" subgraph of Fig. 13a. The logarithmically divergent part of this subgraph is pure s-wave since it contains no momentum dependence. It corresponds to a graph of the type of Fig. 13b and hence must vanish since it couples to the photon which is spin one. Thus no special subtraction is necessary for this subgraph, and it gives rise to no additional asymptotic logarithms.

Carrying out the sum over all orders is therefore simply a matter of counting and keeping track of isotopic factors. We write the series in the form

$$F_{1P(\text{ladder})}(q^2) = \sum_{n=0}^{\infty} \hat{p}_n(-G)^n \frac{1}{n!} \log^n(-q^2/m^2)$$

$$F_{1N(\text{ladder})}(q^2) = \sum_{n=0}^{\infty} \hat{n}_n(-G)^n \frac{1}{n!} \log^n(-q^2/m^2) \tag{IV.4}$$

$$F_{\pi(\text{ladder})}(q^2) = \sum_{n=0}^{\infty} \hat{\pi}_n(-G)^n \frac{1}{n!} \log^n(-q^2/m^2)$$

where $\hat{p}_0 = \hat{n}_0 = \hat{\pi}_0 = 1$. The coefficients $\hat{p}_n$, $\hat{n}_n$, and $\hat{\pi}_n$ obey simple recursion relations which are exhibited graphically in Fig. 12. The relations are

$$\begin{aligned} \hat{p}_n &= \hat{p}_{n-1} + 2\,\hat{n}_{n-1} + 2\,\hat{\pi}_{n-1} \\ \hat{n}_n &= \hat{n}_{n-1} + 2\,\hat{p}_{n-1} - 2\,\hat{\pi}_{n-1} \\ \hat{\pi}_n &= 8\,\hat{p}_{n-1} - 8\,\hat{n}_{n-1} \quad . \end{aligned} \tag{IV.5}$$

These recursion relations can be solved quite easily. The result for the isoscalar combination $\hat{p}_n + \hat{n}_n$ is

$$\hat{p}_n + \hat{n}_n = 3^n \tag{IV.6}$$

which leads to the result (I.2) for the isoscalar form factor. The isovector combination is

$$F^V_{1(\text{ladder})}(q^2) = \sum_{n=0}^{\infty} t_n(-G)^n \frac{\log^n(-q^2/m^2)}{n!} \tag{IV.7}$$

where

$$t_n = \hat{p}_n - \hat{n}_n \quad . \tag{IV.8}$$

Then t_n satisfies the recursion relation

$$t_n = -t_{n-1} + 32\, t_{n-2} \tag{IV.9}$$

with

$$t_0 = 1, \qquad t_1 = 3 \quad . \tag{IV.10}$$

This recursion relation is easily solved by converting (IV.9) to a differential equation. The result is the sum of a growing and a falling exponential term. The dominant asymptotically growing term is

$$F^V_{1(\text{ladder})}(q^2) \sim e^{+\lambda G \log(-q^2/m^2)} = (-q^2/m^2)^{\lambda G} \quad , \tag{IV.11}$$

where $\lambda \approx 5.18$ is the positive root of

$$\lambda^2 + \lambda - 32 = 0 \quad . \tag{IV.12}$$

The recursion relation and the solution for the pion form factor are essentially the same. The fact that the isovector and pion form factors do not fall asymptotically in the ladder approximation makes the significance of the physically more reasonable form (I.2) for the isoscalar form factor quite doubtful. In the next section we will show that the sum of <u>all</u> leading contributions to the form factor in every order of perturbation theory for the pseudoscalar π-N theory also behaves very unphysically.

V. THE GENERAL ITERATION SCHEME

The iterative scheme for performing the complete summation involves the fermion and pion self-energies. In order to avoid calculating these functions, which requires dealing with overlapping divergences, we will relate them to the proton-γ and pion-γ vertices via the Ward identity.

A useful form of the Ward identity is

$$\widetilde{\Lambda}_\mu(p,p) = -\frac{\partial}{\partial p^\mu}\, \widetilde{\Sigma}(p) \quad . \tag{V.1}$$

If we consider only the leading asymptotic ($-p^2 \gg m^2$) term in each order of perturbation theory, we can use expression (III.8) in the right-hand side of (V.1). The asymptotic form of the left-hand side of (V.1) is the same as that of the form factor

$$\Lambda_\mu(p,p) \approx \gamma_\mu \sum_{n=1}^{\infty} p_n \, G^n \log^n(-p^2/m^2) \tag{V.2}$$

where the p_n are the same coefficients that appear in (III.12). This form invariance (of the leading term in each order) under taking various legs asymptotic holds for each graph which contributes a leading logarithm. We show this explicitly for the two second order graphs which are calculated in Appendix B and the general case is treated in the same way. The important point is that in our formalism, the terms which give a leading asymptotic logarithm contain momentum dependence only in the logarithmic factor in the parametric integral. This factor arises from the overall subtraction by use of (A.13).

Inserting (III.8) and (V.2) into (V.1) gives

$$\Sigma_n = -p_n \tag{V.3}$$

so that the proton self-energy to a given order is known as soon as the proton-γ vertex (with any set of legs asymptotic) is known to that order. By isotopic spin symmetry, the neutron self-energy has the same form and the same set of expansion coefficients Σ_n. A similar analysis for the pion self-energy and pion-γ vertex shows that

$$\Pi_n = -\pi_n \quad . \tag{V.4}$$

The sum over all orders of perturbation theory of <u>all</u> leading asymptotic contributions is based on the same ideas as the sum over ladder graphs which was done in the preceeding section. As we explained in Section III, leading asymptotic terms come only from uncrossed ladder graphs with leading self-energy and πNN vertex insertions. Leading vertex insertions are again ladder graphs with leading vertex and self-energy insertions. Leading self-energy insertions are obtained via Eq. (V.3) and (V.4) from leading form factor expressions. Thus we can derive an iteration scheme analogous to (IV.5). It is illustrated in Figs. 14 and 15.

Before describing this iteration scheme in more detail, it will be helpful to summarize our notation. The sums of all leading asymptotic terms in each order for the various vertices are written as follows ($z \equiv \log(-q^2/m^2)$, q being the four-momentum of the leg of the vertex graph with the largest spacelike four-momentum):

$$
\begin{aligned}
F_{1P}(q^2) &\equiv p(z) = \sum_{n=0}^{\infty} p_n (Gz)^n \\
F_{1N}(q^2) &\equiv n(z) = \sum_{n=0}^{\infty} n_n (Gz)^n \\
F_{\pi}(q^2) &\equiv \pi(z) = \sum_{n=0}^{\infty} \pi_n (Gz)^n \\
F_{\pi NN}(q^2) &\equiv v(z) = \sum_{n=0}^{\infty} v_n (Gz)^n \quad .
\end{aligned}
\tag{V.5}
$$

It will also be useful to have a notation for the sum of the leading logarithmic terms in each order of perturbation theory in the renormalized fermion and boson propagators:

$$
\begin{aligned}
-i\not{p}\widetilde{S}'_F(p) &\approx \sum_{n=0}^{\infty} s_n\, G^n \log^n(-p^2/m^2) \equiv s(\log(-p^2/m^2)) \\
-ip^2\widetilde{D}'_F(p^2) &\approx \sum_{n=0}^{\infty} d_n\, G^n \log^n(-p^2/m^2) \equiv d(\log(-p^2/m^2)) \quad .
\end{aligned}
\tag{V.6}
$$

All of the coefficients with subscript zero (p_0, n_0, ...) equal unity.

Finally, we will find it desirable to have a notation for the "rung functions," $\overline{sds}_n$, $\overline{sss}_n$, and $\overline{dsd}_n$. These are illustrated in Fig. 14, and defined by the following equations

$$\overline{sds}_n = -\sum_{\sum_1^5 n_i = n-1} s_{n_1} v_{n_2} d_{n_3} v_{n_4} s_{n_5}$$

$$\overline{sss}_n = -8\sum_{\sum_1^5 n_i = n-1} s_{n_1} v_{n_2} s_{n_3} v_{n_4} s_{n_5} \tag{V.7}$$

$$\overline{dsd}_n = -\sum_{\sum_1^5 n_i = n-1} d_{n_1} v_{n_2} s_{n_3} v_{n_4} d_{n_5} \quad .$$

These quantities describe the effect of adding to a ladder graph another rung, including self-energy and vertex insertions. The factors in front of the summation signs are required by the spinor and isospin algebra.

A proton form factor graph which contributes a leading $\log^n$ term in order G^n can evidently be obtained by attaching a rung with no self energy or vertex insertions to a proton form factor graph giving a leading $G^{n-1} \log^{n-1}$ term, by attaching a rung with a single-leading-log vertex or self-energy insertion to a leading $G^{n-2} \log^{n-2}$ proton form factor graph, and so on. Taking into account also the possibility that in the proton form factor graph the photon may have hit a neutron or pion line, we obtain the following expression for the coefficient p_n in (V.5)

$$p_n = \frac{1}{n} \sum_{i=1} \left(\overline{sds}_i \, p_{n-i} + 2\, \overline{sds}_i \, n_{n-i} + 2\, \overline{dsd}_i \, \pi_{n-i} \right) \quad . \tag{V.8a}$$

The factor of 2 in the neutron contribution is the usual isospin factor. The overall factor of 1/n arises from the integration over the loop created by adding an extra rung:

$$\int_1^{-q^2/m^2} \frac{\log^{n-1} x}{x}\, dx = \frac{1}{n} \log^n(-q^2/m^2) .$$

The following expressions are obtained in the same way:

$$n_n = \frac{1}{n} \sum_{i=1}^{\infty} (2\, \overline{sds}_i\, p_{n-i} + \overline{sds}_i\, n_{n-i} - 2\, \overline{dsd}_i\, \pi_{n-i}) \tag{V.8b}$$

$$\pi_n = \frac{1}{n} \sum_{i=1}^{\infty} (\overline{sss}_i\, p_{n-i} - \overline{sss}_i\, n_{n-i}) \tag{V.8c}$$

$$v_n = \frac{2}{n} \sum_{i=1}^{\infty} \overline{sds}_i\, v_{n-i} \quad . \tag{V.9}$$

These formulas are the generalizations of Eqs. (IV.5). The factor of 2 in (V.9) arises from the replacement of the γ_μ coupling to the photon by the γ_5 coupling to the pion in the $F_{\pi NN}$ graph.

Finally, using (V.3) and (V.4), which express the simple relationship between the proton and pion self energy and electromagnetic vertex leading asymptotic expansion coefficients, we can derive simple expressions for the propagators. We insert the expansions (III.8) and (V.6) into the identity

$$\widetilde{S}'_F = \frac{i}{\not{p} - m - \widetilde{\Sigma}} = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} (-i\widetilde{\Sigma})\, \widetilde{S}'_F$$

and find $s_0 = 1$, and, for $n \geq 1$,

$$s_n = \sum_{i=1}^{n} \Sigma_i\, s_{n-i} = - \sum_{i=1}^{n} p_i\, s_{n-i} \quad . \tag{V.10a}$$

The result for the pion is entirely analogous:

$$d_0 = 1,$$

$$d_n = - \sum_{i=1}^{n} \pi_i \, d_{n-i} , \qquad n \geq 1 . \tag{V.10b}$$

We now have a complete set of equations which, starting with the first-order results obtained in Section III, iteratively determine all of the coefficients p_n, n_n, π_n, etc. Either by hand or more easily by the use of a digital computer[28] one can determine enough of these coefficients to see the general pattern and determine the sum of the series expressions (III.8), (III.9), and (V.5) for the self energies and form factors.

Another method of obtaining these results, one which is hopefully more enlightening as well as more rigorous, will be presented here. It consists in converting the iteration equations written above into a set of integral and functional equations, which are then readily solved.

Let us first note that by explicit calculation $n_0 = n_1 = n_2 = 0$, and that a simple physical argument (given below Eq. (III.16)) — as well as straightforward application of the iteration equations or of the method to be explained shortly — leads to the conclusion that $n_i = 0$ for all i. If we use this fact then we can replace Eqs. (V.8) by the simpler set of equations

$$p_n = \frac{3}{n} \sum_{i=1}^{n} \overline{sds}_i \, p_{n-i} \tag{V.8'a}$$

$$\pi_n = \frac{1}{n} \sum_{i=1}^{n} \overline{sss}_i \, p_{n-i} \; ; \tag{V.8'b}$$

then we no longer require $\overline{dsd}$, either.

We will discuss first the self-energy Eqs. (V.10). It is easy to verify that these iteration equations are equivalent to the equations

$$
\begin{aligned}
s(z) &= \left[1 - \sum_{n=1}^{\infty} \Sigma_n \, (Gz)^n\right]^{-1} = p(z)^{-1} \\
d(z) &= \left[1 - \sum_{n=1}^{\infty} \Pi_n \, (Gz)^n\right]^{-1} = \pi(z)^{-1} .
\end{aligned}
\tag{V.11}
$$

These equations are but a succinct way of stating the result of extending the Ward identity by using the form-invariance of the leading logarithmic contributions to the form factors when different legs or combinations of legs are asymptotic.

The reduction of Eqs. (V.7) to a set of functional equations is again trivial. Defining

$$
\begin{aligned}
\psi(z) &= - \sum_{n=1}^{\infty} \overline{sds}_n \, (Gz)^{n-1} \\
\theta(z) &= - \sum_{n=1}^{\infty} \overline{sss}_n \, (Gz)^{n-1}
\end{aligned}
\tag{V.12}
$$

we have

$$
\begin{aligned}
\psi(z) &= s(z)^2 \, d(z) \, v(z)^2 \\
\theta(z) &= 8 \, s(z)^3 \, v(z)^2 \quad .
\end{aligned}
\tag{V.13}
$$

We finally consider the Eqs. (V.8') and (V.9) for the vertex functions. In (V.8'a) and (V.9) we have expressed the power series coefficients p_n and v_n in terms of $\overline{sds}_n$; our aim is to express the functions p(z) and v(z) in terms of $\psi(z)$. It is easy to verify that the solutions are

$$
\begin{aligned}
p(z) &= \exp\left(-3 \, G \int_0^z \psi \, dz\right) \\
v(z) &= \exp\left(-2 \, G \int_0^z \psi \, dz\right) .
\end{aligned}
\tag{V.14}
$$

Similarly, Eq. (V.8'b) expresses $\pi(z)$ in terms of $\theta(z)$ and $p(z)$:

$$\pi(z) = 1 - G \int_0^z \theta(z)\, p(z)\, dz \quad . \tag{V.15}$$

This completes our set of functional equations. Note that these equations express directly the integral equations implied by Figs. 14 and 15. For example, the integration

$$\int_0^z \psi(z)\, dz = \int_1^{-q^2/m^2} \frac{\psi(\log x)}{x}\, dx$$

adds another "fully dressed" rung to the ladder vertex graph.

In order to solve these equations, we eliminate all the other functions in favor of ψ, thereby obtaining an equation for ψ:

$$\psi = s(z)^2\ v(z)^2\ d(z)$$

$$= \left[\exp\left(-3\,G \int_0^z \psi\, dz\right)\right]^{-2} \left[\exp\left(-2\,G \int_0^z \psi\, dz\right)\right]^{2} \left[1 - G \int_0^z \theta\, p\, dz\right]^{-1} .$$

Noting that the logarithmic derivative of each factor is a function only of ψ, the last factor giving

$$\frac{1}{d(z)}\ \frac{d\,d(z)}{dz} = d(z)\ \frac{d}{dz}\left(G \int_0^z 8\ s^3\ v^2\ \frac{1}{s}\, dz\right) = 8\ G\ s^2\ v^2\ d = 8\ G\psi\ ,$$

we find that

$$\frac{1}{\psi}\ \frac{d\psi}{dz} = 10\ G\psi\ . \tag{V.16}$$

Since the first terms in the power series for s, d, and v are all unity, $\psi(0) = 1$; consequently we can integrate (V.16) to obtain the remarkably simple form

$$\psi(z) = \frac{1}{1 - 10\ Gz}\ . \tag{V.17}$$

The other functions are now readily obtained. The sums of all leading logarithmic contributions to the form factors are

$$
\begin{aligned}
F_{1P}(q^2) &\sim \left[1 - 10\, G \log(-q^2/m^2)\right]^{3/10} \\
F_{\pi}(q^2) &\sim \left[1 - 10\, G \log(-q^2/m^2)\right]^{8/10} \\
F_{\pi NN}(q^2) &\sim \left[1 - 10\, G \log(-q^2/m^2)\right]^{1/5} \quad ,
\end{aligned}
\tag{V.18}
$$

and for the propagators

$$
\begin{aligned}
\widetilde{S}'_F(p) &\sim \frac{i}{\not p} \left[1 - 10\, G \log(-p^2/m^2)\right]^{-3/10} \\
\widetilde{D}'_F(p^2) &\sim \frac{i}{p^2} \left[1 - 10\, G \log(-p^2/m^2)\right]^{-8/10}
\end{aligned}
\tag{V.19}
$$

As we have already pointed out in Section II.E, these results are similar to certain results obtained by use of the renormalization group.[9, 10] In particular, they unfortunately share their deficiencies, including a logarithmic "ghost cut" for large $q^2 < 0$ and unphysical asymptotic behavior.

VI. DISCUSSION

Our investigation has shown that if reasonable asymptotic behavior is to be obtained by summing logarithmically growing terms in each order of perturbation theory, either we must look for a different field theory for which the sum of leading terms in each order does give a falling form factor (possibly of the form (II.17)) or, if we persist in investigating the γ_5 π-N theory, we must face the difficult task of including in the sum non-leading logarithms in each order. A discussion of both of these possibilities is facilitated by first reviewing the connection between our summation techniques and the renormalization group methods discussed in Section II.E.

In applying the renormalization group approach to determine the asymptotic behavior of vertex functions, one is restricted to the completely asymptotic region $(-p^2 = -p'^2 = -q^2 \gg m^2)$ in considering $\Gamma_{\pi NN}$, and, as we have shown using the Ward identity, the determination of the asymptotic behavior of electromagnetic vertex functions is restricted to the region $q=0$, $-p^2 = -p'^2 = m^2$. Our results show that in the γ_5 theory, the renormalization group result for the leading asymptotic logarithms holds also in the form factor region $\not{p} = \not{p}' = m$, $-q^2 \gg m^2$.

A. Summing Non-Leading Terms

For the case of the completely asymptotic vertex or asymptotic propagator, renormalization group methods can be employed to determine the sum of next-to-leading logarithms. It turns out that the unphysical results obtained by summing the leading terms persist when non-leading terms are included.

For example, suppose we consider the asymptotic photon propagator in QED. By renormalization group techniques,[10] it can be shown that the sum of leading logs gives

$$q^2\widetilde{D}'_F(q^2) = \left(1 - \frac{\alpha}{3\pi} \log -q^2/m^2\right)^{-1} \tag{VI.1}$$

which has a logarithmic ghost pole for large spacelike q^2. When next-to-leading logarithms are included, one finds[10]

$$q^2\widetilde{D}'_F(q^2) = \left[1 - \frac{\alpha}{3\pi} \log \frac{-q^2}{m^2} + \frac{3\alpha}{4\pi} \log \left(1 - \frac{\alpha}{3\pi} \log \frac{-q^2}{m^2}\right)\right]^{-1} \tag{VI.2}$$

which again has an analytic behavior inconsistent with the Källen-Lehman representation of the propagator. We have found that similar results are obtained if all logarithms up to the n^{th}-to-leading order are summed, for any finite n.

One finds similar although slightly more complicated results for the γ_5 π-N theory. In this case, the function ψ (cf. Eq. (V.13)) is the "invariant charge"[10] of the theory, corresponding to the numerator of the photon propagator in QED, and the sum of next-to-leading logarithmic contributions to ψ closely resembles (VI.2). The vertex function and propagators including next-to-leading logs are then obtained from ψ by a quadrature.[10] Their behavior is no more reasonable than that of the leading-log results (V.18) and (V.19). Again, inclusion of lower-order logarithmic contributions gives no indication of improving matters.

The above conclusions about the effect of including non-leading contributions apply to the completely asymptotic region which is accessible by renormalization group methods. We must next inquire whether these unpleasant results persist in the form factor region, with the legs of the vertex functions taken on the mass shell. We have begun an investigation of this question by calculating the singly logarithmic term from several second order graphs. In each case we find that the coefficient of $G^2 \log(-q^2/m^2)$ in the Dirac form factor is different from the coefficient of $G^2 \log(-p^2/m^2)$ in the completely asymptotic vertex. However, in each case these terms come from the ultraviolet region of integration and we expect that the next-to-leading logs in the form factor will exhibit the same sort of behavior as the next-to-leading logs in the completely asymptotic vertex.

We have considered in some detail the next-to-leading logarithmic contributions to the sum of ladder graphs discussed in Section IV. Although the sum of ladder graphs is not particularly significant, it is combinatorically simpler than the sum of all graphs; and we hope that our discussion of this case may be illustrative. For simplicity, we avoid the complications of isospin and consider the π^0-p theory, for which the sum of leading logarithms is given by (IV.2). The general expression, including non-leading terms, has the form ($z \equiv \log -q^2/m^2$)

$$F_{1(\text{ladder})}(q^2) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \ell_{nm}\, G^n z^m + O(1/q) \tag{VI.2}$$

$$= L_1(z) + L_2(z) + \ldots$$

where, using the results of Section IV,

$$L_1 = \sum_{n=0}^{\infty} \ell_{nn}\, G^n z^n \ , \qquad \ell_{nn} = (-1)^n/n! \ ,$$

$$L_2 = \sum_{n=1}^{\infty} \ell_{n,n-1}\, G^n z^{n-1} \ , \tag{VI.3}$$

etc.

It is simplest to consider first the completely asymptotic vertex $F_{1(\text{ladder})}(q^2, p^2, p'^2)$ (the same remarks will also apply to the "Ward-identity vertex" $F_{1(\text{ladder})}(0, p^2, p^2)$), since the iteration method requires knowledge of the n-rung vertex function with the fermion legs asymptotic in determining the (n+1)-rung vertex function. We let

$$F_{1(\text{ladder})}(q^2, p^2, p'^2) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \ell'_{nm}\, G^n z^m = L_1'(z) + L_2'(z) + \ldots,$$

where here $z = \log -k^2/m^2$, $|k^2| = \max(|p^2|, |p'^2|, |q^2|)$. Note that the "form-invariance" we have stressed previously applies only to leading terms, unfortunately; thus $\ell'_{nn} = \ell_{nn}$, but $\ell'_{n\,n-1} \neq \ell_{n\,n-1}$.

It is necessary to calculate a new quantity before beginning the iteration, the next-to-leading logarithmic contribution from the graph of Fig. 9a, which is denoted by ℓ'_{21}. It turns out that all the terms in L_2' but the first come from inserting leading-log rungs either above or below the rungs which give this next-to-leading two-rung result. Since there are thus two ways of constructing the three-rung graph (either the top two or the bottom two rungs could have corresponded to the original two-rung next-to-leading term), three ways for the four-rung graph, and so on, the result is

$$
\begin{aligned}
L_2' &= \ell_{10}' G + \ell_{21}' G \sum_{n=1}^{\infty} n\, (\ell_{11})^{n-1} G^n \frac{z^n}{n!} \\
&= \ell_{10}' G + \ell_{21}' G^2 \left(\log \frac{-k^2}{m^2}\right) \left(\frac{-k^2}{m^2}\right)^{-G} \quad .
\end{aligned}
\tag{VI.4}
$$

We have verified the arguments which led to (VI.4) by a calculation similar to that given in Appendix C. Entirely analogous reasoning leads to the following expression for the sum of next-to-leading contributions to the form factor:

$$
\begin{aligned}
L_2 &= \ell_{10} G + \ell_{11}\, \ell_{21} G^2 z + (\ell_{11})^2 (\ell_{21}' + \ell_{21})\, G^3 \frac{z^2}{2!} + (\ell_{11})^3 (\ell_{21}' + 2\ell_{21})\, G^4 \frac{z^3}{3!} + \ldots \\
&= \ell_{10} G + (\ell_{21} - \ell_{21}')\, G \left[\left(\frac{-q^2}{m^2}\right)^{-G} - 1\right] + \ell_{21}' G^2 \left(\log \frac{-q^2}{m^2}\right) \left(\frac{-q^2}{m^2}\right)^{-G} \quad .
\end{aligned}
\tag{VI.5}
$$

The results (VI.4) and (VI.5) cast further doubt on the significance of the leading logarithm expressions (I.2) or (IV.3), and give us no reason for optimism regarding the improvement of the results of Section V by the inclusion of non-leading terms. The sums of next-to-leading terms (VI.4) and (VI.5) are larger by a factor of $\log -q^2/m^2$ than the sums of leading terms, and in addition they contain constant terms which survive in the limit $-q^2 \to \infty$. (It turns out that $\ell_{10} = 0$; however, the difference $\ell_{21} - \ell_{21}'$ does not vanish.)

B. Other Field Theories

Any field theory with pseudoscalar coupling will exhibit the same type of asymptotic behavior as the model considered in this paper. Thus, enlarging the algebraic structure to SU(3), for example, will not change any of our conclusions.

The same is true of models which include scalar particles coupled to the fermions such as the linear σ-model. The logarithmically increasing terms in such theories come from the ultraviolet integration region and give rise to essentially the same type of asymptotic behavior as in the purely pseudoscalar theory.

We have pointed out in Section II that there does exist a renormalizable field theory, namely spinor electrodynamics with a massive photon, for which the leading asymptotic logarithms in the form factor come from the infrared integration region and for which these logarithms sum to the not unreasonable result (II.17). It is our contention, however, that this is very misleading. What is being done in summing these dominant infrared logarithms is to take from this field theory its low energy or statistical properties while ignoring the high energy or short distance properties of the theory. When there are no infrared logarithms present, the dominant ultraviolet logarithms come from the terms in the parametric integrand containing only X_{ij} numerator factors. However, even when the infrared logarithms, which come from terms containing Y_i numerator factors, are dominant, the pure X_{ij} terms are still present and thus the ultraviolet logarithms are still present. Our formalism can then be applied to sum these ultraviolet logs. Even when the infrared logarithms dominate in each order of perturbation theory, the ultraviolet logarithms are expected to sum to give the unreasonable type of asymptotic behavior found in the pseudoscalar meson theory.

C. Conclusions

As we have seen, both experimental evidence and theoretical arguments seem to militate against the sorts of results obtained by summing asymptotically leading logarithms arising from the ultraviolet region in Feynman integrals. We emphasize that this conclusion applies not only to the ultraviolet-dominated γ_5

π-N theory to which we have devoted most of our attention in the above discussion, but also to QED, to the sigma model, and to any other field theory containing ultraviolet divergences. Although our calculations have been confined to the consideration of the asymptotic behavior of propagators and vertex functions, we feel it to be likely that similar strictures apply to those calculations of any scattering amplitude which involve summation of leading logs in perturbation theory.

Confronted with this situation, there appear to be four possible alternatives:

1. Consider a field theory with no ultraviolet divergences. Unfortunately, the only example of such a super-renormalizable field theory in Minkowski four-space, the theory of a scalar ϕ field self-coupled with a ϕ^3 interaction, has certain formal deficiencies. However, even apart from these deficiencies and from the fact that the omission of fermions in this theory prevents it from even approximating a description of reality, there is reason to suppose that the ϕ^3 theory cannot explain the observed rapid asymptotic decrease of the elastic form factor. This plausibility argument is based on a theorem of Weinberg (cf. Section 19.14 of Bjorken and Drell[6]), which relates the asymptotic behavior of a graph to the divergence of its internal loops; specifically, if G is a fully asymptotic vertex graph,

$$\Gamma^G \sim (q^2)^{\alpha_G} (\log |q^2|)^{\beta_G} \tag{VI. 6}$$

where α_G is the largest degree of divergence of various sets of connected lines in G — for the ϕ^3 theory $\alpha_G \leq -1$ — and β_G is determined by more complicated considerations.[7,8] Now consider the graphs contributing to the elastic form factor. The zeroth order graph (with no loops) contributes a constant, which we can regard as being unity; the loops which are part of all higher-order graphs are all convergent (except for the self-energy insertions, which are taken care of by mass renormalization). Thus the Feynman graph expansion for the form factor corresponds to a series, every term of which but the first is asymptotically rapidly falling. Barring a remarkable cancellation, it seems most likely that the series will sum to a constant as $q^2 \rightarrow \infty$.

2. Attempt to modify the large-momentum behavior of conventional field theory. This can probably not be true without tampering with the locality properties of the theory, and there seems to exist no entirely satisfactory way of doing this. Nevertheless, this approach probably deserves intensive investigation. An interesting effort along these lines is the recent treatment by Drell, Levy, and Yan[2] of deep inelastic electron-proton scattering.

3. Abandon field theory. We have discussed in Section II several non-field-theoretic attempts to account for the asymptotic behavior of the form factor. Composite models probably have a kernel of truth in them, but seem rather ad hoc and model-dependent. The self-consistent, or bootstrap, model of Harte,[18] and the statistical model of Mack[19,20] are both guided in part by field theoretic ideas, but lack the serious deficiencies which appear to be associated with the ultraviolet region in field theory. (Loop integrals in Harte's model are dominated by the low-p^2 region on account of his ansatz (II.3a).) Thus we must regard such approaches as promising.

An unfortunate correlative of the adoption of the previous two approaches is the abandonment of the long-standing hope that the measurement of the form factor at large momentum transfer will enable us to probe the very fine scale structure of the nucleon. This hope is not quashed by the final approach we would like to suggest.

4. Abandon the attempt to sum leading terms in perturbation theory, or abandon perturbation theory altogether. It is possible that by summing <u>all</u> of the terms in Eq. (VI.2), for example, and not just leading or next-to-leading

logarithms, we would have obtained satisfactory results. However, our examination of the structure of such sums makes us regard this as an unlikely possibility. If field theory really can say anything about asymptotic behavior, we feel it is more likely that it will do so only through the use of some technique which avoids the perturbation expansion. Redmond,[24] Bogoliubov,[25] and others have given interesting arguments that the S matrix has an essential singularity when the coupling constant vanishes, and that the perturbation series is at best an asymptotic expansion. We would like to adopt this last explanation for our difficulties with perturbation sums. The difficulty, of course, is that in field theory no calculational alternative to the perturbation expansion as yet exists.

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APPENDIX A

PARAMETRIC INTEGRAL NOTATION

All of the perturbation theory amplitudes in this paper are calculated by expressing them directly in terms of integrals over Feynman parameters, using methods developed by Nakanishi[30] and others. A method of performing renormalization subtractions[31] using the parametric integral formalism will also be employed. These techniques are considerably simpler than the straightforward application of Feynman rules.[6] They greatly facilitate the subtraction procedure — particularly for graphs with multiple internal renormalization — and the identification and extraction of asymptotically leading terms.

In this Appendix we give the parametric integral form of the amplitude $W_I^{(G)}$ corresponding to the arbitrary Feynman graph G. The subscript "I" signifies that the expression given is "intermediate renormalized" in that any internal subtractions required are performed with the Lorentz scalars (squared four-momenta) characterizing propagators and the triplet of scalars characterizing vertex functions all taken equal to zero. It will become apparent when various first and second order graphs are computed that the choice of renormalization point is irrelevant in determining the leading asymptotic behavior.

It is necessary to assign a direction to all internal lines of the general graph G so that electrical, baryonic, etc., charge flows continuously through the graph. The internal lines are numbered in any convenient fashion. The amplitude, including all factors of i associated with propagators and $(2\pi)^{-4}$ associated with loop integrals, then has the form

$$W_I^{(G)} = \left(\frac{1}{16\pi^2 i}\right)^n \int_0^\infty dx_1 \ldots dx_N \, \mathcal{S}\mathcal{N} \frac{1}{U^2} e^{iE} \tag{A.1}$$

where

n = number of loops in G

N = number of internal lines in G

and $\mathscr{A}$ is a renormalization operator which will be described in detail below, U and E are parametric functions, and the operator $\mathscr{N}$ is a product of the usual numerator factors $(\hat{\not{p}}_i + m)$ for fermion propagators, $-g_{\mu\nu}$ for photon propagators, $-ie\gamma_\mu$ for photon-fermion vertices, $-ie(\hat{p}_i + \hat{p}_j)_\mu$ for photon-boson vertices, and $g\,\tau_\alpha\,\gamma_5$ for fermion-meson vertices, where the operators $\hat{p}_j$ involve differentiation with respect to auxilliary Lorentz four-vectors ℓ_j (which are then set equal to zero):

$$\hat{p}_j = \frac{1}{ix_j}\,\partial_{\ell_j} \quad . \tag{A.2}$$

Let us first suppose that the graph G does not require any internal subtractions. Then $\mathscr{A}$ can be set equal to unity, and the parametric functions have the form

$$U = \sum_{\{\nu_1, \ldots, \nu_n\}} x_{\nu_1} \cdots x_{\nu_n} \quad , \tag{A.3}$$

where the braces $\{\ldots\}$ indicate that the sum is over all sets $\nu_1, \ldots, \nu_n$ such that $p_{\nu_1}, \ldots, p_{\nu_n}$ is a set of independent integration momenta, and

$$E = V + \sum_{i\in G} \hat{x}_i \ell_i \cdot Y_i - \frac{1}{4} \sum_{i,j\in G} \hat{x}_i \hat{x}_j \ell_i \cdot \ell_j\, X_{ij} - \sum_{i\in G} x_i (m_i^2 - i\epsilon) \ , \tag{A.4}$$

where, as long as no subtractions are required, we can take $\hat{x}_i = x_i$ for all i. Thus E involves several more parametric functions, V, Y_i, and X_{ij}. The first of these, the only one which need be evaluated if all the lines of G are spinless, has the form

$$V = \frac{1}{U} \sum_{r>s} W^{rs}(-k_r \cdot k_s) \quad , \tag{A.5}$$

where all the external momenta $k_r (r = 1, \ldots, \ell)$ of the graph G are regarded as directed inward,[32] and W^{rs} is the U function of the vacuum graph G^{rs} formed from G by connecting the legs r and s and ignoring all other legs, with the direction of this new loop (labeled 0 in Fig. 16) chosen so as to enter the subgraph G with leg r, and with the parameter x_0 set equal to zero at the end

$$W^{rs} = \left[U_{G^{rs}}\right]_{x_0=0} \quad . \tag{A.6}$$

As an example, consider the first-order vertex graph drawn in Fig. 17.[33] There are three independent integration momenta, so

$$U = x_1 + x_2 + x_3 \ ;$$

referring to Fig. 17b, we calculate

$$W^{pp'} = \left[x_3(x_1 + x_2 + x_0) + x_0(x_1 + x_2)\right]_{x_0=0} = x_3(x_1 + x_2) \ .$$

For a form-factor graph, such as Fig. 17 with $p^2 = p'^2 = m^2$, it is convenient to express V in the form

$$V = \frac{1}{U}(q^2 W^{q^2} + m^2 W^{pp'}) \tag{A.7}$$

where

$$W^{q^2} = \frac{1}{2}(W^{qp'} + W^{qp} - W^{pp'}) \ .$$

Nakanishi[30] gave a method for evaluating W^{q^2} directly:

$$W^{q^2} = \sum_k W^{C(AB)} \left(P_k(AB)\right) \tag{A.8}$$

where the sum is over all paths P_k connecting the nodes A and B (where p and -p' enter the graph) but not passing through node C (where q enters the graph), and

$W^{C(AB)}(P_k)$ is the W function for the graph obtained by shrinking all the lines along the path P_k to a point, and connecting that point to C (see Fig. 18). For the first-order vertex graph of Fig. 17, the resulting diagram is Fig. 17c, and

$$W^{q^2} = x_1 x_2 \ .$$

In general, it will be necessary to evaluate some of the Y_i and X_{ij} functions. Let us first define the parametric functions U_C and W_C^{rs}, where C is a closed circuit of lines in G:

$$U_C = \sum_{\substack{\{\nu_1, \ldots, \nu_{n-1}\} \\ \text{with no } \nu_i \in C}} x_{\nu_1} \cdots x_{\nu_{n-1}} \tag{A.9}$$

where the sum is over all sets $\nu_1 \cdots \nu_{n-1}$ corresponding to independent integration momenta, such that none of the ν_i in the set belongs to the set of lines comprising the circuit C; and, corresponding to (A.6),

$$W_C^{rs} = \left[(U_C)_{Grs}\right]_{x_0=0} \ . \tag{A.10}$$

Then

$$X_{ij} = \frac{1}{U} \sum_{C \in C(i,j)} s_C U_C \ ,$$

where the sum runs over all circuits C in G which include the lines i and j, and s_C is a sign factor which is positive if the directions of i and j are parallel along circuit C, and negative if they are antiparallel. (These s_C factors are the only use we make of the directions assigned to the internal lines.) Similarly, recalling that G has ℓ external lines,

$$(Y_i)_\mu = \sum_{r=1}^{\ell-1} (Y_i^{\ell r})_\mu \ , \tag{A.11}$$

and

$$Y_i^{\ell r} = \frac{k_r}{U} \sum_{C \epsilon C(0,i)} s_C W_C^{\ell r} .$$

In the graph illustrated in Fig. 17 there is only one loop; for this case, we must set $U_C = 1$. Then

$$X_{12} = \frac{1}{U}, \quad X_{13} = -\frac{1}{U}, \quad X_{23} = -\frac{1}{U} .$$

Straightforward evaluation of the Y's gives

$$Y_1 = \frac{1}{U}\left[-(x_2 + x_3)\, p + x_2 p'\right]$$

$$Y_2 = \frac{1}{U}\left[x_1 p - (x_1 + x_3)\, p'\right]$$

$$Y_3 = \frac{1}{U}\left[-x_1 p - x_2 p'\right] .$$

We can now write the integral for the Feynman graph Fig. 17 explicitly:[31]

$$\Lambda_\mu^{(17)} = \frac{(-ie)2g^2}{16\pi^2 i} \int_\rho^\infty \frac{dx_1 dx_2 dx_3}{(x_1+x_2+x_3)^2} \left(\frac{1}{ix_1}\partial_{\ell_1} + \frac{1}{ix_2}\partial_{\ell_2}\right)_\mu \gamma_5 \left(\frac{1}{ix_3}\not\partial_{\ell_3} + m\right)\gamma_5 e^{iE}\Big|_{\ell_i=0}$$

$$= \frac{(-ie)2g^2}{16\pi^2 i} \int_\rho^\infty \frac{dx_1 dx_2 dx_3}{(x_1+x_2+x_3)^2} \left[(Y_1+Y_2)_\mu \gamma_5 (\not Y_3 + m)\gamma_5 - \frac{i}{2}(X_{13}+X_{23})\gamma_5\gamma_\mu\gamma_5\right]$$

$$\exp i\left[\frac{W^{q^2} q^2 + W^{pp'} m^2}{U} - (x_1+x_2)\, m_\pi^2 - x_3 m^2\right]$$

$$= \Lambda_{\mu Y}^{(17)} + \Lambda_{\mu X}^{(17)} .$$

The term $\Lambda_{\mu Y}$ does not diverge at the lower limit of integration, so for it we can set $\rho = 0$ and insert the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\, \delta\left(1 - \frac{1}{\lambda}\sum_{i=1}^{3} x_i\right), \qquad (x_i \geq 0) , \tag{A.12}$$

into the integral. Changing variables $x_i \to x_i/\lambda$ $(i = 1, 2, 3)$ and doing the λ integral, we obtain

$$\tilde{\Lambda}^{(17)}_{\mu Y} = \frac{(-ie)2g^2}{16\pi^2} \int_0^\infty \frac{dx_1 dx_2 dx_3\, \delta(1 - x_1 - x_2 - x_3)\,(Y_1 + Y_2)_\mu (m - \not{Y}_3)}{x_1 x_2 q^2 - x_3^2 M^2 - (1-x_3)\, m_\pi^2} \; .$$

It is not difficult to verify that this integral is asymptotically constant as $q^2 \to \infty$. Note also that there is no infrared divergence if we set $m_\pi = 0$.

In working out the term $\Lambda_{\mu X}$, we could use the standard subtraction procedure to be explained below. But we will instead use the identity[34]

$$\int_0^\infty \frac{d\lambda}{\lambda} \left(e^{ia\lambda} - e^{ib\lambda}\right) = \log \frac{b}{a} \quad , \tag{A.13}$$

along with (A.12), and obtain

$$\tilde{\Lambda}^{(17)}_{\mu X} = \frac{(-ie)2g^2}{16\pi^2} (-\gamma_\mu) \int_0^\infty dx_1 dx_2 dx_3\, \delta(1 - x_1 - x_2 - x_3) \log \left[\frac{-x_1 x_2 q^2 + x_3^2 m^2 + (1-x_3) m_\pi^2}{x_3^2 M^2 + (1-x_3)\, m_\pi^2}\right]$$

$$= -\frac{(-ie)2g^2}{32\pi^2}\, \gamma_\mu \log \frac{-q^2}{m^2} + O(1) \quad .$$

Thus this is the dominant term as $q^2 \to \infty$, and it arises from the asymptotic region of integration in momentum space, or the neighborhood of the origin in parametric space. Again note that there is no infrared divergence if $m_\pi = 0$.

Let us now introduce the general renormalization procedure. Previously we assumed that the graph G did not require renormalization; and when it turned out that the graph (Fig. 17) used as an illustration actually does require a subtraction, we made it in the usual way. Now we suppose that G requires renormalization, and introduce the notation[31]

S_i = divergent subgraph of G (possibly S_i = G)

d_i = degree of divergence of S_i

n_i = number of independent circuits in S_i

$\mathscr{S}$ = set of all divergent subgraphs S_i of G.

For each divergent subgraph S_i we introduce an auxiliary parameter ξ_i. For each S_i in turn we then examine the parametric function U (defined, on the assumption that no subtractions are required, by Eq. (A.3)); each term in U which is of order $n_i + m$ in X_k, where $X_k \in S_i$, is multiplied by ξ_i^{2m}. The same rule is applied to W^{rs}, U_C, and W_C^{rs}. We next define

$$\hat{x}_k = \left(\prod_{S_i \in \mathscr{S}_k} \xi_i \right) x_k \tag{A.14}$$

where $\mathscr{S}_k$ is the subset of $\mathscr{S}$ whose members contain the line k; all these functions are then inserted into Eq. (A.4).

Finally, we define the subtraction operator $\mathscr{A}$ required in Eq. (A.1):[35]

$$\mathscr{A} = \left[\prod_{S_i \in \mathscr{S}} \int_0^1 d\xi_i \right] \left[\prod_{S_j \in \mathscr{S}} \frac{(1-\xi_i)^{d_j}}{d_j!} \left(\frac{\partial}{\partial \xi_i} \right)^{d_j+1} \right] \tag{A.15}$$

Let us illustrate these formulas with an example, the double rung ladder graph pictured in Fig. 19. Including the rules for subtractions, we apply formulas (A.3), (A.5), etc., and find

$$U = (x_2+x_3+x_5)(x_1+x_4+x_6) + \xi^2 x_5(x_2+x_3)$$

$$UX_{12} = UX_{13} = UX_{34} = x_5$$

$$UX_{14} = x_2 + x_3 + x_5$$

$$UX_{23} = \frac{1}{\xi^2} (x_1 + x_4 + x_6) + x_5$$

$$W^{q^2} = x_1x_4(x_2+x_3+x_5) + \xi^2 x_2x_3(x_1+x_4+x_6+ \xi^2 x_5)$$

$$+ \xi^2 (x_2x_4 + x_1x_3) x_5$$

$$W^{pp'} = (x_1+x_4)(x_2+x_3+x_5) x_6 + \xi^2 (x_2+x_3) x_5x_6 .$$

Here ξ is the parameter corresponding to the divergent subgraph composed of lines 2, 3, and 5, for which $d = 0$, $n = 1$.

APPENDIX B

CALCULATION OF SOME SECOND ORDER VERTEX GRAPHS

We first calculate the leading asymptotic term ($-q^2 \gg m^2$, $\not{p} = \not{p}' = m$) from the double rung ladder graph of Fig. 9a. The relevant parametric functions are given by (A.16) using the labeling of Fig. 9a. When the $\mathcal{N}$ operator in (A.1) acts on the exponential, it produces three types of terms: one term containing a factor $\not{X}_i - m$ in the numerator for each fermion line, several terms containing two factors of $\not{X}_i - m$ and one factor of X_{jk}, and several terms containing two factors of X_{jk}. A simple analysis shows that it is only a term of the latter type, involving the factor $X_{14}X_{23}$, that gives rise to the leading $\log^2(-q^2/m^2)$ term. The reason for this is that the leading asymptotic term comes from the ultraviolet integration region and it is only the $X_{14}X_{23}$ term which, in the absence of subtractions and inserting a cutoff Λ^2, behaves like $\log^2 \Lambda^2$. Thus it is only this term which behaves like $\log^2(-q^2/m^2)$ after the renormalization subtractions. The dominant asymptotic term is

$$\tilde{\Lambda}_\mu^{(9a)}(q^2, \not{p} = \not{p}' = m) \approx -\frac{g^4}{(2\pi)^8} \frac{\pi^4}{4} \int_0^\infty dx_1 \ldots dx_6 \, \delta(1-x_1 - \ldots - x_6)$$

$$\times \gamma_\lambda \gamma_\sigma \gamma_\mu \gamma^\sigma \gamma^\lambda \int_0^1 d\xi \, \frac{\partial}{\partial \xi} \, \xi^2 \, X_{14}X_{23} \frac{1}{U^2} \tag{B.1}$$

$$\times \log\left(1 - \frac{q^2}{m^2} \, \frac{W^{q^2}}{(x_1 + \ldots + x_4)\, U - W^{pp'}}\right) .$$

The overall subtraction at $\not{p} = \not{p}' = m$, $q = 0$ has been performed using (A.13) while the internal subtraction is effected by the ξ operation at the point $p = p' = m$, $q = 0$. The choice of subtraction point does not affect the leading logarithmic contribution. We have taken the limit $m_\pi \to 0$ since we are interested only in the leading term.

It is important to realize that the above results depend very critically on the γ_5 coupling. In a theory such as massive QED, the terms containing factors of $\not{\ell}_i - m$ would be infrared divergent if the photon mass were taken to zero and in fact these terms would give rise to an asymptotic $\log^4(-q^2)$ behavior coming from the infrared integration region, which would dominate the ultraviolet $\log^2(-q^2)$ factor coming from the $X_{14}X_{23}$ term.

Without the subtraction using the ξ operation, the integral (B.1) would diverge logarithmically in the subintegration $x_2x_3x_5$. This ultraviolet divergence would come at the origin of parameter space. The subtraction ($\xi = 0$) term would be logarithmically divergent in the variables $x_1x_4x_6$ were it not for the presence of the logarithm. Since the logarithm cuts off an otherwise logarithmically divergent integral, we shall see that a $\log^2(-q^2/m^2)$ asymptotic behavior results. The above considerations show that the leading term comes from the integration region

$$x_1, x_4, x_6 \ll 1 \tag{B.2}$$

of the subtraction ($\xi = 0$) term.[36] We insert the factor

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\, \delta\left(1 - \frac{1}{\lambda}\,(x_1+x_4+x_6)\right) \tag{B.3}$$

and rescale $x_1, x_4, x_6 \rightarrow \lambda x_1, \lambda x_4, \lambda x_6$. Then the leading term will come from the integration region $\lambda \ll 1$. Thus

$$\begin{aligned}\tilde{\Lambda}_\mu^{(9a)}(q^2, \not{p}=\not{p}'=m) \approx 4G^2\, \gamma_\mu \int_0^\infty dx_1 \ldots dx_6\; \delta(1-x_1-x_4-x_6)\; \delta(1-x_2-x_3-x_5) \\ \int_0^{\epsilon\ll 1} \frac{d\lambda}{\lambda} \log\left(1 - \frac{q^2}{m^2}\,\lambda\, \frac{x_1x_4}{(x_1+\ldots+x_4) - x_6(x_1+x_4)}\right) .\end{aligned} \tag{B.4}$$

Doing the λ integration gives

$$\tilde{\Lambda}_\mu^{(9a)}(q^2, \not p = \not p' = m) \approx 2G^2 \gamma_\mu \log^2(-q^2/m^2) \int_0^\infty dx_1 dx_4 dx_6 \; \delta(1-x_1-x_4-x_6)$$

$$\int_0^\infty dx_2 dx_3 dx_5 \; \delta(1-x_2-x_3-x_5) \tag{B.5}$$

$$= \gamma_\mu G^2 \frac{1}{2!} \log^2(-q^2/m^2)$$

which is the result stated in Section III.

Regardless of which leg is taken asymptotic, the dominant term will be the one containing the factor $X_{14}X_{23}$. The general form of this term is

$$-4G\gamma_\mu \int_0^\infty dx_1 \ldots dx_6 \; \delta(1-x_1-\ldots-x_6) \int_0^1 d\xi \frac{\partial}{\partial \xi} \xi^2 X_{14}X_{23} \frac{1}{U^2}$$
$$\times \log\left(\frac{V(q^2,p^2,p'^2) - (x_1+\ldots+x_4)\,m^2}{V(0,m^2,m^2) - (x_1+\ldots+x_4)\,m^2}\right) \tag{B.6}$$

with V defined by (A.5). Since momentum dependence is found only in the argument of the logarithm, it can easily be seen that the leading term is independent of which legs are taken asymptotic. Thus for the kinematic region $-p^2 = -p'^2 \gg m^2$, $q = 0$ — which is the region in which the vertex can be related to the asymptotic propagator using the Ward identity — the leading term is

$$\gamma_\mu G^2 \frac{1}{2!} \log^2(-p^2/m^2) \; .$$

We next consider the graph of Fig. 10d with the fermion self-energy subgraph. We reproduce and label this graph in Fig. 19. An analysis similar to the above shows that the leading asymptotic term is

$$\tilde{\Lambda}_\mu^{(19)}(q^2, \not p = \not p' = m) \approx \int_0 dx_1 \ldots dx_6\ \delta(1 - x_1 - \ldots - x_6) \int_0^1 d\xi \left(\frac{\partial}{\partial \xi}\right)^2 \xi^2 \frac{x_5(x_3+x_5)}{U^4}$$

$$\log\left(1 - \frac{q^2}{m^2} \frac{W^{q^2}}{(x_1 + \ldots + x_4)\ U - W^{pp'}}\right) \tag{B.7}$$

where

$$U = \xi^2\ x_3x_5 + (x_3+x_5)\ (x_1+x_2+x_4+x_6)$$

$$W^{q^2} = x_1\left[(x_3+x_5)\ (x_2+x_4) + \xi^2\ x_3x_5\right] \tag{B.8}$$

$$W^{pp'} = x_6\left[\xi^2\ x_2x_5 + (x_3+x_5)\ (x_1 + x_2 + x_4)\right]$$

The internal mass and wave function subtractions are performed at the origin of momentum space using the ξ operations.

We use the fact that

$$\int_0^1 d\xi\ (1-\xi)\ \left(\frac{\partial}{\partial \xi}\right)^2 \xi\, f(\xi) = \int_0^1 d\xi\ \frac{\partial}{\partial \xi}\ f(\xi) \tag{B.9}$$

and again find that the dominant asymptotic term comes from the subtraction ($\xi = 0$) term and from the integration region $x_3, x_5 \ll 1$. We find

$$\tilde{\Lambda}_\mu^{(19)}(q^2, \not p = \not p' = m) \approx -6\ \gamma_\mu\, G^2 \log^2(-q^2/m^2) \int_0^\infty dx_1dx_2dx_4dx_6\ \delta(1 - x_1 - x_2 - x_4 - x_6)$$

$$= -\gamma_\mu\, G^2 \log^2(-q^2/m^2) \tag{B.10}$$

When an isotopic factor of three is included for the isospin-symmetric theory, we get the result indicated in Fig. 10d. It is again found that this form (B.10) holds in any asymptotic region.

APPENDIX C

We will first prove the relation (IV. 2) using the notation and techniques of Appendix A. We express the renormalized amplitude $\tilde{\Gamma}_{\mu,n}(q^2, \not{p}=\not{p}'=m)$ corresponding to the n^{th} order ladder graph of Fig. 11 by using the operator $\mathscr{A}$ defined by A.15 to perform the internal subtractions. The only term which will give rise to a leading $\log^n(-q^2/m^2)$ contribution for $q^2 \gg m^2$ is the one which, without the subtractions, would be logarithmically divergent in each of the n vertex subgraphs. This term contains only the n numerator factors $X_{x_i y_i}$, $i=1,2,\ldots n$, and no factors of Y_i. For this term, the Dirac algebra is trivial and for $-q^2 \gg m^2$,

$$\tilde{\Gamma}_{\mu,n}(q^2, \not{p}=\not{p}'=m) \approx (2G)^n \gamma_\mu \int_0^\infty \left(\prod_{i=1}^{n} dx_i dy_i dz_i\right) \delta\left(1 - \sum_{i=1}^{n} (x_i+y_i+z_i)\right)$$

$$\times \mathscr{A}_n \frac{1}{U_n^2} \left(\prod_{i=1}^{n} X_{x_i y_i, n}\right) \tag{C.1}$$

$$\times \log\left(1 - \frac{q^2}{m^2} \frac{W_n^{q^2}}{U_n \sum_{i=1}^{n} (x_i+y_i) - W_n^{pp'}}\right) .$$

The subscript n on the parametric functions indicates that they refer to the n^{th} order ladder graph. We have explicitly performed the overall subtraction at $q = 0$, $\not{p} = \not{p}' = m$ using the relations (A.12) and (A.13). The operator $\mathscr{A}_n$ effects only the internal subtractions (at the point $p = p' = q = 0$).

The parametric functions are defined according to Appendix A with the subtraction parameter ξ_i assigned to the vertex subgraph composed of the lines labeled $x_i, y_i, z_i, x_{i+1}, y_{i+1}, z_{i+1}, \ldots x_n, y_n, z_n$. For each of these graphs, $d_i = 0$

and thus

$$\mathscr{D}_n = \left(\prod_{i=2}^{n} \int_0^1 d\xi_i \right) \left(\prod_{j=2}^{n} \frac{\partial}{\partial \xi_i} \right) . \tag{C.2}$$

Let us consider the subgraph $x_n y_n z_n$. The integration over these parameters is now finite due to the subtraction involving the parameter ξ_n. By the same arguments as in Appendix B, it can be seen that the leading asymptotic term will come from the integration region

$$x_i, y_i, z_i \ll 1 , \qquad i=1 \ldots n-1 \tag{C.3}$$

of the subtraction ($\xi_n=0$) term.[36] By using the properties of the parametric functions,[31] it can be shown that

$$\begin{aligned}
&U_n(\xi_n=0) = (x_n + y_n + z_n)\, U_{n-1} \\
&X_{x_n y_n, n}(\xi_n=0) = (x_n + y_n + z_n)^{-1} \\
&X_{x_i y_i, n}(\xi_n=0) = X_{x_i y_i, n-1} , \quad i=1, \ldots, n-1 , \\
&W_n^{q^2}(\xi_n=0) = (x_n + y_n + z_n)\, W_{n-1}^{q^2} \\
&W_n^{pp'}(\xi_n=0) = (x_n + y_n + z_n)\, W_{n-1}^{pp'} .
\end{aligned} \tag{C.4}$$

We realize the restriction (C.3) by inserting into (C.1) the factor

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\, \delta\left(1 - \frac{1}{\lambda} \sum_{i=1}^{n-1} (x_i + y_i + z_i)\right) \tag{C.5}$$

and rescaling $x_i, y_i, z_i \rightarrow \lambda x_i, \lambda y_i, \lambda z_i$, $i=1, \ldots, n-1$. We then restrict the λ integration such that $\lambda \ll 1$. Using (C.4), we find

$$\tilde{\Gamma}_{\mu n}(q^2, \not p = \not p' = m) \approx -(2G)^n \int_0^\infty dx_n dy_n dz_n \ \delta(1 - x_n - y_n - z_n)$$

$$\int_0^\infty \left(\prod_{i=1}^{n-1} dx_i dy_i dz_i \right) \delta\left(1 - \sum_{i=1}^{n-1} (x_i + y_i + z_i)\right) \int_0^\epsilon \frac{d\lambda}{\lambda} \mathscr{A}_{n-1}$$

$$\times \frac{1}{U_{n-1}^2} \log\left(1 - \lambda \frac{q^2}{m^2} \frac{W_{n-1}^{q^2}}{U_{n-1} \sum_{i=1}^{n} (x_i + y_i) - W_{n-1}^{pp'}}\right) \tag{C.6}$$

with $\epsilon \ll 1$.

The λ integration gives

$$\int_0^\epsilon \frac{d\lambda}{\lambda} \log(1 + \lambda a) \approx \frac{1}{2} \log^2(1 + a) \tag{C.7}$$

while

$$\int_0^\infty dx_n dy_n dz_n \ \delta(1 - x_n - y_n - z_n) = \frac{1}{2} \quad . \tag{C.8}$$

Inserting these results into (C.6) and comparing with (C.1), we have the expression for $\Gamma_{\mu, n-1}(q^2)$ but with the logarithm replaced by $\frac{1}{2!} \log^2$.

This technique can be continued. The next step is to subtract the innermost vertex subgraph of the reduced graph corresponding to $\Gamma_{\mu, n-1}(q^2)$. The result will be the integral for $\Gamma_{\mu, n-2}(q^2)$ but with the logarithm factor replaced by $\frac{1}{3!} \log^3$. Clearly the final result will be exactly (IV.2)

$$\Gamma_{\mu, n}(q^2) = (-1)^n G^n \frac{\log^n(-q^2/m^2)}{n!} \quad . \tag{IV.2}$$

The general iteration scheme (V.8) relies on results similar to those for the ladder graph. It must be shown that for an arbitrary vertex graph which contributes a leading logarithmic factor in its order of perturbation theory, the amplitude has a form similar to (IV.2). The proof is carried out just as for the ladder graph by working from the inside out. One starts with the parametric integral representation of the amplitude with only the overall subtraction performed explicitly. The form will be similar to (C.1). The subtractions for the innermost vertex or self-energy subgraphs (those which do not themselves contain such subgraphs) can be carried out in any order using the above method. In the case of overlapping vertex subgraphs, the subtractions can be performed in either order. Once these subgraphs have been dealt with, the remaining amplitude will correspond to the graph formed by shrinking all of these innermost graphs to a point except that the single log will be replaced by $\log^{n+1}$ multiplied by factors arising from the Dirac algebra where n is the number of innermost graphs. This procedure can then be continued as with the ladder graphs to give the results (V.5) with the spinor and isotopic factors kept track of by (V.8).

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2. S. D. Drell, Donald J. Levy, and Tung-Mow Yan, Phys. Rev. Letters 22, 744 (1969), and Report No. SLAC-PUB-606; J. D. Bjorken and E. A. Paschos, Report No. SLAC-PUB-572, Stanford Linear Accelerator Center, Stanford University, Stanford, California (1969).
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4. For the reader's convenience we summarize here the definition of the theoretically more convenient F form factors,

$$\langle p' | J_\mu(0) | p \rangle = \bar{u}(p')\, e\, \Gamma_\mu(p',p)\, u(p)$$

$$= e\, \bar{u}(p') \left[\gamma_\mu F_1(q^2) + i\sigma_{\mu\nu} q^\nu \frac{\kappa}{2m} F_2(q^2) \right] u(p)$$

and the relation of F_1 and F_2 to the Sachs form factors,

$$F_1(q^2) = \left[\frac{q^2}{4m^2} G_M(q^2) + G_E(q^2) \right] \Big/ \left(1 + \frac{q^2}{4m^2} \right)$$

$$F_2(q^2) = \left[G_M(q^2) - G_E(q^2) \right] / \kappa \left(1 + \frac{q^2}{4m^2} \right) \quad .$$

We call F_1 and F_2 "form factors" when the external fermion lines are on the mass shell ($p'^2 = p^2 = m^2$), and "vertex functions" otherwise. Off the mass shell, there is a third form factor F_3, which we take to be the coefficient of q_μ.

5. We will omit explicit reference to many well-known papers, and instead direct the interested reader to the several excellent reviews, including the Rapporteur's talks on Electromagnetic Interactions by S. D. Drell at the SLAC Conference (1967), and by N. M. Kroll at the Vienna Conference (1968).

6. We use the metric $a^2 = a_0^2 - \underset{\sim}{a}^2$ and the other conventions adopted by J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw-Hill, New York, 1965). In Eq. (I. 1), λ_0 is chosen so as to exactly cancel the logarithmic divergence associated with the fermion "box" graph. We do not really expect that Eq. (I. 1) will provide an adequate description of physical reality — at very least, the interaction Lagrangian should probably include SU_3. Nevertheless, we feel that our investigation of (I. 1) will indicate the general features to be expected from a field theoretic approach.

7. The simplest argument in that mentioned by D. R. Yennie in the Brandeis Lectures, edited by K. W. Ford (Gordon and Breach, New York, 1963), Vol. 1, p. 221. In QED, the infrared divergences in any cross section (as the mass μ of the virtual photons tends to zero) are exactly cancelled by the infrared divergences of the cross section for the same process, but with the emission of very low energy real photons. The details of this cancellation were spelled out by D. R. Yennie, S. C. Frautshi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961). In γ_5 π-N theory, however, there is no divergence as $m_\pi \to 0$ in the cross section for a process involving emission of pions since the infrared region is supressed by the p-wave coupling:

$$\bar{u}(p')\,\gamma_5\,u(p) \sim \chi^+ \underset{\sim}{\sigma} \cdot (\underset{\sim}{p}' - \underset{\sim}{p})\,\chi \quad .$$

Hence, since the physically observable cross section must be finite, there can be no infrared divergence associated with virtual pions either.

This conclusion can also be obtained directly. It is straightforward to show — see the appendices — that lowest order graphs are not infrared divergent as $m_\pi \to 0$. We now inductively assume that an amplitude $A(p,p',\ldots)$, corresponding to a graph with n-1 virtual pions, is not infrared divergent, and form the amplitude A' associated with a graph containing an additional virtual pion. By an argument of Yennie et al., it follows that if a given Feynman amplitude is not infrared divergent as $m_\pi \to 0$, then the only way a meson line can be added to it in order to produce an infrared divergence is to have both ends of the meson line end on external lines of the corresponding Feynman graph. We obviously need not consider self-energy insertions on external legs. The amplitude A' will thus have the form

$$A' = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_\pi^2} \bar{u}(p')\gamma_5 \frac{i}{\not p' - \not k - m} A(p-k, p'-k, \ldots) \frac{i}{\not p - \not k - m} \gamma_5 u(p)$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 - m_\pi^2} \frac{\bar{u}(p') \not k \gamma_5 A(p-k, p'-k, \ldots) \gamma_5 \not k u(p)}{(k^2 - 2p'\cdot k)(k^2 - 2p\cdot k)}$$

which is not divergent as $m_\pi \to 0$. This completes the inductive proof.

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26. The notation $\not{p}' = \not{p} = m$ indicates, as usual, that the renormalized vertex $\Lambda_{\mu 1}$ is regarded as sandwiched between Dirac spinors $\bar{u}(p')$ and $u(p)$.

27. The on-mass-shell pion form factor is defined by the equation

$$\langle \pi(p') \,|\, J_\mu(0) \,|\, \pi(p) \rangle = e(p' + p)_\mu \; F_\pi(q^2) \;.$$

Off the mass shell we add another term proportional to q_μ.

28. In using a digital computer to do the iteration indicated by the equations (V.7) - (V.10), it is most convenient to include a factor of n! with each of the coefficients $s_n, v_n, d_n, \overline{sds}_n$, etc., so that they are all integers.

29. See Refs. 9 and 10. Results analogous to Eqs. (V.18) and (V.19) have also been obtained for the π^0-p theory. Defining $Q = 1 - 10\,G \log(-q^2/m^2)$, we find that asymptotically $F_{1p} \sim Q^{1/10}$, $F_{\pi NN} \sim Q^{-1/5}$, $\widetilde{S}'_F \sim (i/\not{p})\, Q^{-/10}$, and $\widetilde{D}'_F \sim (i/p^2)\, Q^{-2/5}$.

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The reader should consult this paper for further explanations and for derivations of the formulas quoted without proof in this Appendix.

32. It must be emphasized that the directions of momenta assigned to external lines need bear no relation at all to the directions assigned to the internal lines. The actual values of the external momenta are irrelevant in computing the parametric functions U, W, etc.

33. This graph is ultraviolet divergent and requires a subtraction. For the time being we can regard the integral (A.1) to be regularized by changing the lower limit of integration from 0 to a small number $\rho > 0$. In the parametric formalism, ultraviolet divergences always appear at the lower limit of integration, and this is a general method of regularizing them.

34. We generally use this identity in doing the outermost, or "overall," subtraction. In evaluating the general vertex graph, the use of (A.13) allows us to subtract on mass shell ($p^2 = p'^2 = M^2$, $q^2 = 0$) instead of at the origin of momentum space ($p^2 = p'^2 = q^2 = 0$).

35. The efficacy of this operator is based on the identity

$$f(x) - f(0) - \ldots - \frac{f^{(n)}(0)}{n!} x^n = \int_0^1 d\xi \, \frac{(1-\xi)^n}{n!} \left(\frac{\partial}{\partial \xi}\right)^{n+1} f(\xi x) \, .$$

36. In practice, the integral over the subtraction parameter ξ is usually replaced by the formula quoted in footnote 35. We refer to $f(0) + f'(0) + \ldots$ as the "subtraction terms."

FIGURE CAPTIONS

1. Form factor model of Ball and Zachariasen (Ref. 13). Note the absence of an inhomogeneous contact term in (b), the presence of which would correspond to an elementary "core" within the particle represented by the double line and would change the asymptotic behavior from $F(q^2) \sim 1/q^4$ to $F(q^2) \sim 1/q^2$.
2. Bootstrap model for the form factor.
3. Graphs contributing leading logarithms to the electromagnetic form factor in massive QED: ladders and crossed-ladders.
4. Integral equation used by Landau and collaborators (Ref. 9) in conjunction with Dyson's equations for the propagators D_F and S_F for the determination of the asymptotic behavior of Γ_5, D_F, and S_F.
5. First order proton-γ vertex graphs.
6. First order neutron-γ vertex graphs.
7. First order π-γ and N-π vertex graphs.
8. First order self-energy graphs.
9. Second order ladder and crossed-ladder proton-γ vertex graphs.
10. All second order proton-γ vertex graphs. The number under each graph is the contribution of that graph $\left(\text{times a common factor } (G^2/2) \log(-q^2/m^2)\right)$.
11. The n^{th} order vertex ladder graph.
12. The n^{th} order ladder graphs.
13. Third order ladder graph with fermion "box" subgraph.
14. The "rung functions" $\overline{sds}_n$, $\overline{sss}_n$, and $\overline{dsd}_n$, which describe the effect of adding onto a vertex graph another rung with $\log^{n-1}$ self-energy and vertex corrections (and a single logarithm from the divergent loop integral for the new rung). Thus the sum ranges over all partitions $n_1+n_2+n_3+n_4+n_5=n-1$.

15. The iteration equations for determining the n^{th} order leading contribution to the various vertex functions.

16. Calculation of the parametric function W^{rs}.

17. Calculation of a first-order vertex graph.

18. Method of calculating W^{q^2}.

19. Second order vertex graph with fermion self-energy insertion.

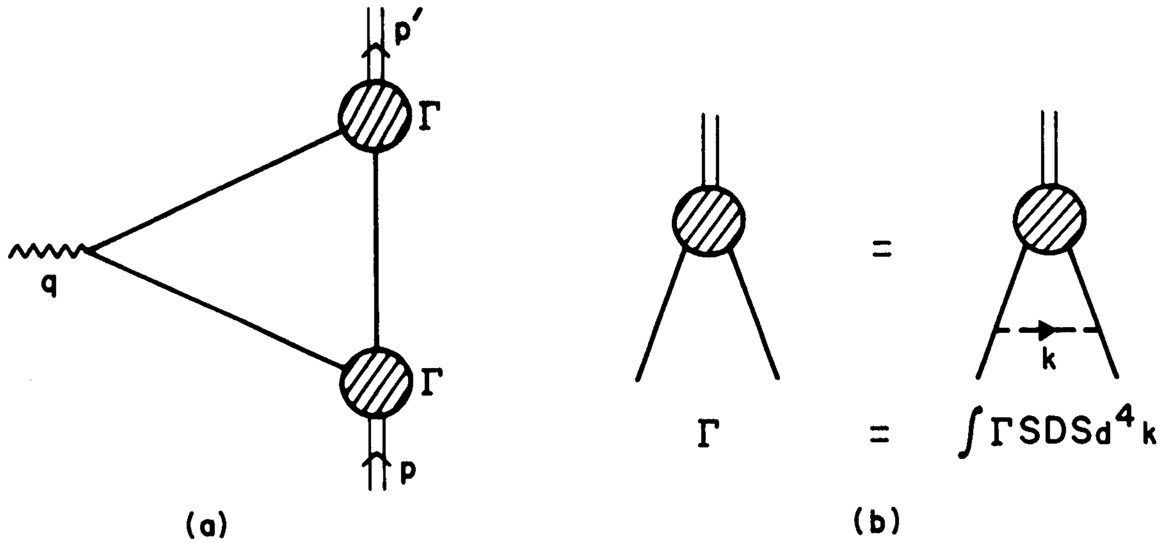

Fig 1

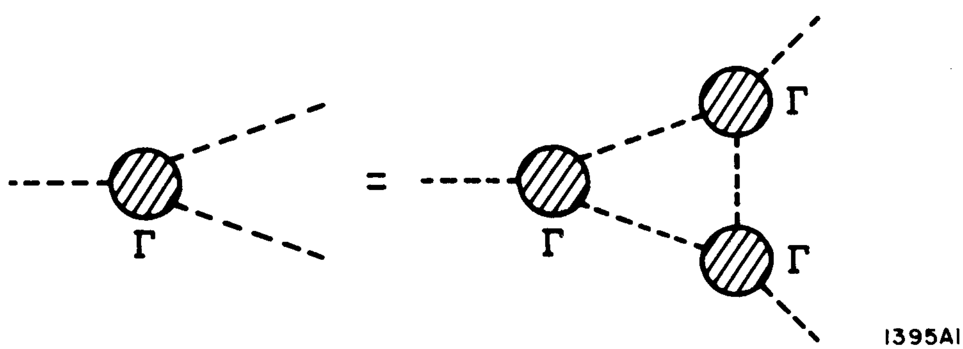

Fig. 2

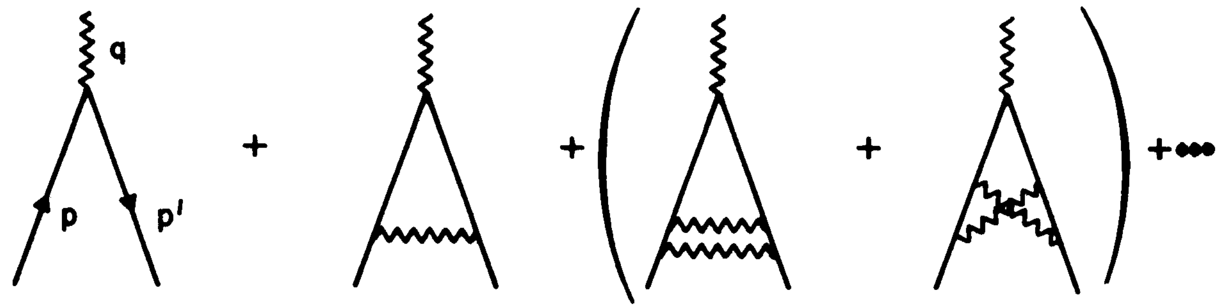

Fig. 3

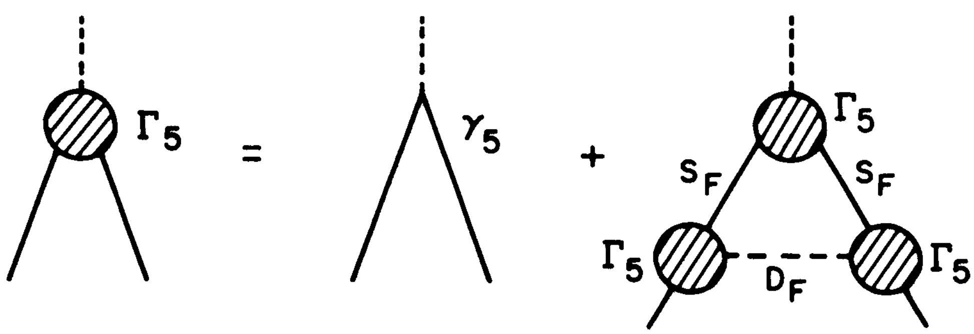

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Fig. 4

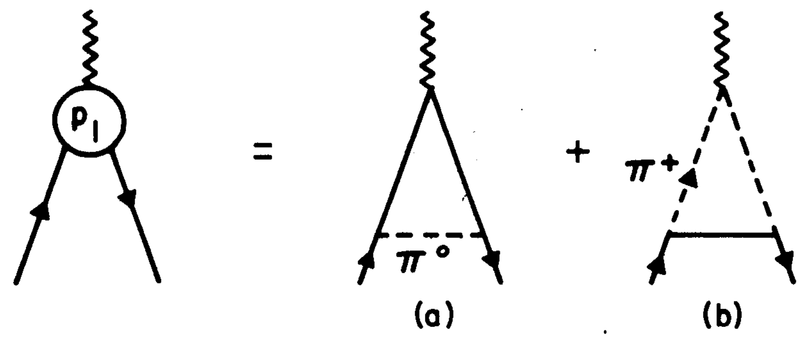

Fig. 5

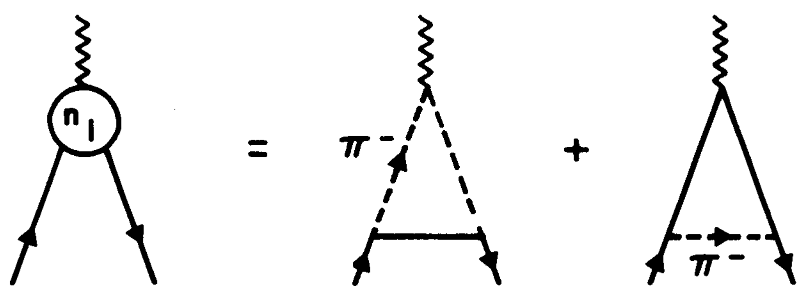

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Fig. 6

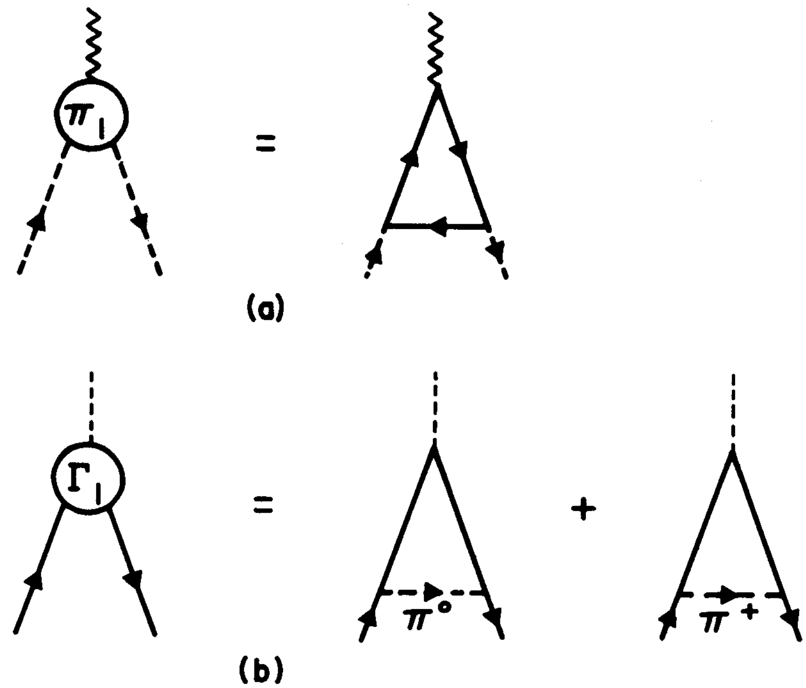


Fig. 7

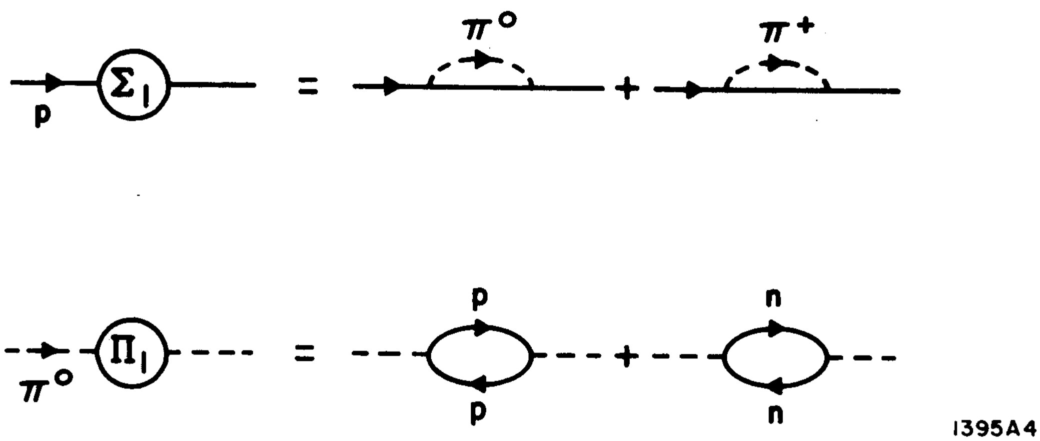
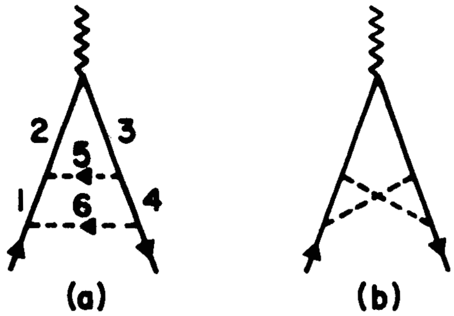
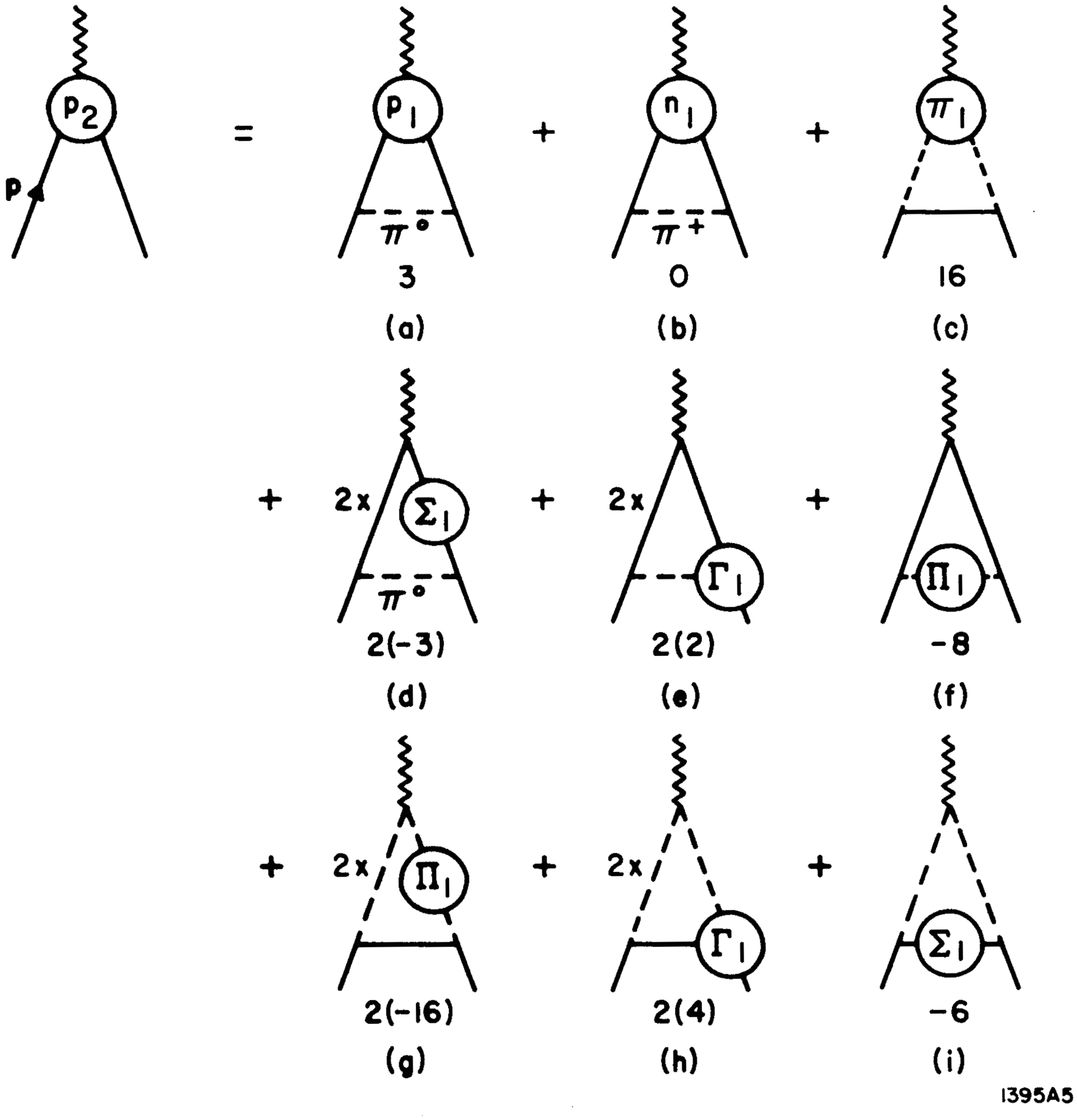


Fig. 8



2
3
5
1
6
4
(a)
(b)

Fig. 9



p_2
p
=
p_1
π^0
3
(a)
+
n_1
π^+
0
(b)
+
π_1
16
(c)
+
2x
Σ_1
π^0
2(-3)
(d)
+
2x
Γ_1
2(2)
(e)
+
Π_1
-8
(f)
+
2x
Π_1
2(-16)
(g)
+
2x
Γ_1
2(4)
(h)
+
Σ_1
-6
(i)
1395A5

Fig. 10

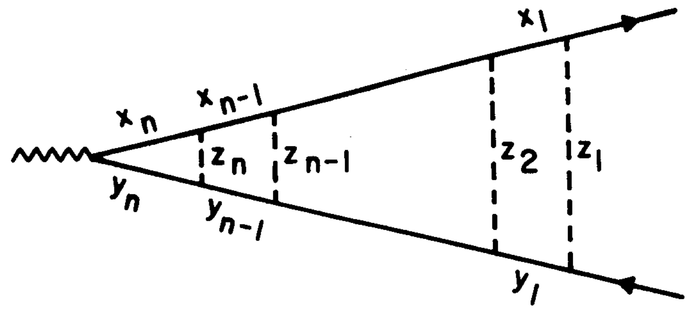


Fig. 11

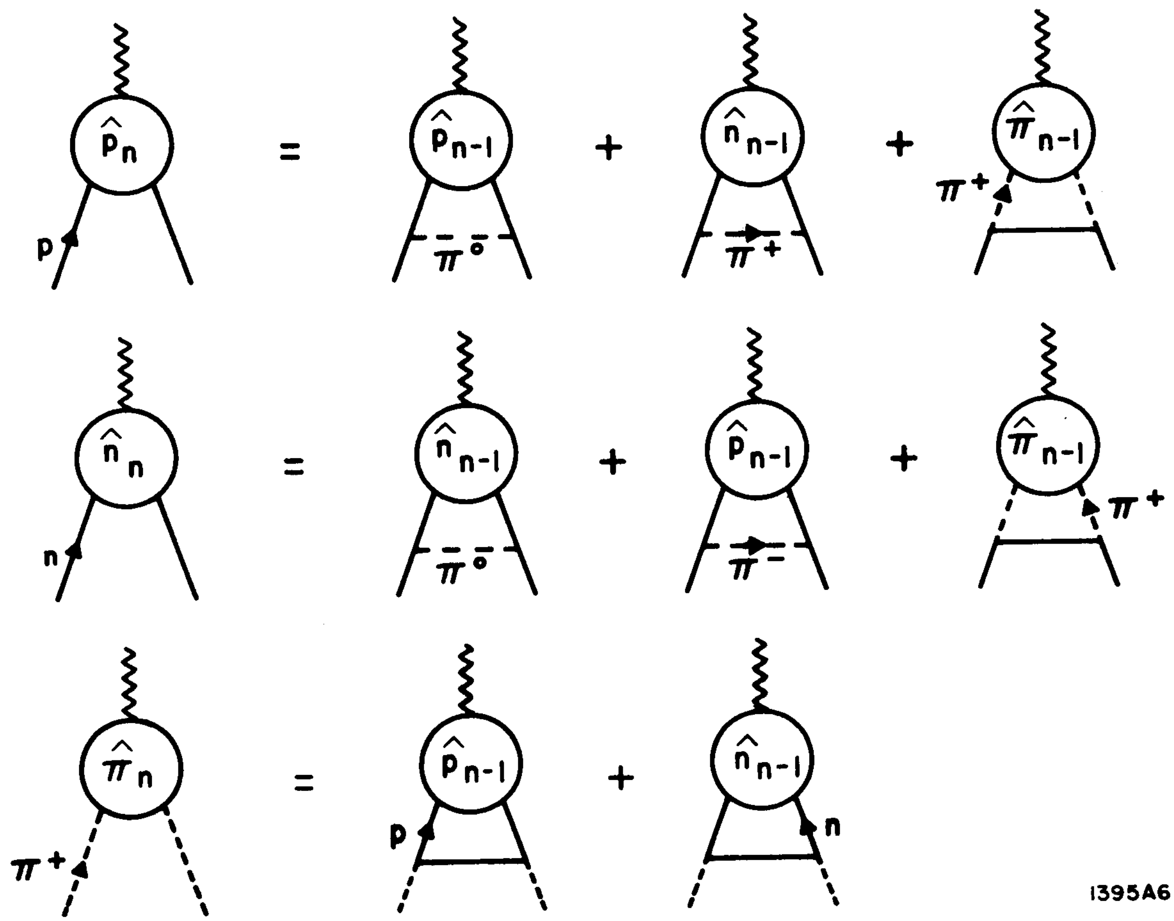


Fig. 12

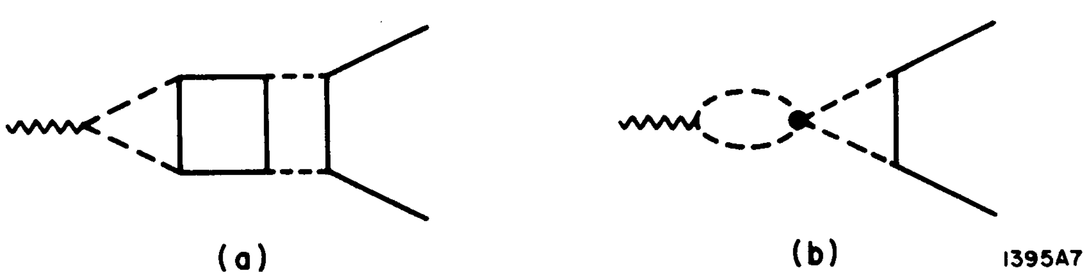
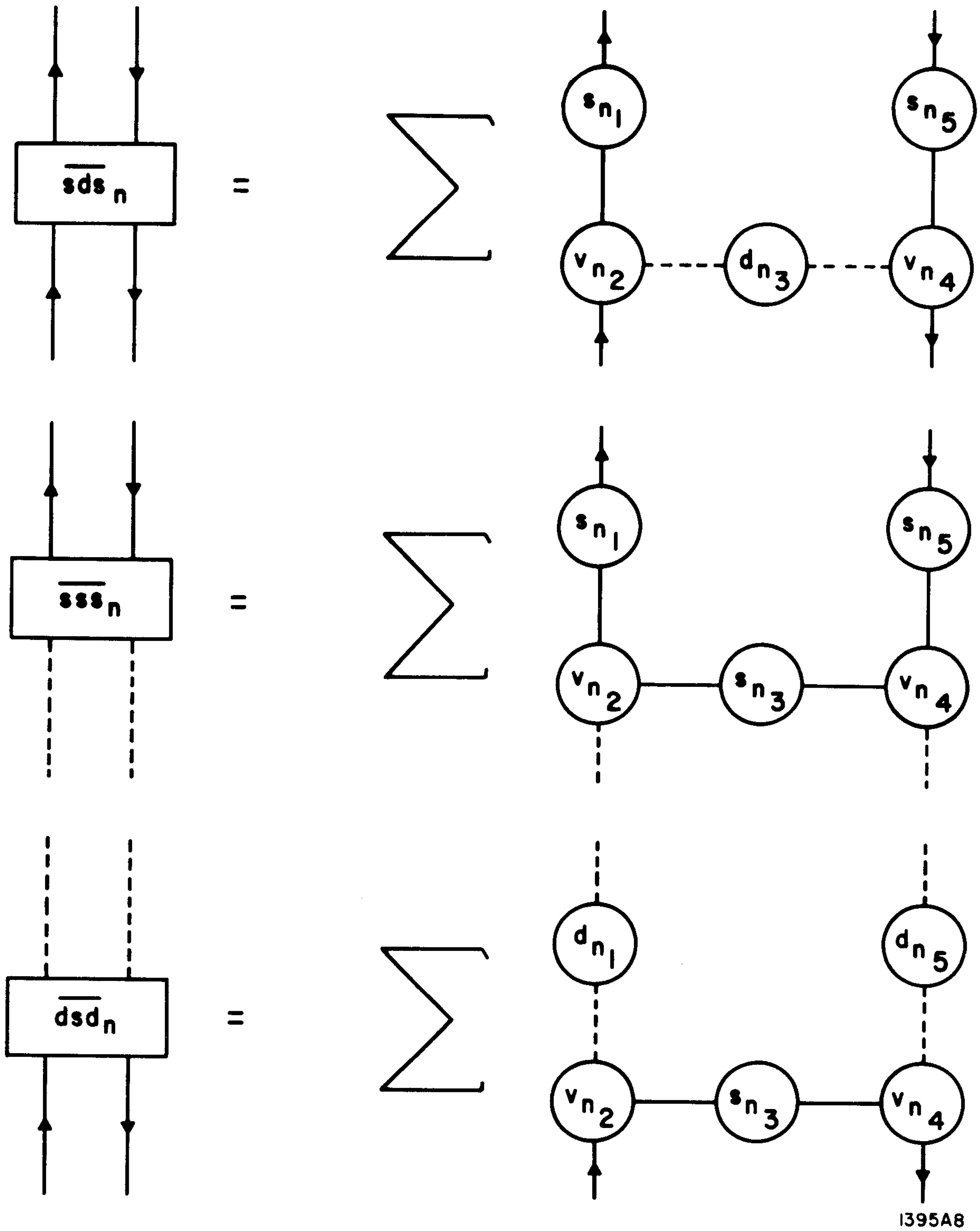
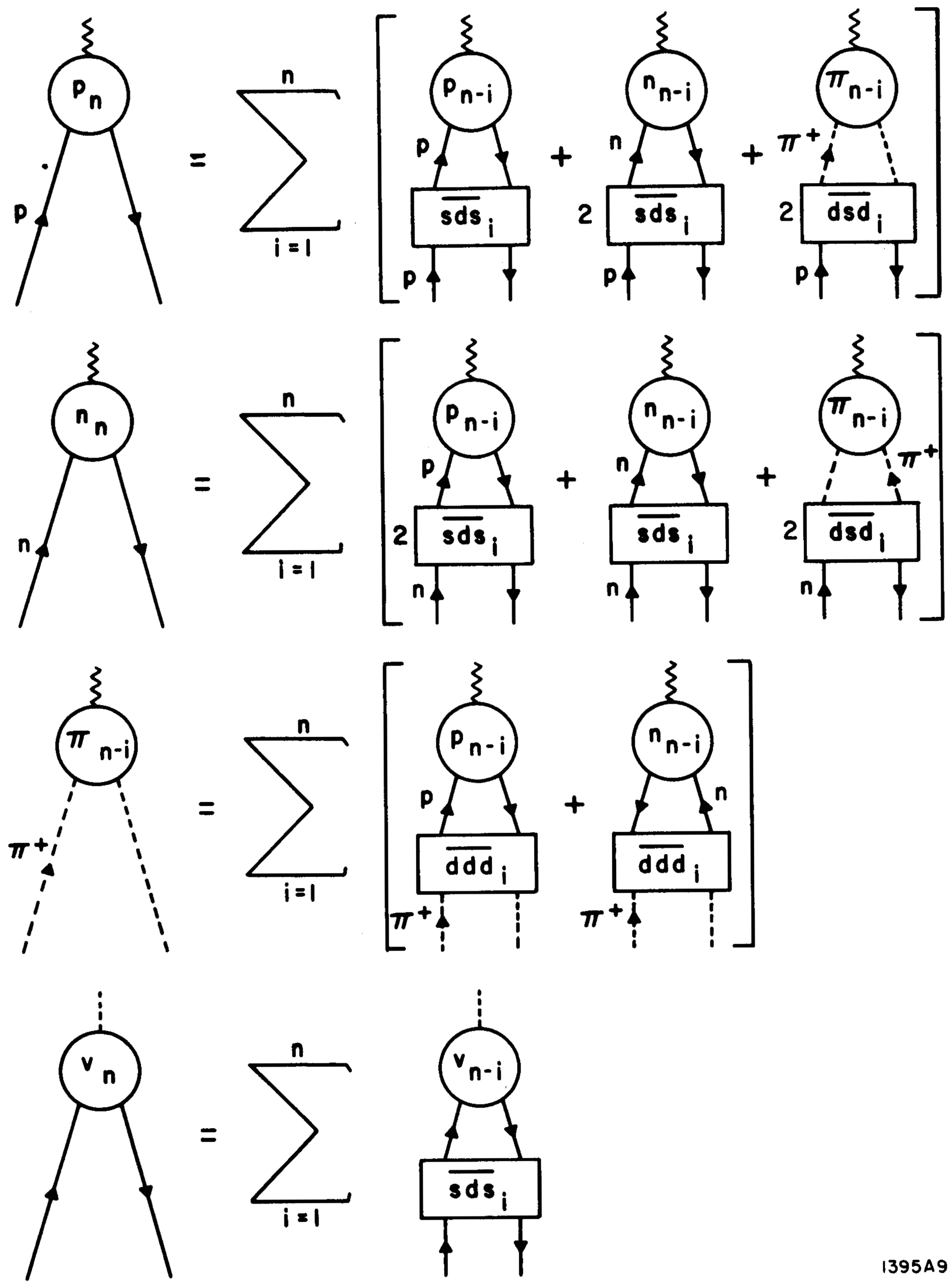


Fig. 13



$\overline{sds}_n$
=
$\sum$
s_{n_1}
s_{n_5}
v_{n_2}
d_{n_3}
v_{n_4}
$\overline{sss}_n$
=
$\sum$
s_{n_1}
s_{n_5}
v_{n_2}
s_{n_3}
v_{n_4}
$\overline{dsd}_n$
=
$\sum$
d_{n_1}
d_{n_5}
v_{n_2}
s_{n_3}
v_{n_4}
1395A8

Fig 14



p n
p
n
i=1
p n-i
p
sds i
p
n n-i
n
2
sds i
p
π n-i
π+
2
dsd i
p
n n
n
n
i=1
p n-i
p
2
sds i
n
n n-i
n
sds i
n
π n-i
π+
2
dsd i
n
π n-i
π+
n
i=1
p n-i
p
ddd i
π+
n n-i
n
ddd i
π+
v n
n
i=1
v n-i
sds i
1395A9

Fig. 15

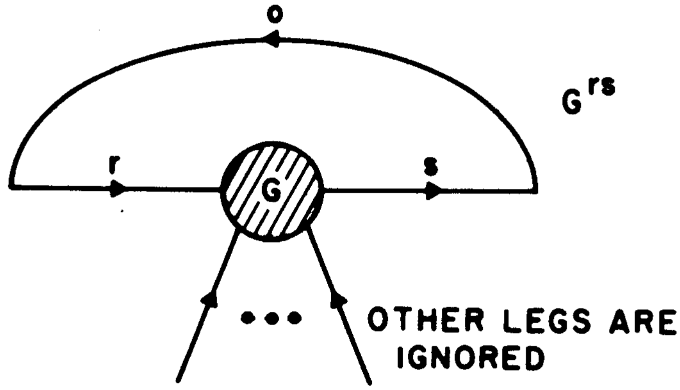

o
G^{rs}
r
G
s
OTHER LEGS ARE IGNORED

Fig. 16

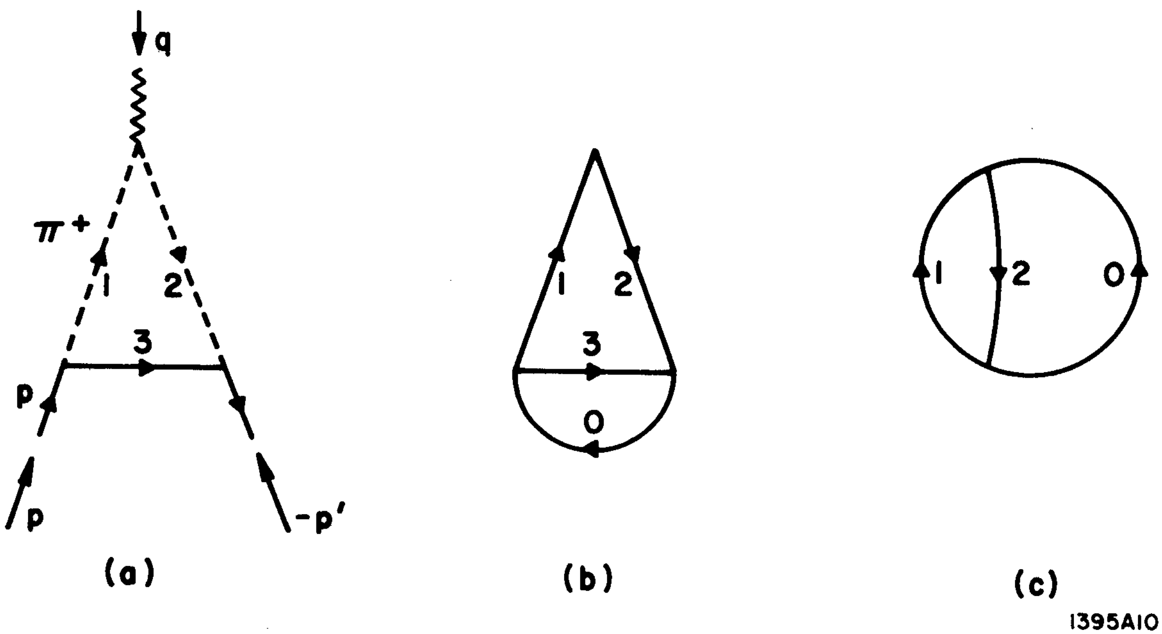

q
π^+
1
2
3
p
p
-p'
(a)
1
2
3
0
(b)
1
2
0
(c)
1395A10

Fig. 17

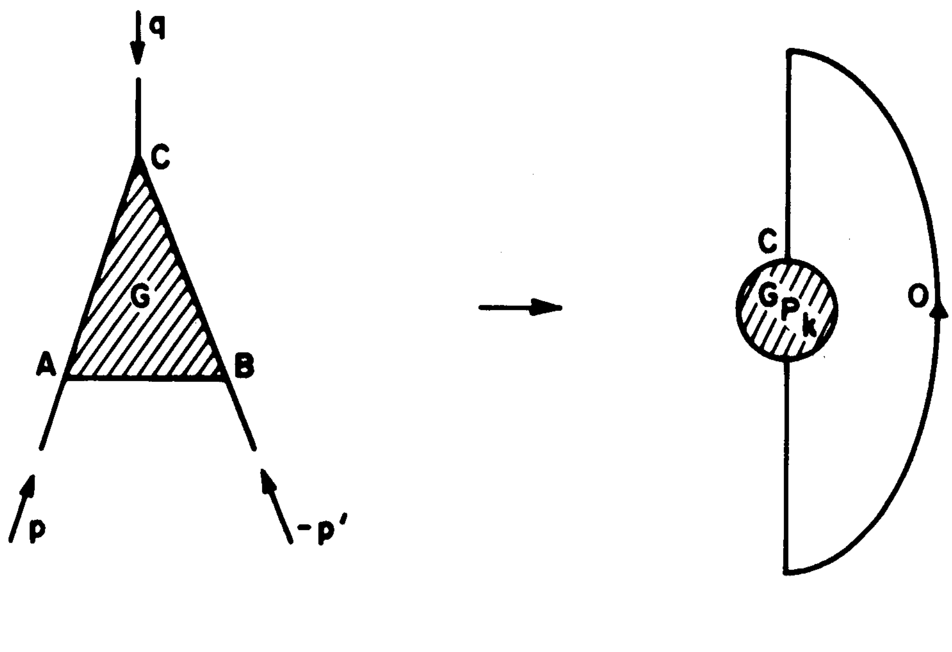
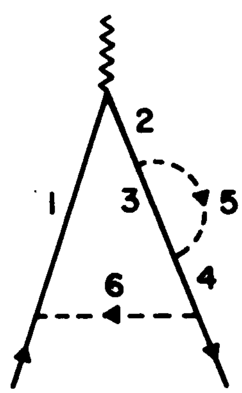


Fig. 18



1395A11

Fig. 19