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CURRENT ALGEBRAS AND UNIVERSAL DIVERGENT RADIATIVE CORRECTIONS*†

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ABSTRACT

We examine, using current algebras, the ultraviolet divergences occurring in the calculation of electromagnetic radiative corrections to any lowest order weak process at arbitrary momentum transfer. We consider all orders in perturbation theory in the fine structure constant α. The divergent parts of the radiative corrections are expressed in terms of matrix elements of equal-time current commutators by using the Bjorken expansion of time-ordered products of currents at large momenta. We assume the validity of this expansion and of the use of "naive" current commutation relations in discussing various current algebra models. We impose the condition that the divergences contribute only to an unobservable, universal weak coupling constant renormalization. It is shown that, in models with operator Schwinger terms in the current commutators, this condition cannot be satisfied for non-zero momentum transfer. Also, it is not satisfied for a weak interaction theory mediated by a vector boson. Two current algebra models are exhibited which are satisfactory if the weak Hamiltonian has a local current-current form. For these models, the weak and electromagnetic currents of both the hadrons and the leptons obey the same commutation relations, and the Schwinger terms are c-numbers. One, a quark model of hadrons with integrally charged quarks together with the conventional lepton currents, gives finite radiative corrections. The second, the algebra of fields model for the total electromagnetic and weak currents, including leptons, contains only a universal divergent factor. These two results are shown to hold to all orders in α. In obtaining these results, divergent contributions to electromagnetic mass shifts and to electromagnetic renormalization effects in strong interaction processes are isolated and removed by adding a counter term to the interaction Hamiltonian. These divergences may thus be treated as a separate problem, which we do not discuss in detail.

I. INTRODUCTION

Our current experimental knowledge of leptonic and semileptonic weak interactions is well described by the familiar universal current-current form[1,2] of the phenomenological interaction Lagrangian. One of the most remarkable features of this Lagrangian is that it predicts, for μ-decay and neutron β-decay, the approximate equality of the respective vector coupling constants, which is consistent with experimental observations. Specifically, by using the conserved vector current hypothesis,[1] one may infer from the equality of the appropriate bare coupling constants that the renormalized coupling constants are equal even after the inclusion of strong interaction effects. The fact that the bare coupling constants may be chosen to be equal leads us to believe that the effective Lagrangian may have a more fundamental significance.

To test this "universality" of the weak interactions we must also include the electromagnetic radiative corrections. We expect these to be small corrections at least of the order of α, where $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$ is the fine structure constant. However, for semileptonic processes a serious problem arises in their calculation since divergent momentum integrals occur. It is these divergences we wish to study, to all orders in perturbation theory in the fine structure constant.

The most straightforward resolution of this problem would be to construct a theory in which all these divergences cancelled out so that the amplitude for any weak process would be finite to all orders in α. A less stringent condition would nevertheless be satisfactory since all that is required of a consistent theory is that measurable quantities, such as ratios of coupling constants, masses, or form factors, be finite. Hence, it is enough to impose the condition that any divergences arising in the calculation of electromagnetic radiative corrections to weak processes contribute only to an overall (possibly cut-off dependent) constant factor times a finite matrix element. Of course, this factor must be universal, i.e., it must be the same for all weak processes so that ratios of any measurable parameters will be finite. The divergent factor can then be absorbed into the definition of the weak coupling constant, since its overall scale is undetermined. Note that it is not sufficient merely to require that ratios of the coupling constants defined at zero momentum transfer be finite. The ratios of various form factors occurring for non-zero momentum transfers are measurable and therefore must also be finite and calculable.

We are interested here in the implications of current algebra[3] for the problem of divergences in radiative corrections to weak interactions. It is our purpose in this paper to develop a technique for discussing these divergences which does not depend upon the particular weak process considered and which is valid for arbitrary momentum transfers. Furthermore, we wish to examine the corrections to all orders in e^2, not merely to second order, as in all previous investigations.

Since this topic has received considerable attention in the past by various authors,[4-12] let us first briefly review the previous work before detailing our contribution to the subject. The status of the radiative corrections problem before the advent of current algebra was summarized by Berman and Sirlin.[13] They observed that the radiative corrections to μ-decay could be shown to be finite to all orders in α by performing a Fierz transformation[14] on the weak Hamiltonian. However, for decays involving bare hadrons, the corrections were in general logarithmically divergent. It was generally conjectured that when strong interactions were taken into account, they would provide the convergence factors necessary to make the semileptonic amplitudes finite, although Berman and Sirlin provided some qualitative arguments to the contrary.

Bjorken[4] pointed out that the assumption of Gell-Mann's current algebra postulate[3] implied that matrix elements of the exact hadronic currents behaved at large momenta like those of point particles, thus nullifying the above conjecture. Specifically, he showed that in the simple quark model with fractionally charged quarks, the second order (in e) radiative corrections to the vector part of the neutron β-decay amplitude at zero momentum transfer are logarithmically divergent, treating the strong interactions exactly. Abers, Norton, Dicus, and Quinn[5] generalized Bjorken's result and emphasized that certain contributions to the divergence depended only on the relatively model-independent commutators of the time components of the electromagnetic and weak currents. However, the divergent corrections as a whole were model-dependent, and several models for the commutators of the space components of the hadronic currents were constructed[6,7] so that the radiative corrections to neutron β-decay would be finite in second order.

Using basically the same techniques as Bjorken and Abers et al., Callan[9] and Preparata and Weisberger[10] generalized their work to include any semileptonic process. These authors did not restrict the discussion to only zero momentum transfer. Both papers considered only models constructed from renormalizable theories of strong interactions. They concluded that the models mentioned above, involving hadronic currents constructed from integrally charged quarks, gave finite second order radiative corrections to a general semileptonic process. Preparata and Weisberger further noted that currents containing bilinear products of spin zero fields yielded additional divergent corrections for non-zero momentum transfer.

Sirlin[8] and Abers et al.,[5] studied the second order radiative corrections to the vector part of μ-decay and neutron β- decay in a weak interaction theory mediated by a vector boson. They showed that, at zero momentum transfer, only a universal divergence occurred. That is, the divergent part was merely a constant factor times the uncorrected matrix element, and this factor was the same for both μ-decay and neutron β-decay. Furthermore, this result depended only on model independent current commutators involving time components of currents. For the weak boson theory, the order $-e^2$ corrections to G_A/G_V, the ratio of the axial vector and vector coupling constants in neutron β-decay, were shown by Sirlin[11] to be finite in the algebra of fields model, although this was not true in general. He used a technique which is very similar to ours although there are some differences in detail. In particular, our interpretation of electromagnetic mass shift contributions is somewhat different from his.

The tool which we shall use to discuss divergent radiative corrections is the expansion of time-ordered products of operators at large momenta in terms of equal-time commutators. The relevance of this technique to current algebra was first pointed out by Bjorken[4] and Johnson and Low.[15] All of the above mentioned papers at some point used this device. Many also employed Ward identities to handle external line wave function renormalization and to exhibit cancellations of certain divergent contributions. Since the end result is that the divergences involve highly model-dependent current commutators, we shall make this explicit by applying the Bjorken expansion in a straightforward manner. We shall assume the expansion is justified for time-ordered products containing an arbitrary number of currents.[(1)] For point particles there is no problem, but for the exact hadronic currents it is by no means obvious that this is a valid assumption. Nevertheless, we shall take it as our starting point without further ado, since it is certainly impossible to justify it rigorously.

A related assumption we shall make is that we may use the "naive" current commutation relations obtained from the canonical commutators and the equations of motion for any particular model of the hadronic currents. Our point of view here is that these models should not necessarily be taken seriously, but that perhaps the current algebra should be. For, if we assume that Bjorken expansion is valid, then the requirement that no divergences occur in the calculation of physically measurable quantities restricts the form that the current commutators may take. Several recent investigations[17-19] have shown that, when simple strong interaction models are treated in perturbation theory, the naive commutators no longer hold. We shall comment on this point in the conclusion.

In extending the results on second order radiative corrections to all orders in e^2, we shall see that our method allows us to isolate only those divergences due to momentum loops containing virtual photons. Thus, we ignore any divergences arising from momentum loop integrations in the hadron or lepton "blobs" which the photon lines enter. In fact, we <u>must</u> neglect any such divergences to be consistent with our use of naive commutation relations, as we discuss in the conclusion. The underlying physical assumption is that the basic hadronic and leptonic theory of matter, whatever it is, must be sufficiently convergent at high momenta that such divergences, if they occur at all, do not affect the current commutators.

Lest it be misunderstood, we should state that throughout this paper we shall use the manipulations of "naive quantum field theory". Thus, we ignore any singularities of local products of quantum field operators except those which are explicit in the use of equations of motion and canonical commutation relations. These ambiguities in local products of operators provide a possible escape from the divergence difficulties, but we are interested in the more conventional solutions to the problem.

In Section II we begin our discussion by considering second order electromagnetic radiative corrections. We first illustrate the Bjorken expansion for the time-ordered product of three operators which occurs there. We treat the hadron and lepton currents on the same footing so that no special weak process is singled out. An important part of the discussion of the divergent corrections to the weak coupling constant is the removal of divergent contributions to electromagnetic mass shifts and to radiative corrections to strong interaction parameters. We identify these terms and argue that they are removed by an appropriate counter term in the interaction Hamiltonian. Because of this, these divergences may be considered as a separate problem, which we shall not study here since it has been examined considerably by others.[20] In removing these contributions some care is required in making the covariant generalization of the Bjorken limit, an explicitly non-covariant procedure. This is illustrated for several models.

Next we discuss the possibility that the remaining divergences contribute only to a universal constant factor times the uncorrected matrix element. We show that this is not possible if there are operator (q-number) Schwinger terms in the current commutators by considering, as an example, currents constructed from a bilinear product of spin-zero fields. These terms produce, for non-zero momentum transfer, contributions which are manifestly not proportional to the lowest order matrix element.[10] If q-number Schwinger terms are absent, the current commutators involving time components of currents are the same for hadrons and leptons, independent of specific models. We point out that the divergent radiative corrections will then be universal if the commutators of the space components of the currents are also the same for hadrons as for leptons. Two models where this condition is satisfied are exhibited. These are the integrally charged quark models[6,7] mentioned above for hadrons together with the conventional lepton currents, and the algebra of fields model[21] for total electromagnetic and weak currents, including leptons, as proposed by T. D. Lee.[22] In the former model the radiative corrections are finite and in the latter a non-vanishing, but universal, divergence is found.[11,23]

We conclude Section II with a discussion of second order radiative corrections in a weak interaction theory mediated by a vector boson. We show that the positive result of Sirlin[8,11] and Abers _et al._,[5] is not maintained for non-zero momentum transfer. Non-universal divergent terms are found, and a counter term having a local current-current form would have to be added to the interaction Hamiltonian to make the radiative corrections finite.

We consider the generalization to these second order results to all orders in e^2 in Section III. After discussing the additional assumptions, we take up the two models which were satisfactory in second order with respect to universality. For the algebra of fields model we consider the fourth order calculation in some detail as an illustrative example. Here an important point to be mentioned is that in order to avoid ambiguities in making the covariant generalization of the Bjorken limit, we must let only one photon loop momentum go to infinity at a time holding all others fixed. This offers no problem since we are dealing with only logarithmically divergent integrals.

Then we show that, for the algebra of fields model, to any given order in e^2 the amplitude for any weak process may be expressed, once the divergent mass renormalization terms are removed, as a divergent constant factor times the finite part of the matrix element to the next lower order. This is precisely the condition for the divergences not to have any observable effects. It contributes simply to an overall rescaling of the weak coupling constant. In summing the series for all orders in e^2 the divergent coefficient in second order exponentiates.

Next we consider the quark model which led to finite radiative corrections in second order and show, examining the fourth order case in detail, that the corrections are finite to all orders. It is pointed out that this result could have been anticipated, knowing the same is true for μ-decay, since our technique is independent of any particular weak process.

In conclusion we present a critical discussion of our assumptions, in particular the use of naive commutation relations. We also point out the difficulties encountered in attempting to apply the Bjorken expansion to discuss divergences in non-renormalizable field theories. In the light of our results, we summarize the current status of the problem of radiative corrections to weak interactions.

An appendix examines certain details concerning divergent contributions to external line wave function renormalization.

II. SECOND ORDER ELECTROMAGNETIC RADIATIVE CORRECTIONS

We wish to show that the ultraviolet divergent part of the second order electromagnetic radiative corrections to leptonic and semileptonic weak processes may be expressed in terms of matrix elements of equal-time commutators of the weak Hamiltonian density $H_{wk}(x)$ and the electromagnetic current $\mathscr{J}_{\mu}^{e.m.}(x)$, and of the electromagnetic current with its first time derivative. The method we use was first proposed by Bjorken[4] and by Johnson and Low.[15] The specific technique we describe here was applied recently by Sirlin[11] to discuss divergences in radiative corrections to G_A/G_V, the ratio of the axial vector and vector coupling constants in neutron β-decay. We show here that it can be used to analyze divergences in any weak process at arbitrary momentum transfer. Although a direct test of universality of the weak coupling constant can only be made through measurements of leptonic and semileptonic weak processes, the discussion of divergences applies equally well to non-leptonic weak decays.

In the calculation of electromagnetic radiative corrections, infrared divergences also occur due to the massless nature of the photon. This difficulty is not serious and its resolution is well known,[24] so we shall not consider it here.

We shall throughout this paper work in the interaction representation with respect to electromagnetic and weak interactions. Weak interactions will be treated only in lowest order. In the absence of electromagnetic corrections, the amplitude for a leptonic or semileptonic weak process $A \longrightarrow B\ell\nu_\ell$ is

$$\mathscr{M}_0 = \langle B\ell\nu_\ell | H_{wk}(0) | A \rangle .$$

We assume that the weak Hamiltonian density has the usual local current-current form in the following discussion. Later we shall treat the case where the weak interaction is mediated by a vector boson.

The second order electromagnetic radiative corrections are given by

$$\mathcal{M}_1 = \frac{(-i)^2}{2!}\int d^4x \int d^4y \langle B\ell\nu_\ell | T\left\{H_{e.m.}(x), H_{e.m.}(y), H_{wk}(0)\right\}|A\rangle$$

where

$$H_{e.m.}(x) = e\mathcal{J}_\mu^{e.m.}(x)\mathcal{A}^\mu(x) .$$

Here $-e$ denotes the bare electron charge ($e > 0$) and $\mathcal{A}^\mu(x)$ is the electromagnetic field. The total electromagnetic current $\mathcal{J}_\mu^{e.m.}(x)$ consists of a hadron piece, denoted by $J_\mu^{e.m.}(x)$, a lepton part, denoted by $j_\mu^{e.m.}(x)$, and perhaps other terms for W-bosons, etc. The lepton current in conventional quantum electrodynamics is

$$j_\mu^{e.m.}(x) = - \left[\bar{e}(x)\gamma_\mu e(x) + \bar{\mu}(x)\gamma_\mu \mu(x)\right]$$

where $e(x)$ and $\mu(x)$ are the electron and muon fields, respectively. We do not want to restrict ourselves to only this form of the lepton current since we wish to consider, among others, a model in which the total electromagnetic current obeys algebra of fields commutation relations.[22] The electromagnetic Hamiltonian density may, in general, also contain contact terms which will affect the second order radiative corrections. For the moment we shall ignore this possibility, although we shall come back to it later when we consider a model of hadron currents constructed from a bilinear product of spin-zero fields. [Our notation for Dirac matrices and Lorentz four-vectors and scalar products follows that of Bjorken and Drell.[24]]

Properly, we should consider, rather than $\mathcal{M}_1$, the appropriate expression obtained from the reduction formula. The added complications arising from this do not affect the discussion of divergences in second order, as we show in an appendix. All matrix elements we write down will, of course, be understood to mean the connected part only.

A. The Bjorken Expansion

We may rewrite $\mathcal{M}_1$ above as

$$\mathcal{M}_1 = \frac{i}{2} e^2 \int \frac{d^4k}{(2\pi)^4} D^{\mu\nu}(k)\, T_{\mu\nu}(k)$$

where

$$D_{\mu\nu}(k) = \left(g_{\mu\nu} - \lambda \frac{k^\mu k^\nu}{k^2}\right) \frac{1}{k^2 + i\epsilon}$$

is the photon propagator in an arbitrary gauge, and

$$T_{\mu\nu}(k) = \int d^4x \int d^4y\, e^{-ik(x-y)} \langle B\ell\nu_\ell \left| T\left\{ \mathcal{J}_\mu^{e.m.}(x),\ \mathcal{J}_\nu^{e.m.}(y),\ H_{wk}(0) \right\} \right| A \rangle .$$

In order to find the logarithmically divergent contribution to $\mathcal{M}_1$ we need to know the part of $T_{\mu\nu}(k)$ which goes as $1/k^2$ for large k. To extract this part we take the limit as $k^0 \longrightarrow \infty$ with $\vec{k}$ fixed, since in this limit the behavior of time-ordered products (as opposed to covariant T* products) is particularly simple. Consider the following object X:

$$X \equiv \int d^4x \frac{d}{dx^0} \left\{ e^{-ikx} \int d^4y e^{iky} \langle B\ell\nu_\ell \left| T \left\{ \mathcal{J}_\mu^{e.m.}(x),\ \mathcal{J}_\nu^{e.m.}(y),\ H_{wk}(0) \right\} \right| A \rangle \right\} .$$

Our basic assumption is that, in the limit as $k^0 \longrightarrow \infty$ with $\vec{k}$ fixed, the above matrix element is sufficiently well behaved that the surface terms at $x^0 = \pm\infty$ may be ignored and hence that $X = 0$. If one inserts complete sets of states in the time-ordered product, one sees that this means that the high mass intermediate states must be unimportant so that the oscillations of the factor e^{-ikx} will dominate.

By taking the time derivatives d/dx_0 and setting $X = 0$, we obtain an expression for $T_{\mu\nu}(k)$:

$$T_{\mu\nu}(k) \xrightarrow[k^0 \longrightarrow \infty]{} \frac{i}{k^0} \left[L_{\mu\nu}^{(1)} + L_{\mu\nu}^{(2)}(k) + L_{\mu\nu}^{(3)}(k) \right]$$

where

$$L^{(1)}_{\mu\nu} = \int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \langle B\ell\nu_\ell \left| T\left\{ \left[\mathcal{J}^{e.m.}_\mu(x), \mathcal{J}^{e.m.}_\nu(y) \right] \delta(x_0 - y_0), H_{wk}(0) \right\} \right| A\rangle$$

$$L^{(2)}_{\mu\nu}(k) = \int d^4x \int d^4y\, e^{-ik(x-y)} \langle B\ell\nu_\ell \left| T\left\{ \left[\mathcal{J}^{e.m.}_\mu(x), H_{wk}(0) \right] \delta(x_0), \mathcal{J}^{e.m.}_\nu(y) \right\} \right| A\rangle$$

$$L^{(3)}_{\mu\nu}(k) = \int d^4x \int d^4y\, e^{-ik(x-y)} \langle B\ell\nu_\ell \left| T\left\{ \partial_0 \mathcal{J}^{e.m.}_\mu(x), \mathcal{J}^{e.m.}_\nu(y), H_{wk}(0) \right\} \right| A\rangle .$$

The equal-time commutators come, of course, from differentiating the θ-functions in the time-ordered product. Note that $L^{(1)}_{\mu\nu}$ is independent of k^0. Hence it is the leading term in the expansion of $T_{\mu\nu}(k)$.

We shall further assume that the above procedure can be continued with $L^{(2)}_{\mu\nu}(k)$ and $L^{(3)}_{\mu\nu}(k)$ to obtain sufficient terms in the expansion of $T_{\mu\nu}(k)$ in powers of $1/k^0$ to isolate the divergent contribution to $\mathcal{M}_1$. It should perhaps be mentioned that such manipulations are manifestly justified in renormalizable perturbation theory calculations for point particles. To assume that it is true for matrix elements of the exact currents is, needless to say, a very strong assumption, which we discuss further in the conclusion. However, by making this assumption we shall see that we can make some very general statements concerning the divergences.

Here it suffices to consider the $1/(k^0)^2$ terms. We insert a derivative d/dy^0 in $L^{(2)}_{\mu\nu}(k)$ and $L^{(3)}_{\mu\nu}(k)$ to obtain the following result for $T_{\mu\nu}(k)$:

$$T_{\mu\nu}(k) \xrightarrow[k^0\to\infty]{} \frac{i}{k^0} L^{(1)}_{\mu\nu} + \frac{1}{(k^0)^2}\left(M_{\mu\nu} + N_{\mu\nu}\right) + T^f_{\mu\nu}(k)$$

where

$$M_{\mu\nu} = \int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \langle B\ell\nu_\ell \left| T\left\{ \left[\mathcal{J}^{e.m.}_\nu(y), \partial_0 \mathcal{J}^{e.m.}_\mu(x) \right] \delta(x^0 - y^0), H_{wk}(0) \right\} \right| A\rangle$$

and

$$N_{\mu\nu} = \int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \langle B\ell\nu_\ell \Big| \Big[\mathscr{J}_\nu^{e.m.}(y), \big[\mathscr{J}_\mu^{e.m.}(x), H_{wk}(0) \big] \Big] \delta(x_0)\delta(y_0) \Big| A \rangle$$

are independent of k_0 since the equal-time δ-functions have eliminated the factors $e^{-ik^0 x_0}$. The term $T^f_{\mu\nu}(k)$ is still dependent on k^0 so we assume it goes as $1/(k^0)^{2+\epsilon}$, $\epsilon > 0$, as $k^0 \to \infty$ and hence leads to finite contributions to $\mathscr{M}_1$. $T^f_{\mu\nu}(k)$ contains three terms which are

$$T^f_{\mu\nu}(k) = \frac{1}{k_0^2} \int d^4x \int d^4y\, e^{-ik(x-y)} \langle B\ell\nu_\ell \Big| \Big[T\Big\{ \partial_0 \mathscr{J}_\mu^{e.m.}(x), \partial_0 \mathscr{J}_\nu^{e.m.}(y), H_{wk}(0) \Big\}$$

$$+ T\Big\{ \partial_0 \mathscr{J}_\mu^{e.m.}(x), \big[\mathscr{J}_\nu^{e.m.}(y), H_{wk}(0) \big] \delta(y^0) \Big\}$$

$$+ T\Big\{ \partial_0 \mathscr{J}_\nu^{e.m.}(y), \big[\mathscr{J}_\mu^{e.m.}(x), H_{wk}(0) \big] \delta(x^0) \Big\} \Big] \Big| A \rangle .$$

Clearly a sufficient condition for $T^f_{\mu\nu}(k)$ to go faster than $(1/k^0)^2$ is for an additional partial integration of a derivative to be justified, in which case it goes at least as $(1/k^0)^3$ as $k^0 \to \infty$.

The expansion above is, of course, explicitly not covariant. After taking the various time derivatives inside the time-ordered product so that they are acting on current operators, we may, by then expressing the currents as functions of field operators, use the equations of motion of these fields (in the interaction representation) to write $\partial_0 \mathscr{J}_\mu^{e.m.}(x)$ and higher time derivatives in terms of spatial derivatives $\partial/\partial x^i$ ($i = 1, 2, 3$) of fields. If we then continue the expansion in powers of $1/k^0$ we will thus obtain equal-time commutators which contain Schwinger terms,[25] i.e., spatial derivatives of four dimensional δ-functions. After partial integration these derivatives will give rise to factors of k^i. We implicitly assume that such Schwinger terms in higher terms in the expansion will yield the powers of k^i/k^0 necessary to maintain the covariance of $T_{\mu\nu}(k)$.

We extract the covariant amplitudes by the following procedure. After evaluating the equal-time commutators, all the terms in $T_{\mu\nu}(k)$ will have a tensor structure

$$\mathscr{T}_{\mu\nu\,00\ldots i\,j\ldots}^{\qquad\qquad i\,j\ldots} \;.$$

In the limit as $k_0 \longrightarrow \infty$ with $\vec{k}$ fixed, we have

$$\frac{k^\mu}{k^2} \longrightarrow g^{\mu 0}\,\frac{1}{k^0}\,, \quad \frac{k^\mu k^\nu}{k^2} \longrightarrow g^{\mu 0} g^{\nu 0}\,,$$

and

$$g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \longrightarrow g^{\mu k} g^{\nu}{}_{k} \;.$$

Hence we see that by reversing this limit we can construct the appropriate covariant tensor; e.g.,

$$k^0 k^0 \ldots \mathscr{T}_{\mu\nu 00\ldots i\,j\ldots}^{\qquad\quad i\,j\ldots}$$

$$\longrightarrow k^\rho k^\sigma \ldots \mathscr{T}_{\mu\nu\rho\sigma\ldots\,\alpha\beta\gamma\delta\ldots} \left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right)\left(g^{\gamma\delta} - \frac{k^\gamma k^\delta}{k^2}\right) \ldots \;.$$

Note that any terms containing a factor k^i will thus be eliminated since

$$k^i \mathscr{S}_i \longrightarrow \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) k_\mu \mathscr{S}_\nu = 0 \;.$$

B. "Mass" Renormalization Effects

We next wish to argue that when the proper covariant contributions to the terms $L^{(1)}_{\mu\nu}$ and $M_{\mu\nu}$ have been constructed, they will be cancelled by appropriate counter terms added to the interaction Hamiltonian to remove the divergent part of lepton mass renormalization and of electromagnetic renormalization effects (i.e., mass and coupling constant shifts) in strong interaction processes of the hadrons. In fact, the second order electromagnetic correction to any hadronic amplitude $\langle f\ \text{out} | i\ \text{in} \rangle$ is given by the expression

$$\frac{i}{2}\, e^2 \int \frac{d^4k}{(2\pi)^4}\, D^{\mu\nu}(k)\, t_{\mu\nu}(k)$$

where

$$t_{\mu\nu}(k) = \int d^4x \int d^4y\, e^{-ik(x-y)} \langle f\ \text{out} \,|\, T\left\{ J_\mu^{e.m.}(x),\ J_\nu^{e.m.}(y) \right\} |\, i\ \text{in} \rangle \,.$$

Making an expansion of $t_{\mu\nu}(k)$ in powers of $1/k^0$ analogous to the one for $T_{\mu\nu}(k)$, we find

$$t_{\mu\nu}(k) \xrightarrow[k^0 \to \infty]{} \frac{i}{k^0} \int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \langle f\ \text{out} \left| \left[J_\mu^{e.m.}(x),\ J_\nu^{e.m.}(y) \right] \delta(x^0-y^0) \right| i\ \text{in} \rangle$$

$$+ \left(\frac{1}{k^0}\right)^2 \int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \langle f\ \text{out} \left| \left[J_\nu^{e.m.}(y),\ \partial_0 J_\mu^{e.m.}(x) \right] \delta(x^0-y^0) \right| i\ \text{in} \rangle + t^f_{\mu\nu}(k)$$

where $t^f_{\mu\nu}(k)$ contains the finite part of the radiative corrections. The first two terms we see will give divergent contributions to these processes. These divergences must be cancelled by adding counter terms to the interaction Hamiltonian. Since the operators

$$\left[J_\mu^{e.m.}(x),\ J_\nu^{e.m.}(y) \right] \delta(x^0-y^0) \text{ and } \left[J_\nu^{e.m.}(y),\ \partial_0 J_\mu^{e.m.}(x) \right] \delta(x^0-y^0)$$

are completely determined by their matrix elements between arbitrary hadronic states, we may choose the counter terms to contain precisely these operators.

These divergent counter terms are not directly relevant to universality of the weak interactions; however, for the theory of radiative corrections to be completely consistent we would have to show that they did not contribute divergences to the calculation of any observable hadronic parameters, such as electromagnetic mass differences. This is a difficult problem in itself, but since it has been discussed considerably by others[20] we shall not consider it here.

This same argument can obviously be applied to the lepton part of the electromagnetic current. In the case where the leptons are just free fields in the interaction representation, the necessary counter term is just the divergent part of the electromagnetic mass shift of the leptons. For a model where the total electromagnetic current obeys a field-current identity the counter term will, in general, contain other contributions. In discussing such models, we shall for simplicity refer to these additional contributions as "strong interaction" effects.

One should note that the counter terms we are adding remove only certain divergent terms. We do not make all the subtractions necessary to carry out the renormalization program, i.e., to express everything in terms of physical masses and coupling constants. In fact, using our technique this would be very unwieldy. We specifically do not want to add wave function renormalization counter terms, since this would make the question of using the equations of motion to evaluate time derivatives of operators a very delicate one. We shall need to use equations of motion in the interaction representation to calculate certain commutators.

There is a difficulty with the above argument which is related to the non-covariance of our procedure. This is best illustrated by considering lepton currents in conventional quantum electrodynamics, where the fields satisfy free field equations of motion.[2] First, however, observe that the photon propagator in an arbitrary gauge

$$D_{\mu\nu}(k) = \left(g_{\mu\nu} - \lambda \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{k^2 + i\epsilon}$$

becomes, in the limit as $k^0 \longrightarrow \infty$ with $\vec{k}$ fixed,

$$D_{\mu\nu} \xrightarrow[k_0 \to \infty]{} \left[(1-\lambda)\, g_{\mu 0} g_{\nu 0} + g_{\mu k} g_\nu^{\ k}\right] \frac{1}{k_0^2 + i\epsilon} \ .$$

Referring to $\mathcal{M}_1$, we see that we need consider only the combinations

$$(1-\lambda)L_{00}^{(1)} + L_i^{(1)\,i} \quad \text{and} \quad (1-\lambda)\,M_{00} + M_i^{\,i}\ .$$

Since the electromagnetic current commutes with itself, $L_{00}^{(1)} = L_i^{(1)\,i} = 0$ in the general case, so we may henceforth ignore $L_{\mu\nu}^{(1)}$.

The usual lepton electromagnetic current is

$$j_\mu^{e.m.}(x) = -\,\overline{e}(x)\gamma_\mu e(x) - \overline{\mu}(x)\gamma_\mu \mu(x)$$

where the electron and muon fields e(x) and $\mu(x)$ respectively obey equations of motion

$$\left(i\gamma^\mu\partial_\mu - m_e\right)e(x) = 0 \quad \text{and} \quad (i\gamma^\mu\,\partial_\mu - m_\mu)\,\mu(x) = 0$$

and the canonical anticommutation relations,

$$\left\{e^\dagger(x'),\ e(x)\right\}\delta(x^0 - x'^0) = \left\{\mu^\dagger(x'),\ \mu(x)\right\}\delta(x^0 - x'^0) = \delta^4(x - x')$$

$$\left\{e(x'),\ e(x)\right\}\delta(x^0 - x'^0) = \left\{\mu(x'),\ \mu(x)\right\}\delta(x^0 - x'^0) = 0\ .$$

By using current conservation

$$\partial^\mu\, j_\mu^{e.m.}(x) = 0$$

we see that

$$\left[j_0^{e.m.}(y),\ \partial_0 j_0^{e.m.}(x)\right]\delta(x^0 - y^0) = 0$$

provided we use the naive commutation relations and ignore the well-known ambiguity[25] of the Schwinger term in $\left[j_0^{e.m.}(y),\ j_i^{e.m.}(x)\right]\delta(x^0 - y^0)$. We shall make the conventional assumption that it is a c-number and therefore does not contribute to connected amplitudes. Thus $M_{00} = 0$, so there are no gauge dependent contributions to the divergent mass shift, as must be the case.

Using the above anticommutation relations and the equations of motion, it is easily verified, after a short calculation, that

$$\left[j^i_{e.m.}(y),\ \partial_0 j_i^{e.m.}(x)\right]\delta(x^0-y^0) = 12i\ m_e\ \bar{e}(x)e(x)\ \delta^4(x-y)$$

$$-4\left[\bar{e}(y)\ \gamma^i e(x) - \bar{e}(x)\gamma^i e(y)\right]\frac{\partial}{\partial x^i}\ \delta^4(x-y) + (e\rightarrow\mu)\ .$$

We note that the first term already has the form of a mass counter term; however, we must construct the covariant generalization of the second term, which requires some care. The leptonic part of M_i^i we denote by $M_i^{(\ell)\,i}$. It may now be written as

$$M_i^{(\ell)\,i} = \int d^4x\int d^4y e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\langle B\ell\nu_\ell\left|\left[12i\,m_e\ \delta^4(x-y)\,T\left\{\bar{e}(x)\,e(x),\ H_{wk}(0)\right\}\right.\right.$$

$$\left.\left.-4\ \frac{\partial}{\partial x^i}\ \delta^4(x-y)\,T\left\{\bar{e}(y)\gamma^i e(x) - \bar{e}(x)\gamma^i e(y),\ H_{wk}(0)\right\} + (e\rightarrow\mu)\right]\right|A\rangle\ .$$

The covariant generalization of the second term in $M_i^{(\ell)\,i}$ is

$$-4\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\int d^4x\int d^4y e^{-ik(x-y)}\ \frac{\partial}{\partial x^\nu}\ \delta^4(x-y)$$

$$\times\ \langle B\ell\nu_\ell\left|T\left\{\bar{e}(y),\ \gamma_\mu\, e(x),\ H_{wk}(0)\right\} - T\left\{\bar{e}(x),\ \gamma_\mu\, e(y),\ H_{wk}(0)\right\}\right|A\rangle$$

which reduces to the original expression when $k^0\rightarrow\infty$. We now perform a partial intergration with $\partial/\partial x^\nu$ to put the derivative on e(x) and $\bar{e}(x)$. Differentiating the exponential gives a term proportional to k_ν, which vanishes since

$$k_\nu\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) = 0\,.$$

Also, realize that under symmetrical k-integration we may make the replacement

$$g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\longrightarrow\frac{3}{4}\,g^{\mu\nu}\ .$$

We are thus led to the covariant generalization of $M_i^{(\ell)\,i}$, which is

$$M^{(\ell)}{}_{\mu}{}^{\mu} = \int d^4x \int d^4y\, \delta^4(x-y) \langle B\ell\nu_\ell \Big| \Big[12_i\, m_e T\Big\{\bar{e}(x)\, e(x),\ H_{wk}(0)\Big\}$$

$$+3\, \frac{\partial}{\partial x^\mu}\Big(T\Big\{\bar{e}(y),\ \gamma^\mu e(x),\ H_{wk}(0)\Big\} - T\Big\{\bar{e}(x),\ \gamma^\mu e(y),\ H_{wk}(0)\Big\}\Big) + (e \to \mu)\Big] \Big| A \rangle .$$

In taking the derivative $\partial/\partial x^\mu$ inside the time-ordered product we pick up additional equal-time commutators from differentiating the θ-functions. The terms involving the anticommutator

$$\Big\{ e^\dagger(x),\ e(y)\Big\}\, \delta(x^0 - y^0)$$

actually cancel, but in any case they do not contribute to the connected amplitude since they are c-numbers. One thus obtains

$$M^{(\ell)}{}_{\mu}{}^{\mu} = \delta M^{(\ell)}{}_{\mu}{}^{\mu} + \overline{M}^{(\ell)}{}_{\mu}{}^{\mu}$$

where

$$\delta M^{(\ell)}{}_{\mu}{}^{\mu} = 6i \int d^4x \langle B\ell\nu_\ell \Big| T\Big\{ m_e \bar{e}(x)\, e(x) + m_\mu \bar{\mu}(x)\, \mu(x),\ H_{wk}(0)\Big\} \Big| A \rangle$$

and

$$\overline{M}^{(\ell)}{}_{\mu}{}^{\mu} = -3 \int d^4x\, \delta(x^0) \langle B\ell\nu_\ell \Big| \Big\{ \Big[e^\dagger(x),\ H_{wk}(0)\Big] e(x) + e^\dagger(x) \Big[H_{wk}(0),\ e(x)\Big] + (e \to \mu)\Big\} \Big| A \rangle$$

after using the equations of motion.

The careful reader will have noticed that the above manipulations were somewhat of a swindle. If we had performed the partial integration with $\partial/\partial x^i$ before making the covariant generalization, we would not have obtained the term $\overline{M}^{(\ell)}{}_{\mu}{}^{\mu}$. It is easily verified that if we combine $M^{(\ell)}{}_{\mu}{}^{\mu}$ with the contribution of $N_{\mu\nu}$ to $\mathcal{M}_1$, we obtain a result in agreement with a manifestly covariant Feynman diagram calculation only if the term $\overline{M}^{(\ell)}{}_{\mu}{}^{\mu}$ is included. We shall therefore assume this method of handling these terms is also valid in the general case when strong interactions are present. This seems very reasonable as time ordered products not containing operator derivatives are presumably less singular objects.

The contribution of $M^{(\ell)\,\mu}_{\mu}$ to $\mathcal{M}_1$ is

$$\frac{i}{2}\, e^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k^2+i\epsilon)^2}\, \left[\delta M^{(\ell)\,\mu}_{\mu} + \overline{M}^{(\ell)\,\mu}_{\mu}\right] .$$

We recognize $\delta M^{(\ell)\,\mu}_{\mu}$ as simply the divergent part of the lepton electromagnetic mass shift. It is removed by the appropriate counter term. However, $\overline{M}^{(\ell)}{}_{\mu}{}^{\mu}$ contributes to weak coupling constant renormalization. We note that whenever the commutator

$$\left[j^i_{e.m.}(y), \partial_0 j_i^{e.m.}(x)\right] \delta(x^0 - y^0)$$

contains q-number Schwinger terms, $\overline{M}^{(\ell)\,\mu}_{\mu}$ will in general be non-vanishing.

Turning now to the hadron part of the electromagnetic current, we may perform a completely analogous calculation if the currents are constructed from spin-1/2 fields. In the simplest quark model[26] the hadronic electromagnetic current is

$$J^{e.m.}_{\mu}(x) = V^3_{\mu}(x) + \frac{1}{\sqrt{3}}\, V^8_{\mu}(x) = \bar{q}(x)\, Q_h \gamma_{\mu} q(x)$$

where the octet of vector currents is given by

$$V^a_{\mu}(x) = \bar{q}\,(x)\, \frac{1}{2}\lambda^a \gamma_{\mu}\, q(x) \qquad a = (1, \ldots, 8)$$

in terms of the quark fields q(x). Here the λ matrices and the indices (a) use the conventional SU(3) notation (in Adler and Dashen,[27] for example). We have denoted the hadronic charge matrix by Q_h. If the quark fields obey an equation of motion

$$i\gamma^{\mu} \partial_{\mu}\, q(x) = F(x)\, q(x)$$

for some unspecified[3] hadronic operator F(x), we may repeat the above calculation for leptons, to obtain the hadronic contributions to $\delta M^{(h)}{}_{\mu}{}^{\mu}$ and $\overline{M}^{(h)}{}_{\mu}{}^{\mu}$ which are

$$\delta M^{(h)}{}_{\mu}{}^{\mu} = 6i\int d^4x \, \langle B\ell\nu_\ell \left| T\left\{ \bar{q}(x) \frac{1}{2} \left\{ Q_h^2, F(x) \right\} q(x), H_{wk}(0) \right\} \right| A \rangle$$

and

$$\overline{M}^{(h)}{}_{\mu}{}^{\mu} = -3\int d^4x \, \delta(x^0) \langle B\ell\nu_\ell \left| \left\{ \left[q^\dagger(x), H_{wk}(0) \right] Q_h^2 q(x) + q^\dagger(x) Q_h^2 \left[H_{wk}(0), q(x) \right] \right\} \right| A \rangle .$$

To make the analogy more explicit, we may define lepton fields

$$\psi_e(x) = \begin{pmatrix} \nu_e(x) \\ e(x) \end{pmatrix} \qquad \psi_\mu(x) = \begin{pmatrix} \nu_\mu(x) \\ \mu(x) \end{pmatrix}$$

and charge matrices

$$Q_e = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad Q_\mu = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then the lepton expression for $\overline{M}^{(\ell)}{}_{\mu}{}^{\mu}$ takes the form

$$\overline{M}^{(\ell)}{}_{\mu}{}^{\mu} = -3\int d^4x \, \delta(x^0) \langle B\ell\nu_\ell \Big| \Big\{ \left[\psi_e^\dagger(x), H_{wk}(0) \right] Q_e^2 \psi_e(x) + \psi_e^\dagger(x) Q_e^2 \left[H_{wk}(0), \psi_e(x) \right] + (e \longrightarrow \mu) \Big\} \Big| A \rangle$$

It is very instructive to consider as a further example currents constructed from spin-zero bosons, as occurs in a perturbation theory calculation of radiative corrections to pion decays. It provides an example where q-number Schwinger terms appear. Suppose then that the octet of vector currents $V_\mu^a(x)$ is constructed from a bilinear product of scalar fields $\phi_a(x)$, $a = (1, \ldots, 8)$. They have the form

$$V_\mu^a(x) = f^{abc} \phi_b(x) \partial_\mu \phi_c(x) .$$

The scalar fields obey the canonical commutation relations

$$\left[\partial_0 \phi_a(x), \phi_b(y) \right] \delta(x^0 - y^0) = -i\delta_{ab} \delta^4(x - y)$$

$$\left[\phi_a(x), \phi_b(y) \right] \delta(x^0 - y^0) = \left[\partial_0 \phi_a(x), \partial_0 \phi_b(y) \right] \delta(x^0 - y^0) = 0.$$

The electromagnetic current is, as usual,

$$J_\mu^{e.m.}(x) = V_\mu^3(x) + \frac{1}{\sqrt{3}} V_\mu^8(x) .$$

As is well known,[24] the electromagnetic interaction Hamiltonian of spin-zero fields contains a contact term. If we assume a model hadronic Lagrangian density

$$\mathscr{L}_h(x) = \frac{1}{2} \partial^\mu \phi^a(x) \partial_\mu \phi^a(x) - \frac{1}{2} F_{ab}(x) \phi^a(x) \phi^b(x)$$

where $F_{ab}(x)$ is free of operator derivatives but otherwise unspecified, then the minimal electromagnetic substitution

$$\partial_\mu \phi^a(x) \longrightarrow \partial_\mu \phi^a(x) + e\left(f^{3ab} + \frac{1}{\sqrt{3}} f^{8ab}\right) \mathscr{A}_\mu(x) \phi^b(x)$$

yields, in the usual fashion, the interaction representation Hamiltonian density $H = H_0 + H_{e.m.}$, where

$$H_{e.m.}(x) = eJ_\mu^{e.m.}(x) \mathscr{A}^\mu(x) - \frac{1}{2} e^2 C^{ab} \phi_a(x) \phi_b(x) \mathscr{A}^i(x) \mathscr{A}_i(x) .$$

Here $\mathscr{A}_\mu(x)$ is the electromagnetic potential, $J_\mu^{e.m.}(x)$ is defined above, and

$$C_{ab} \equiv \left(f^{3ac} + \frac{1}{\sqrt{3}} f^{8ac}\right)\left(f^{3bc} + \frac{1}{\sqrt{3}} f^{8bc}\right) .$$

Note that if we write

$$J_\mu^{e.m.}(x) = i\phi^a(x) Q_h^{ab} \partial_\mu \phi^b(x) ,$$

where the hadronic charge matrix Q_h is

$$Q_h^{ab} = -i\left(f^{3ab} + \frac{1}{\sqrt{3}} f^{8ab}\right) ,$$

then C_{ab} is simply

$$C_{ab} = \left(Q_h^2\right)^{ab} .$$

In discussing second order radiative corrections we must in this case consider

$$\mathcal{M}_1 = \frac{i}{2} e^2 \int \frac{d^4k}{(2\pi)^4} D^{\mu\nu}(k) \left[T_{\mu\nu}(k) + T^c_{\mu\nu} \right]$$

where the additional term $T^c_{\mu\nu}$ is given by

$$T^c_{\mu\nu} = g_{\mu k} g_\nu{}^k \int d^4x \langle B \ell \nu_\ell \left| T \left\{ C^{ab} \phi_a(x) \phi_b(x), H_{wk}(0) \right\} \right| A \rangle .$$

$T^c_{\mu\nu}$ will give a quadratically divergent contribution to $\mathcal{M}_1$. However, we immediately note that it will be cancelled by the mass counter term since it is only a "tadpole" contribution. Of course, in higher orders this will no longer be the case.

We are now ready to calculate the commutators in $M_{\mu\nu}$. Using current conservation and the canonical commutation relations one finds

$$\left[J_0^{e.m.}(y), \partial_0 J_0^{e.m.}(x) \right] \delta(x^0 - y^0) = -i C^{ab} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^i} \left[\phi_a(x) \phi_b(x) \delta^4(x-y) \right] .$$

The commutator $\left[J^i, \partial_0 J_i \right]$ may be calculated without using the equations of motion, simply from the canonical commutation relations. After a brief calculation we find that it may be written as

$$\left[J^i_{e.m.}(y), \partial_0 J_i^{e.m.}(x) \right] \delta(x^0 - y^0) = i C^{ab} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^i} \left[\phi_a(x) \phi_b(y) \delta^4(x-y) \right]$$

$$+ 2i C_{ab} \left(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i} \right) \left[\phi_a(x) \phi_b(y) \frac{\partial}{\partial x^i} \delta^4(x-y) \right]$$

$$- 4i C^{ab} \phi_a(x) \phi_b(y) \frac{\partial^2}{\partial x^i \partial x_i} \delta^4(x-y) .$$

By combining these two commutators we obtain for $(1-\lambda) M_{00}^{(h)} + M^{(h)\,i}_{\;\;i}$,

$$(1-\lambda) M_{00}^{(h)} + M^{(h)\,i}_{\;\;i} = \int d^4y x \int d^4 e^{i\vec{k}\cdot(\vec{x}-\vec{y})} iC_{ab} \langle B\ell\nu_\ell | X_{ab} | A\rangle ,$$

where

$$X_{ab} = \lambda \frac{\partial}{\partial x^i}\frac{\partial}{\partial y_i}\left[\delta^4(x-y)\, T\left\{\phi_a(x)\phi_b(x), H_{wk}(0)\right\}\right]$$

$$+2\left(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i}\right)\left[\frac{\partial}{\partial x^i}\ \delta^4(x-y)\, T\left\{\phi_a(x)\ \phi_b(y), H_{wk}(0)\right\}\right]$$

$$-4\frac{\partial^2}{\partial x^i \partial x_i}\ \delta^4(x-y)\, T\left\{\phi_a(x)\ \phi_b(y), H_{wk}(0)\right\} .$$

In the first two of these terms we find, after partial integrations, that they are respectively proportional to $k^i k_i$ and k^i. In making the covariant generalization they do not contribute since

$$k^i \longrightarrow k_\nu \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) = 0 .$$

Note that this eliminates the gauge dependent term, as expected.

The Schwinger term in the third expression gives rise to the same ambiguity that we noted in the spin-1/2 field example when we make the covariant generalization. Again, by referring to the Feynman diagram calculation in the free field case when $F_{ab}(x) = \delta_{ab} m^2$, we find that we must use the same prescription for handling this term in order to get the correct answer for the divergent part of the radiative corrections. Realizing this, it is straightforward to show that the covariant generalization of $(1-\lambda)M_{00}^{(h)} + M^{(h)\,i}_{\;\;i}$ is

$$M^{(h)\,\mu}_{\;\;\mu} = \delta M^{(h)\,\mu}_{\;\;\mu} + \overline{M}^{(h)\,\mu}_{\;\;\mu}$$

where

$$\delta M^{(h)\,\mu}_{\;\;\mu} = 3i\int d^4x \langle B\ell\nu_\ell | T\left\{C^{ab}\phi_a(x) F_{bc}(x)\phi_c(x),\ H_{wk}(0)\right\} | A\rangle$$

and

$$\overline{M}^{(h)\,\mu}_{\mu} = 3i\int d^4x\,\delta(x^0)\,\langle B\ell\nu_\ell\Big|\Big\{\Big[\phi_a(x),\,H_{wk}(0)\Big]\Big(Q_h^2\Big)^{ab}\partial_0\phi_b(x) + \phi_a(x)\Big(Q_h^2\Big)^{ab}\Big[H_{wk}(0),\,\partial_0\phi_b(x)\Big]\Big\}\Big|A\rangle\,.$$

In writing these expressions we have used the equations of motion

$$\partial^\mu\partial_\mu\,\phi^a(x) = -F^{ab}(x)\phi^b(x)$$

and have dropped a term involving the c-number commutator

$$\Big[\phi_b(y),\,\partial_0\,\phi_a(x)\Big]\,\delta(x^0-y^0)$$

since it contributes only to the disconnected piece.

The expression for $\overline{M}^{(h)\,\mu}_{\mu}$ has a familiar structure. To make this even more explicit we note that in both the spin-1/2 and spin-zero field examples, the charge density operator had the form

$$\mathscr{J}_0^{e.m.}(x) = -i\,\Pi(x)\,Q\phi(x)$$

where the $\phi(x)$'s were the canonical fields, the $\Pi(x)$'s the canonically conjugate momenta, and Q the appropriate charge matrix. For this type of current, we found

$$\overline{M}_\mu^{\ \mu} = -3i\int d^4x\,\delta(x^0)\,\langle B\ell\nu_\ell\Big|\Big\{\Big[\Pi(x),\,H_{wk}(0)\Big]Q^2\phi(x) + \Pi(x)Q^2\Big[H_{wk}(0),\,\phi(x)\Big]\Big\}\Big|A\rangle\,.$$

A final type of model we shall consider is the algebra of fields.[21] This is obtained by constructing a model Lagrangian which yields a field-current identity,[28] i.e.,

$$V_\mu^a\,(x) = \frac{m_0^2}{g_0}\;\phi_\mu^a(x) \qquad a=(1,\,\ldots,\,8).$$

The massive vector fields $\phi_\mu^a(x)$ have a bare mass m_0 and a Yang-Mills[29] type self-interaction with a bare coupling constant g_0. Such a field-current identity can hold for the total electromagnetic and weak currents, as proposed by T. D. Lee.[22] In this model, the commutation relations relevant to the calculation of $M^{(h)}_{\mu\nu}$ are

$$\left[J_0^{e.m.}(y), \partial_0 J_0^{e.m.}(x)\right] \delta(x^0 - y^0) = \frac{-4}{3} \frac{i}{m_0^2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x_i} \delta^4(x-y)$$

and

$$\left[J^i_{e.m.}(y), \partial_0 J_i^{e.m.}(x)\right] \delta(x^0 - y^0) = -i \frac{4}{3} \left(g_i^{\ i} - \frac{1}{m_0^2} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x^i}\right) \delta^4(x-y)$$

$$+ i \frac{g_0^2}{m_0^2} C^{ab} V_i^a(x) V_b^i(x) \delta^4(x-y) .$$

Here, as usual, the hadronic electromagnetic current is

$$J_\mu^{e.m.}(x) = V_\mu^3(x) + \frac{1}{\sqrt{3}} V_\mu^8(x) ,$$

and also

$$C_{ab} = \left(f^{3ac} + \frac{1}{\sqrt{3}} f^{8ac}\right) \left(f^{3bc} + \frac{1}{\sqrt{3}} f^{8bc}\right) .$$

The c–number parts of these commutators do not contribute to connected amplitudes. The remaining term does not contain spatial derivatives of operators, so $\overline{M}^{(h)\,\mu}_{\ \ \mu} = 0$. The covariant contribution to $M^{(h)\,\mu}_{\ \ \mu}$ is thus simply

$$\delta M^{(h)\,\mu}_{\ \ \mu} = \frac{3}{4} i \frac{g_0^2}{m_0^2} C^{ab} \int d^4x \langle B\ell\nu_\ell \left| T\left\{V^a_\mu(x) V^\mu_b(x), H_{wk}(0)\right\}\right| A\rangle$$

and is completely cancelled by the mass counter term.

C. Divergent Corrections and Universality

We have just seen how to consistently remove the divergent part of mass renormalization and of electromagnetic renormalization effects in strong interaction processes. The remaining contribution to $\mathcal{M}_1$, which we denote by $\widetilde{\mathcal{M}}_1$, still contains divergent terms. We wish to inquire in which current algebra models does the divergent part of $\tilde{\mathcal{M}}_1$ give only a universal weak coupling constant renormalization. This divergent part $\tilde{\mathcal{M}}_1^{\text{div.}}$ is

$$\tilde{\mathcal{M}}_1^{\text{div.}} = \frac{i}{2} e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+i\epsilon)^2} \left[\overline{M}_\mu^{\ \mu} + \left(g^{\mu\nu} - \lambda \frac{k^\mu k^\nu}{k^2} \right) N_{\mu\nu} \right] .$$

Here $\overline{M}_\mu^{\ \mu}$ are the model-dependent contributions from $M_{\mu\nu}$ which we found above, and

$$N_{\mu\nu} = \int d^4x \int d^4y \, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \langle B \ell \nu_\ell \Big| \Big[\mathcal{J}_\nu^{e.m.}(y) \Big[\mathcal{J}_\mu^{e.m.}(x), H_{wk}(0) \Big] \Big] \delta(x^0)\,\delta(y^0) \Big| A \rangle .$$

The condition for universal divergences in second order is

$$\tilde{\mathcal{M}}_1^{\text{div.}} = \gamma \int \frac{d^4k}{(k^2+i\epsilon)^2} \mathcal{M}_0$$

where γ is some constant independent of the particular process and $\mathcal{M}_0$ is the uncorrected matrix element.

We first consider the case of a local current-current Hamiltonian density

$$H_{wk}(x) = \frac{G}{\sqrt{2}} \mathcal{J}_\rho^{wk(+)}(x) \, \mathcal{J}_{wk}^{\rho(-)}(x)$$

with

$$\mathcal{J}_\rho^{wk(+)}(x) = J_\rho^+(x) + j_\rho^+(x), \qquad \mathcal{J}_\rho^{wk(-)}(x) = \left(\mathcal{J}_\rho^{wk(+)}(x) \right)^\dagger .$$

Here $J_\rho^\pm(x)$ and $j_\rho^\pm(x)$ are the hadronic and leptonic weak currents, respectively. We assume the hadronic current has the usual Cabibbo[2] form

$$J_\rho^+(x) = \left[J_\rho^1(x) + i\, J_\rho^2(x) \right] \cos\theta + \left[J_\rho^4(x) + i\, J_\rho^5(x) \right] \sin\theta$$

where θ is the Cabibbo angle and the SU(3) indices (1, 2, 4, 5) use the conventional notation. The weak currents $J_\rho^a(x)$ all have a V-A structure

$$J_\rho^a(x) = V_\rho^a(x) - A_\rho^a(x) \qquad a = (1,2,4,5) .$$

Following Gell-Mann,[3] we assume that the normalization of the octets of hadronic vector and axial vector currents is fixed by requiring the space integrals of their time components

$$F^a = \int d^3x\, V_0^a(x) \qquad F_5^a = \int d^3x\, A_0^a(x) \qquad a = (1, \ldots, 8)$$

to generate, at equal times, an SU(3) × SU(3) algebra. This implies, apart from Schwinger terms (which we abbreviate as S. T.), the current algebra

$$\left[V_0^a(x),\ V_\mu^b(y)\right] \delta(x^0 - y^0) = \left[A_0^a(x),\ A_\mu^b(y)\right] \delta(x^0 - y^0)$$

$$= i\, f^{abc}\, V_\mu^c(x)\, \delta^4(x - y) + \text{S.T.}$$

$$\left[A_0^a(x),\ V_\mu^b(y)\right] \delta(x^0 - y^0) = \left[V_0^a(x),\ A_\mu^b(y)\right] \delta(x^0 - y^0)$$

$$= i\, f^{abc}\, A_\mu^c(x)\, \delta^4(x-y) + \text{S.T.}$$

The appropriate commutators of the electromagnetic and weak hadronic currents are thus

$$\left[J_0^{e.m.}(x),\ J_\mu^\pm(y)\right] \delta(x^0 - y^0) = \left[J_\mu^{e.m.}(x),\ J_0^\pm(y)\right] \delta(x^0 - y^0)$$

$$= \pm\, J_\mu^\pm(x)\ \delta^4(x - y) + \text{S.T.}$$

The leptonic weak current $j_\rho^+(x)$ consists of electron and muon pieces

$$j_\rho^+(x) = j_\rho^{+(e)}(x) + j_\rho^{+(\mu)}(x)\,.$$

Throughout we shall assume ($\mu \longleftrightarrow e$) symmetry, which, of course, guarantees ($\mu \longleftrightarrow e$) universality. With the usual point interaction the lepton currents are

$$j_\rho^{+(e)}(x) = \overline{\nu}_e(x)\gamma_\rho(1 - \gamma_5)\, e(x)$$

$$j_\rho^{+(\mu)}(x) = \overline{\nu}_\mu(x)\gamma_\rho(1 - \gamma_5)\, e(x).$$

As mentioned before, we do not restrict ourselves to this model, although we shall assume that the leptonic electromagnetic and weak currents obey the usual commutation relations

$$\left[j_0^{e.m.}(x),\, j_\mu^{\pm}(y)\right]\delta(x^0 - y^0) = \left[j_\mu^{e.m.}(x),\, j_0^{\pm}(y)\right]\delta(x^0 - y^0)$$
$$= \pm\, j_\mu^{\pm}(x)\,\delta^4(x-y) + \text{S.T.}$$

With these preliminaries out of the way, we may now begin our discussion of the divergent corrections, contained in $\tilde{\mathscr{M}}_1^{\text{div.}}$. We start with a treatment of the case where the hadronic currents are constructed from a bilinear product of spin-zero fields. This example illustrates the difficulty with theories with q-number Schwinger terms. We must now include both scalar fields $s_a(x)$ and pseudoscalar fields $p_a(x)$, with vector and axial vector currents respectively given by

$$V_\mu^a(x) = f^{abc}\left[s_b(x)\partial_\mu s_c(x) + p_b(x)\partial_\mu p_c(x)\right]$$

and

$$A_\mu^a(x) = d^{abc}\left[s_b(x)\partial_\mu p_c(x) - p_b(x)\partial_\mu s_c(x)\right].$$

The non-vanishing canonical commutators are

$$\left[\partial_0 s_a(x),\, s_b(y)\right]\delta(x^0 - y^0) = \left[\partial_0 p_a(x),\, p_b(y)\right]\delta(x^0 - y^0) = -i\,\delta_{ab}\,\delta^4(x-y).$$

Using these one easily obtains

$$\left[J_0^{e.m.}(x),\ J_0^{\pm}(y)\right]\delta(x^0-y^0) = \pm J_0^{\pm}(x)\ \delta^4(x-y)$$

$$\left[J_0^{e.m.}(x),\ J_i^{\pm}(y)\right]\delta(x^0-y^0) = \pm J_i^{\pm}(x)\,\delta^4(x-y) + i\,\frac{\partial}{\partial x^i}\left[\mathscr{S}^{\pm}(x)\,\delta^4(x-y)\right]$$

$$\left[J_i^{e.m.}(x),\ J_0^{\pm}(y)\right]\delta(x^0-y^0) = \pm J_i^{\pm}(x)\,\delta^4(x-y) - i\,\frac{\partial}{\partial y^i}\left[\mathscr{S}^{\pm}(x)\,\delta^4(x-y)\right]$$

where $\mathscr{S}^{\pm}(x) \equiv \left[\mathscr{S}^1(x) \pm i\mathscr{S}^2(x)\right]\ \cos\theta + \left[\mathscr{S}^4(x) \pm i\mathscr{S}^5(x)\right]\ \sin\theta$ and

$$\mathscr{S}^a(x) = f^{abc}\left(f^{3bd} + \frac{1}{\sqrt{3}}\ f^{8bd}\right)\left[s_c(x)\ s_d(x) + p_c(x)\ p_d(x)\right]$$

$$+ d^{abc}\left(f^{3bd} + \frac{1}{\sqrt{3}}\ f^{8bd}\right)\left[\ s_c(x)\ p_d(x) - p_c(x)\ s_d(x)\right].$$

Note also that the commutators of the spatial components of the currents vanish; in particular

$$\left[J_i^{e.m.}(x),\ J_j^{\pm}(y)\right]\delta(x^0-y^0) = 0.$$

We wish to show for this model that the presence of q-number Schwinger terms in the current commutators destroys any possibility that the divergences give simply a universal constant factor. For this purpose, let us consider a semileptonic process, i.e., choose $|A\rangle$ and $|B\rangle$ to be hadronic states. $\overline{M}_{\mu}^{\ \mu}$ clearly does not involve Schwinger terms so we need not consider it further in this model. In the remaining term $N_{\mu\nu}$, we need to calculate $(1-\lambda)N_{00} + N_i^{\ i}$ in the limit as $k^0 \longrightarrow \infty$. It easily verified that the gauge dependent term N_{00} actually vanishes. For a semileptonic process $N_i^{\ i}$ is given by either

$$N_i^{\ i} = \frac{G}{\sqrt{2}}\int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\langle B\ell\nu_\ell\left|\left[\mathscr{J}_i^{e.m.}(y),\left[\mathscr{J}^i_{e.m.}(x),\ J_\mu^{+}(0)\,j_-^{\mu}(0)\right]\right]\delta(x^0)\delta(y^0)\right|A\rangle$$

or the corresponding form with the Hermitean conjugate piece of the weak Hamiltonian. For the moment let us suppose the lepton current commutators contain no Schwinger terms. Then the only part of $N_i^{\ i}$ yielding a Schwinger term is

$$\frac{G}{\sqrt{2}}\int d^4x\int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\delta(x^0)\delta(y^0)\Big\{\langle B\Big|\Big[J_i^{e.m.}(x),\, J_0^{\pm}(0)\Big]\Big|A\rangle\langle \ell\nu_\ell\Big|\Big[j^i_{e.m.}(y), j_0^{\mp}(0)\Big]\Big|0\rangle$$

$$+\langle B\Big|\Big[J_i^{e.m.}(y),\, J_0^{\pm}(0)\Big]\Big|A\rangle\langle \ell\nu_\ell\Big|\Big[j^i_{e.m.}(x), j_0^{\mp}(0)\Big]\Big|0\rangle\Big\}$$

$$= \mp\frac{G}{\sqrt{2}}\int d^4x\,\delta(x^0)\left(e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}}\right)\langle B\Big|\Big[J_i^{e.m.}(x),\, J_0^{\pm}(0)\Big]\Big|A\rangle\langle \ell\nu_\ell\Big|j^i_{\pm}(0)\Big|0\rangle .$$

To evaluate the contribution of the Schwinger term we translate the hadronic matrix element through $-x$, yielding a factor $e^{i\vec{q}\cdot\vec{x}}$, where $q = p_A - p_B$ is the lepton momentum transfer. After performing a partial integration, this part of the above matrix element gives

$$\mp\frac{G}{\sqrt{2}}\, 2q_i\langle B|\mathscr{S}^{\pm}(0)|A\rangle\langle \ell\nu_\ell|j^i_{\mp}(0)|0\rangle .$$

The covariant generalization is made in the standard way

$$q_i j^i_{\mp}(0) \longrightarrow \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) q_\mu j^{\mp}_\nu(0) \longrightarrow \frac{3}{4}\, q^\mu j^{\mp}_\mu(0) ,$$

the last replacement being made under symmetrical k-integration. The resulting term is manifestly not proportional to the lowest order matrix element

$$\mathscr{M}_0 = \frac{G}{\sqrt{2}}\langle B|J^{\pm}_\mu(0)|A\rangle\langle \ell\nu_\ell|j^\mu_{\mp}(0)|0\rangle .$$

Thus we conclude that the divergent part of the radiative corrections in this model is not universal in the sense that we have used it.

If the lepton currents contained similar Schwinger terms they would, of course, have to be included. However, for them to cancel with the hadronic terms would be virtually impossible as it would imply precise equality of certain hadronic and leptonic matrix elements. Thus, spin-zero boson models seem to be unsatisfactory. This has been noted by Preparata and Weisberger,[10] who used a perturbation calculation of pion β-decay as an illustrative example. A popular model which is ruled out on this basis is the σ-model.[30] Here the currents contain both spin-1/2 and spin-zero parts, so the above Schwinger terms will occur. It is difficult to draw any general conclusions about Schwinger terms for models with higher spin fields since the commutator

$$\left[J^{i}_{e.m.}(y), \left[J_{i}^{e.m.}(x), J_{j}^{\pm}(0)\right]\right] \delta(x^0)\,\delta(y^0)$$

could conceivably cancel the type of terms found above, although this does not occur in the simplest examples of currents bilinear in spin-1/2 or spin-one fields. On the other hand, for the algebra of fields model the Schwinger terms are c-numbers, so they do not contribute to connected amplitudes.

Let us now turn to models where the currents are constructed from spin-1/2 fields. We use the naive commutators, assuming any Schwinger terms are c-numbers. In the quark model the vector and axial vector currents are

$$V_{\mu}^{a}(x) = \bar{q}(x)\,\frac{1}{2}\lambda^{a}\gamma_{\mu}q(x) \text{ and } A_{\mu}^{a}(x) = \bar{q}(x)\,\frac{1}{2}\lambda^{a}\gamma_{\mu}\gamma_5\, q(x) \qquad a = (0, 1, \ldots, 8),$$

respectively. The canonical anticommutation relations

$$\left\{q^{\dagger}(x), q(y)\right\}\delta(x^0-y^0) = \delta^4(x-y), \quad \left\{q(x), q(y)\right\}\delta(x^0-y^0) = \left\{q^{\dagger}(x), q^{\dagger}(y)\right\}\delta(x^0-y^0) = 0$$

yield the current algebra postulated above and, in addition, the space-space commutators

$$\left[V_i^a(x), V_j^b(y)\right]\delta(x^0-y^0) = \left[A_i^a(x), A_j^b(y)\right]\delta(x^0-y^0)$$
$$= i\left[-g_{ij}\,f^{abc}\,V_0^c(x) + \epsilon_{ijk}\,d^{abc}\,A_c^k(x)\right]\delta^4(x-y)$$

and

$$\left[V_i^a(x), A_j^b(y)\right]\delta(x^0-y^0) = \left[A_i^a(x), V_j^b(y)\right]\delta(x^0-y^0)$$
$$= i\left[-g_{ij}\,f^{abc}\,A_0^e(x) + \epsilon_{ijk}\,d^{abc}\,V_c^k(x)\right]\delta^4(x-y).$$

In the simplest quark model[26] the electromagnetic current is given by

$$J_\mu^{e.m.}(x) = V_\mu^3(x) + \frac{1}{\sqrt{3}} V_\mu^8(x) = \bar{q}(x) Q_h \gamma_\mu q(x)$$

where

$$Q_h = \frac{1}{2}\left(\lambda^3 + \frac{1}{\sqrt{3}} \lambda^8\right) = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} .$$

The space-space commutators of the electromagnetic and weak currents are thus

$$\left[J_i^{e.m.}(x), J_j^\pm(y)\right] \delta(x^0 - y^0) = \mp g_{ij} J_0^\pm(x)\, \delta^4(x-y) - i \frac{1}{3} \epsilon_{ijk} J_\pm^k(x)\, \delta^4(x-y) .$$

We shall assume in this example that the lepton weak and electromagnetic currents have the conventional form given by

$$j_\mu^+(x) = \bar{\nu}_e(x) \gamma_\mu (1-\gamma_5) e(x) + \bar{\nu}_\mu(x) \gamma_\mu (1-\gamma_5) \mu(x)$$

$$j_\mu^{e.m.}(x) = -\bar{e}(x)\gamma_\mu e(x) - \bar{\mu}(x) \gamma_\mu \mu(x)$$

with

$$j_\mu^-(x) = j_\mu^+(x)^\dagger .$$

The appropriate space-space commutators are easily verified to be

$$\left[j_i^{e.m.}(x), j_j^\pm(y)\right] \delta(x^0 - y^0) = \mp g_{ij} j_0^\pm(x)\, \delta^4(x-y) + i\epsilon_{ijk} j_\pm^k(x)\, \delta^4(x-y) .$$

To discuss the divergences in the radiative corrections for this model we need to know $\overline{M}_\mu^{\ \mu}$ and, in the limit as $k^0 \to \infty$, $(1-\lambda) N_{00} + N_i^{\ i}$. As usual, it is easily checked that N_{00} vanishes. $N_i^{\ i}$ contains a term involving the double commutator

$$\left[J^{i}_{e.m.}(y)\ \left[J_{i}^{e.m.}(x),\ J^{a}_{\mu}(0)\right]\right]\ \delta(x^0)\quad \delta(y^0)$$

$$= -\,3\ g_{\mu 0}\,\overline{q}(x)\left[Q_h,\left[Q_h,\ \tfrac{1}{2}\lambda^a\right]\right]\gamma_0(1-\gamma_5)\ q(x)\ \delta^4(x)\ \delta(y)$$

$$-\ g_{\mu j}\,\overline{q}(x)\left(\left[Q_h,\left[Q_h,\tfrac{1}{2}\lambda^a\right]\right]+2\left\{Q_h,\left\{Q_h,\ \tfrac{1}{2}\lambda^a\right\}\right\}\right)\gamma^j(1-\gamma_5)\,q(x)\,\delta^4(x)\,\delta^4(y).$$

The covariant generalization of this expression is constructed by the replacements

$$g_{\mu j}\gamma^j \longrightarrow g_{\mu\alpha}\left(g^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2}\right)\gamma_\beta \longrightarrow \frac{3}{4}\gamma_\mu$$

$$g_{\mu 0}\gamma^0 \longrightarrow g_{\mu\alpha}\,\frac{k^\alpha k^\beta}{k^2}\ \gamma_\beta \longrightarrow \frac{1}{4}\gamma_\mu$$

where, once again, we have used the fact that a symmetrical k-integration is to be performed. We thus obtain for the right-hand side of the above expression

$$-\frac{3}{2}\overline{q}\,(0)\left(\left[Q_h,\ \left[Q_h,\ \tfrac{1}{2}\lambda^a\right]\right]+\left\{Q_h,\ \left\{Q_h,\ \tfrac{1}{2}\lambda^a\right\}\right\}\right)\gamma_\mu\,(1-\gamma_5)\ q(0)\ \delta^4(x)\ \delta^4(y)$$

$$= -\ 3\ \overline{q}(0)\left\{Q_h^2,\ \tfrac{1}{2}\lambda^a\right\}\ \gamma_\mu\,(1-\gamma_5)\,q(0)\ \delta^4(x)\ \delta^4(y).$$

A considerable simplification ensues if we note that the contribution of this double commutator is exactly cancelled by the corresponding term in $\overline{M}_\mu^{\ \mu}$. Indeed, one finds there

$$-\ 3\int d^4x\,\delta(x^0)\left\{\left[q^\dagger(x),\ J^a_\mu(0)\right]Q_h^2\,q(x)+q^\dagger(x)\,Q_h^2\left[J^a_\mu(0),\ q(x)\right]\right\}$$

$$=\ 3\ \overline{q}(0)\ \left\{Q_h^2,\ \tfrac{1}{2}\lambda^a\right\}\ \gamma_\mu(1-\gamma_5)\ q(0).$$

Clearly, a similar cancellation also occurs for the lepton commutators.

Thus, the only remaining divergent contributions are the terms in $N_i^{\ i}$ of the form

$$\frac{G}{\sqrt{2}}\int d^4x \int d^4y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\delta(x^0)\,\delta(y^0)\langle B\ell\nu_\ell \Big| \Big\{ \Big[\mathscr{J}_i^{e.m.}(x), \mathscr{J}_\mu^{wk(+)}(0)\Big]\Big[\mathscr{J}_{e.m.}^{i}(y), \mathscr{J}_{wk}^{\mu(-)}(0)\Big]$$

$$+\Big[\mathscr{J}_{e.m.}^{i}(y), \mathscr{J}_\mu^{wk(+)}(0)\Big]\Big[\mathscr{J}_i^{e.m.}(x), \mathscr{J}_{wk}^{\mu(-)}(0)\Big]\Big\}\Big| A\rangle .$$

Except for possible Schwinger terms, the commutators

$$\Big[\mathscr{J}_i^{e.m.}(x), \mathscr{J}_0^{wk(\pm)}(0)\Big]\,\delta(x^0)$$

are model independent. However, the above expression also involves the commutators

$$\Big[\mathscr{J}_i^{e.m.}(x), \mathscr{J}_j^{wk(\pm)}(0)\Big]\delta(x^0),$$

which are highly model dependent. We immediately see that for the simple quark model the divergent radiative corrections cannot be universal since the antisymmetric part of the space-space hadronic current commutator differs by a factor of -1/3 from the of the leptonic commutator. This is the result originally obtained by Bjorken.[4]

Clearly a sufficient condition for the divergent corrections to be universal is for the space-space commutators of the hadron and lepton currents to be the same. It was noted by several authors[5,6,7] that the hadron currents could be made to satisfy

$$\Big[J_i^{e.m.}(x), J_j^{\pm}(0)\Big]\delta(x^0) = \mp g_{ij}\, J_0^{\pm}(0)\,\delta^4(x) + i\epsilon_{ijk}J_{\pm}^{k}(0)\,\delta^4(x)$$

in agreement with the usual lepton commutators, by adding an SU(3) singlet piece to the hadronic electromagnetic current. In fact, a quark model[6,7] with integrally charged quarks having a charge matrix

$$Q_h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{or} \quad Q_h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

does the trick. A special feature of this model is that the divergences actually cancel so the second order radiative corrections are finite. One easily verifies that

$$\left[\mathscr{J}_i^{e.m.}(x),\ \mathscr{J}_\mu^{wk(+)}(0)\right] \delta(x^0) \left[\mathscr{J}_{e.m.}^{i}(y), \mathscr{J}_{wk}^{\mu(-)}(0)\right] \delta(y^0)$$

$$= \left[\mathscr{J}_k^{wk(+)}(0)\ \mathscr{J}_{wk}^{k(-)}(0) - 3\,\mathscr{J}_0^{wk(+)}\ \mathscr{J}_{wk}^{0(-)}\right]\delta^4(x)\ \delta^4(y).$$

The covariant generalization has the form

$$\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\mathscr{J}_\mu^{wk(+)}\ \mathscr{J}_\nu^{wk(-)} - 3\,\frac{k^\mu k^\nu}{k^2}\ \mathscr{J}_\mu^{wk(+)}\ \mathscr{J}_\nu^{wk(-)}$$

which vanishes after symmetrical integration.

A final model we shall consider is the algebra of fields. Since in this case the space-space current commutators vanish, universality can clearly only be satisfied if the total electromagnetic and weak currents obey the field-current identity.[22] The commutators relevant to our discussion are then

$$\left[\mathscr{J}_0^{e.m.}(x), \mathscr{J}_\mu^{wk(\pm)}(y)\right]\delta(x^0 - y^0) = \left[\mathscr{J}_\mu^{e.m.}(x), \mathscr{J}_0^{wk(\pm)}(y)\right]\delta(x^0 - y^0) = \pm\mathscr{J}_\mu^{wk(\pm)}(x)\delta^4(x-y)$$

$$\left[\mathscr{J}_i^{e.m.}(x), \mathscr{J}_j^{wk(\pm)}(y)\right]\delta(x^0 - y^0) = 0.$$

As mentioned before, $\overline{M}_\mu{}^\mu = 0$ in this model. N_{00} also vanishes, so we need consider only

$$\left[\mathscr{J}_{e.m.}^{i}(y), \left[\mathscr{J}_i^{e.m.}(x),\ H_{wk}(0)\right]\right]\delta(x^0)\delta(y^0)$$

$$= -2\,\frac{G}{\sqrt{2}}\,\mathscr{J}_i^{wk(+)}(0)\mathscr{J}_{wk}^{i(-)}(0)\ \delta^4(x)\delta^4(y)$$

as is easily verified. By making the covariant generalization

$$\mathscr{J}_i^{wk(+)} \mathscr{J}_{wk}^{i(-)} \longrightarrow \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \mathscr{J}_\mu^{wk(+)} \mathscr{J}_\nu^{wk(-)} \longrightarrow \frac{3}{4} \mathscr{J}_\mu^{wk(+)} \mathscr{J}_{wk}^{\mu(-)} ,$$

we find for the divergent part of the second order radiative corrections

$$\mathscr{M}_1^{div.} = -i \frac{3}{4} e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i\epsilon)^2} \langle B\ell\nu_\ell | H_{wk}(0) | A \rangle ,$$

a result previously obtained by Sirlin[11] and, for a slightly different model, by Schwinger.[23]

We conclude this section with a brief discussion of second order radiative corrections in a weak interaction theory which is mediated by a vector boson[31] rather than a local current-current Hamiltonian. We shall show that the divergent part of the corrections does not satisfy the universality requirement except at q = 0. The weak Hamiltonian density in such a theory is

$$H_{wk}(x) = g\left[\mathscr{J}_\mu^{wk(+)}(x)\, W^\mu_{(-)}(x) + \mathscr{J}_\mu^{wk(-)}(x)\, W^\mu_{(+)}(x)\right] + \frac{g^2}{m_W^2} \mathscr{J}_0^{wk(+)}(x)\, \mathscr{J}_0^{wk(-)}(x)$$

where the vector fields $W_\mu^{(\pm)}(x)$ are free fields of mass m_W. The coupling constant g is related to the Fermi constant G by

$$\frac{g^2}{m_W^2} = \frac{G}{\sqrt{2}} .$$

The lowest order weak interaction matrix element is given by

$$\mathscr{M}_0 = g^2 \int d^4z \langle B\ell\nu_\ell \Big| \Big[T\left\{\mathscr{J}_\alpha^{wk(+)}(z), \mathscr{J}_\beta^{wk(-)}(0)\right\} \langle 0 | T \left\{W^\alpha_{(-)}(z), W^\beta_{(+)}(0)\right\} | 0 \rangle - \frac{i}{m_W^2} \mathscr{J}_0^{wk(+)}(z) \mathscr{J}_0^{wk(-)}(z)\, \delta^4(z)\Big] \Big| A \rangle .$$

The manifestly covariant expression is obtained by observing that the non-covariant part of

$$\langle 0 \left| T \left\{ W^{\alpha}_{(-)}(z),\ W^{\beta}_{(+)}(0) \right\} \right| 0 \rangle$$

cancels the contact term. However, for our purposes the above form is more useful since it is the time-ordered products which are well behaved in the limit as $k^0 \longrightarrow \infty$.

The second order electromagnetic radiative corrections[32] to $\mathcal{M}_0$ are

$$\mathcal{M}_1 = \frac{i}{2} e^2 \int \frac{d^4k}{(2\pi)^4} D^{\mu\nu}(k) \left[T_{\mu\nu}(k) + T^c_{\mu\nu} \right]$$

where $D^{\mu\nu}(k)$ is the photon propagator and

$$T_{\mu\nu}(k) = g^2 \int d^4x \int d^4y \int d^4z\, e^{-ik(x-y)}$$

$$\times \langle B\ell\nu_\ell \left| \left[T\left\{ \mathcal{J}^{e.m.}_\mu(x), \mathcal{J}^{e.m.}_\nu(y), \mathcal{J}^{wk(+)}_\alpha(z) W^{\alpha}_{(-)}(z), \mathcal{J}^{wk(-)}_\beta(0)\ W^{\beta}_{(+)}(0) \right\} \right.\right.$$

$$\left.\left. - \frac{i}{m_W^2} T\left\{ \mathcal{J}^{e.m.}_\mu(x),\ \mathcal{J}^{e.m.}_\nu(y),\ \mathcal{J}^{wk(+)}_0(z) \mathcal{J}^{wk(-)}_0(z)\, \delta^4(z) \right\} \right] \right| A \rangle .$$

$T^c_{\mu\nu}$ is an additional contact term in the electromagnetic interaction of the W-bosons which we need not consider since radiative corrections to the W-boson propagator are necessarily universal. Here the total electromagnetic current $\mathcal{J}^{e.m.}_\mu(x)$ is

$$\mathcal{J}^{e.m.}_\mu(x) = J^{e.m.}_\mu(x) + j^{e.m.}_\mu(x) + \frac{i}{2} e \left[W^{(+)}_{\mu\nu}(x)\, W^{\nu}_{(-)}(x) - W^{(-)}_{\mu\nu}(x)\, W^{\nu}_{(+)}(x) \right]$$

where

$$W^{(\pm)}_{\mu\nu}(x) = \partial_\mu W^{(\pm)}_\nu(x) - \partial_\nu W^{(\pm)}_\mu(x).$$

The divergent contribution to $\mathcal{M}_1$ is obtained as before by isolating the part of $(1-\lambda) T_{00} + T_i^{\ i}$ which goes as $1/k_0^2$ as $k^0 \longrightarrow \infty$. The contact term in the weak interaction will yield the following commutator in $T_i^{\;i}$:

$$-i\,\frac{1}{k_0^2}\,\frac{g^2}{m_w^2}\int d^4x\int d^4y\,\delta(x^0)\,\delta(y^0)\,\langle B\ell\nu_\ell\Big|\Big[\mathscr{J}^{\,i}_{e.m.}(y),\Big[\mathscr{J}_i^{\,e.m.}(x),\mathscr{J}_0^{\,wk(+)}(0)\,\mathscr{J}_0^{\,wk(-)}(0)\Big]\Big]\Big|A\rangle$$

$$= 2i\,\frac{1}{k_0^2}\,\frac{g^2}{m_w^2}\,\langle B\ell\nu_\ell\big|\mathscr{J}_i^{\,wk(+)}(0)\,\mathscr{J}^{\,i(-)}_{wk}(0)\big|A\rangle .$$

Since this contact term contains spatial components of the weak current, it will no longer be cancelled by the non-covariant term in the W-boson time-ordered products. Furthermore, it is easily verified that the remaining contributions to $T_{\mu\nu}(k)$ will not have this form. Clearly, non-minimal electromagnetic interactions of the W-boson will not change this result.

Thus, we conclude that in a W-boson theory we connot satisfy the universality requirement

$$\mathscr{M}_1^{div.} \propto \mathscr{M}_0$$

for arbitrary momentum transfers q, although as was first shown by Sirlin,[8,11] it is satisfied at q = 0. In fact, it is interesting to note that in order to make the radiative corrections finite we would have to cancel the above contribution by adding a counter term to the interaction Hamiltonian having a local current-current form. Hence we should have included such a term in the Hamiltonian from the start. However, in doing that we lose the original motivation for introducing a W-boson interaction.

III. GENERALIZATION TO HIGHER ORDERS IN e^2

Most of the results of the previous section are not new, although they were originally obtained by slightly different techniques. The method we have illustrated above for second order radiative corrections can be used to discuss divergences occuring in higher orders in e^2. We wish to investigate, using current algebras, whether the divergences in radiative corrections to leptonic and semileptonic weak processes to any given order in e^2 can be absorbed into only a universal coupling constant renormalization.

Our method allows us to isolate only those divergences due to momentum loop integrations containing at least one virtual photon line. Thus, in diagrams such as Fig. 1 we ignore all divergences due to closed loops inside the "blobs" containing only hadrons or leptons (or perhaps other types of particles), but no photons.

A few remarks are in order concerning this assumption. In conventional quantum electrodynamics[24] of leptons the only such divergence arises in the second order correction to the photon propagator, i.e., the diagram of Fig. 2, and through its insertion in higher order graphs. However, corrections to the photon propagator even in the general case can only give divergences which are universal, since they contribute alike to all processes. A more serious difficulty was first noted by Adler,[33] concerning the triangle graph involving an axial vector vertex, illustrated in the diagram of Fig. 3. Adler studied this diagram in great detail and showed that it gives rise to divergences in higher order electromagnetic radiative corrections that cannot be removed by simply an overall rescaling of the coupling constants. We shall sweep such problems under the rug by taking the point of view that a proper discussion of divergences

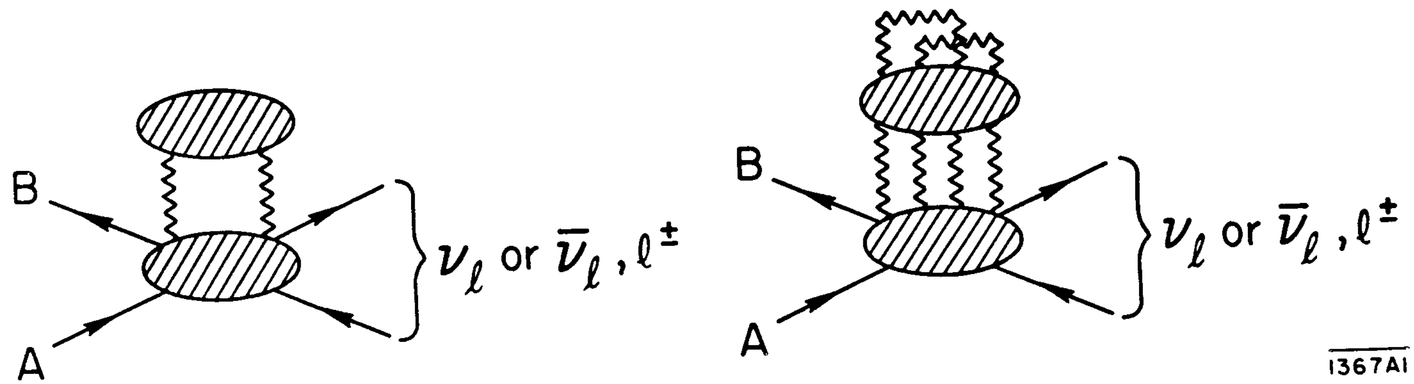

Fig. 1--Examples of contributions to higher order radiative corrections.

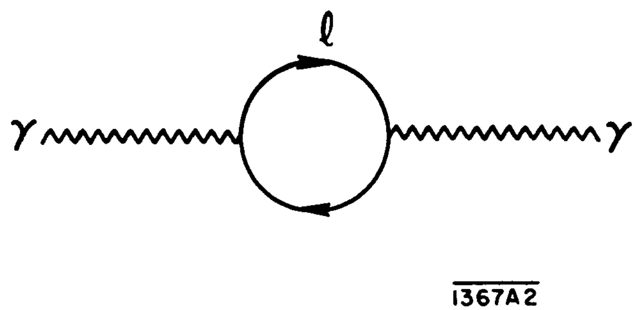

Fig. 2--The second order leptonic correction to the photon propagator.

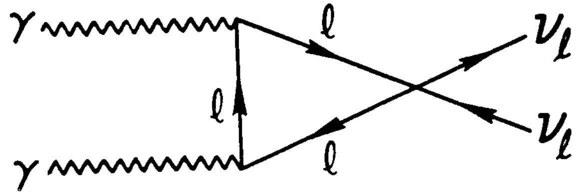

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Fig. 3--The triangle graph in lepton electrodynamics involving an axial vector vertex.

in closed lepton loops requires a satisfactory theory of higher order weak interactions. As for hadrons, a treatment of similar diagrams involves a more detailed commitment to a strong interaction theory than is contained in the statement of current commutation relations. We expressly wish to avoid this, so we shall treat the blobs as black boxes and use only their general expression in terms of currents as matrix elements of

$$T\left\{\mathscr{J}_{\mu_1}^{e.m.}(x_1), \ldots, \mathscr{J}_{\mu_n}^{e.m.}(x_n)\right\}$$

for the case where n photons are attached.

We defer until the conclusion a discussion of whether it is physically reasonable to neglect any divergences inside the blobs. Note, however, that to do so is consistent with our use of naive commutation relations for the various models we shall consider. In fact, these two problems are intimately related, as we shall see later.

In the previous section we found that only two of the commonly discussed current algebra models gave satisfactory results in second order, in the sense that the divergences were universal (or zero). We shall restrict our attention in analyzing higher order radiative corrections to these two models, namely, algebra of fields commutation relations for the total (hadron + lepton) electromagnetic and weak currents, and the quark models with integrally charged quarks for hadrons together with the conventional point interaction for leptons.

First, we shall consider the case of the algebra of fields, since it is less complicated. The electromagnetic radiative corrections in order $(e^2)^n$ to a lowest order weak process are given by

$$\mathscr{M}_n = \frac{(-i)^{2n}}{(2n)!} \prod_{i=1}^{2n} \int d^4x_i \left\langle B\ell\nu_\ell \right| T\left\{H_{e.m.}(x_1), \ldots, H_{e.m.}(x_{2n}), H_{wk}(0)\right\} \left| A \right\rangle$$

plus additional terms if there are contact terms in the electromagnetic interaction.

Here

$$H_{e.m.}(x) = e\mathscr{J}_\mu^{e.m.}(x)\mathscr{A}^\mu(x) = eJ_\mu^{e.m.}(x)\mathscr{A}^\mu(x) + ej_\mu^{e.m.}(x)\mathscr{A}^\mu(x)$$

where the notation is as before.

As in second order, we should properly consider the appropriate vacuum expectation value obtained from the reduction formula. This introduces only inessential complications which obscure the basic argument so we have relegated a discussion of this point to an appendix. We defer the question of contact terms momentarily.

The above expression for $\mathscr{M}_n$ leads to

$$\mathscr{M}_n = \frac{(-ie)^{2n}}{(2n)!} \frac{(-i)^n(2n)!}{n!\, 2^n} \prod_{i=1}^{n} \left[\int d^4k_i\, D^{\mu_i\nu_i}(k_i)\right] T_{\mu_1\nu_1\ldots\mu_n\nu_n}(k_1, \ldots, k_n)$$

where the

$$D^{\mu_i\nu_i}(k_i) = \frac{1}{k_i^2 + i\epsilon}\left(g^{\mu_i\nu_i} - \lambda \frac{k_i^{\mu_i} k_i^{\nu_i}}{k_i^2}\right)$$

are the n photon propagators and

$$T_{\mu_1\nu_1\ldots\mu_n\nu_n}(k_1, \ldots, k_n) = \prod_{i=1}^{n}\left[\int d^4x_i \int d^4y_i\, e^{-ik_i(x_i - y_i)}\right]$$

$$\times \langle B\ell\nu_\ell \left| T\left\{\mathscr{J}_{\mu_1}^{e.m.}(x_1), \mathscr{J}_{\nu_1}^{e.m.}(y_1), \ldots, \mathscr{J}_{\mu_n}^{e.m.}(x_n), \mathscr{J}_{\nu_n}^{e.m.}(y_n), H_{wk}(0)\right\}\right| A\rangle.$$

Here the factor

$$\frac{(-i)^n(2n)!}{n!\, 2^n} = (-i)^n(2n-1)\cdot(2n-3)\ldots 3\cdot 1$$

comes from contracting

$$\langle 0\left|T\left\{\mathscr{A}^{\mu_1}(x_1), \mathscr{A}^{\nu_1}(y_1), \ldots, \mathscr{A}^{\mu_n}(x_n), \mathscr{A}^{\nu_n}(y_n)\right\}\right|0\rangle$$

to form n photon propagators.

In order to isolate the ultraviolet divergent contribution of $\mathcal{M}_n$ we need to find the part of $T_{\mu_1\nu_1\cdots\mu_n\nu_n}(k_1, \ldots, k_n)$ which goes as $\frac{1}{(k_i^0)^2}$ for large k_i^0 in each of the integration variables k_i. These terms will give logarithmic divergences when, after constructing the covariant generalization, the appropriate integration is performed.

We shall assume that the various partial integrations necessary to express these terms as matrix elements of equal-time commutators are justified for the (2n + 1)-point function as we did in second order for the 3-point function. We shall also assume that the prescription for constructing the covariant amplitude, illustrated in second order, generalizes for each of the n momentum variables. Such assumptions are clearly justified for quantum electrodynamics, so it is not unreasonable to assume them here. Our purpose is not to justify these assumptions but, having made them, to see how much current algebras can tell us about ultraviolet divergences.

Before considering radiative corrections in an arbitrary order $(e^2)^n$, we illustrate in order e^4 the new features not occuring in order e^2. We begin by observing that

$$k_1^2 k_2^2\, T_{\mu_1\nu_1\mu_2\nu_2}(k_1, k_2) \xrightarrow[k_1^0,\, k_2^0 \to \infty]{} R_{\mu_1\nu_1\mu_2\nu_2}(k_1, k_2),$$

where

$$R_{\mu_1\nu_1\mu_2\nu_2}(k_1, k_2) = \int d^4x_1 \int d^4y_1 \int d^4x_2 \int d^4y_2\, e^{-ik_1(x_1-y_1)-ik_2(x_2-y_2)}$$

$$\times \frac{\partial}{\partial x_1^\alpha}\frac{\partial}{\partial y_{1\alpha}}\frac{\partial}{\partial x_2^\beta}\frac{\partial}{\partial y_{2\beta}} \langle B\ell\nu_\ell \left| T\left\{\mathcal{J}_{\mu_1}^{e.m.}(x_1), \mathcal{J}_{\nu_1}^{e.m.}(y_1), \mathcal{J}_{\mu_2}^{e.m.}(x_2), \mathcal{J}_{\nu_2}^{e.m.}(y_2), H_{wk}(0)\right\}\right| A\rangle\,,$$

after four partial integrations. The integrations on the time variables are, by assumption, valid in the limit as $k_1^0, k_2^0 \longrightarrow \infty$ and those on the space variables true in any case. To isolate the divergent contribution to $\mathcal{M}_2$, we need the part of $R_{\mu_1\nu_1\mu_2\nu_2}$ which behaves as a constant in either or both of the variables k_1^0 and k_2^0 as $k_1^0, k_2^0 \rightarrow \infty$.

We may simplify matters slightly by using the Landau gauge for the photons so that in the limit as $k_1^0, k_2^0 \longrightarrow \infty$ with $\vec{k}_1, \vec{k}_2$ fixed we need consider only $R_{i_1\ i_2}^{i_1\ i_2}$ (where sums from 1 to 3 are implied over i_1 and i_2). Actually, it is easily verified that gauge dependent terms do not contribute to divergences anyway because as $k_i^0 \rightarrow \infty$ with $\vec{k}_i$ fixed

$$g^{\mu\nu} - \lambda \frac{k_i^\mu k_i^\nu}{k_i^2} \longrightarrow g^{\mu k} g^{\nu}{}_{k} + (1-\lambda) g^{\mu 0} g^{\nu 0} ,$$

and also

$$\left[\mathcal{J}_0^{e.m.}(x), \mathcal{J}_\mu^{e.m.}(y)\right] \delta(x^0 - y^0) = \text{c-number}$$

$$\left[\mathcal{J}_0^{e.m.}(x), H_{wk}(y)\right] \delta(x^0 - y^0) = 0 ,$$

since

$$H_{wk}(y) = \frac{G}{\sqrt{2}} \mathcal{J}_\mu^{wk(+)}(y) \mathcal{J}_{wk}^{\mu(-)}(y) .$$

Considering now $R_{i_1\ i_2}^{i_1\ i_2}$, we first take the derivatives $\frac{\partial}{\partial x_1^\alpha} \frac{\partial}{\partial y_{1\alpha}}$ inside the time-ordered product and use the commutation relations

$$\left[\mathcal{J}_i^{e.m.}(x), \mathcal{J}_j^{e.m.}(y)\right] \delta(x^0 - y^0) = 0$$

$$\left[\mathcal{J}_i^{e.m.}(x), \mathcal{J}_\mu^{wk(\pm)}(y)\right] \delta(x^0 - y^0) = \pm g_{\mu 0} \mathcal{J}_i^{wk(\pm)}(x) \delta^4(x - y)$$

of the algebra of fields model.[21] One obtains

$$\frac{\partial}{\partial x_1^\alpha}\frac{\partial}{\partial y_{1\alpha}}\langle B\ell\nu_\ell\left|T\left\{\mathscr{J}_{i_1}^{e.m.}(x_1),\mathscr{J}_{e.m.}^{i_1}(y_1),\mathscr{J}_{i_2}^{e.m.}(x_2),\mathscr{J}_{e.m.}^{i_2}(y_2),H_{wk}(0)\right\}\right|A\rangle$$

$$= -\langle B\ell\nu_\ell\left|T\left\{\left[\partial_0\mathscr{J}_{i_1}^{e.m.}(x_1),\mathscr{J}_{e.m.}^{i_1}(y_1)\right]\delta(x_1^0-y_1^0)\mathscr{J}_{i_2}^{e.m.}(x_2),\mathscr{J}_{e.m.}^{i_2}(y_2),H_{wk}(0)\right\}\right|A\rangle$$

$$+\langle B\ell\nu_\ell\left|T\left\{\mathscr{J}_{i_2}^{e.m.}(x_2),\mathscr{J}_{e.m.}^{i_2}(y_2),\left[\mathscr{J}_{e.m.}^{i_1}(y_1),\left[\mathscr{J}_{i_1}^{e.m.}(x_1),H_{wk}(0)\right]\right]\delta\left(x_1^0\right)\delta\left(y_1^0\right)\right\}\right|A\rangle$$

$$+\langle B\ell\nu_\ell\left|T\left\{\partial_0\mathscr{J}_{i_1}^{e.m.}(x_1),\mathscr{J}_{i_2}^{e.m.}(x_2),\mathscr{J}_{e.m.}^{i_2}(y_2),\left[\mathscr{J}_{e.m.}^{i_1}(y_1),H_{wk}(0)\right]\delta\left(y_1^0\right)\right\}\right|A\rangle$$

$$+\langle B\ell\nu_\ell\left|T\left\{\partial_0\mathscr{J}_{e.m.}^{i_1}(y_1),\mathscr{J}_{i_2}^{e.m.}(x_2),\mathscr{J}_{e.m.}^{i_2}(y_2),\left[\mathscr{J}_{i_1}^{e.m.}(x_1),H_{wk}(0)\right]\delta\left(x_1^0\right)\right\}\right|A\rangle$$

$$+\langle B\ell\nu_\ell\left|T\left\{\partial_\alpha\mathscr{J}_{e.m.}^{i_1}(x_1),\partial^\alpha\mathscr{J}_{i_1}^{e.m.}(y_1),\mathscr{J}_{i_2}^{e.m.}(x_2),\mathscr{J}_{e.m.}^{i_2}(y_2),H_{wk}(0)\right\}\right|A\rangle .$$

The first two of the five terms in this expression will give contributions to $R_{i_1\ i_2}^{i_1\ i_2}$ which are independent of k_1 since the δ-functions in x_1 and y_1 will eliminate the factor $e^{-ik_1(x_1-y_1)}$. The last three terms will still contain a k_1-dependence after the δ-functions are integrated out, so we shall assume they give only finite contributions to the k_1 momentum integration. In the spirit of our technique, we could make further partial integrations of derivatives with respect to x_1 or y_1 to show that these terms give contributions to $T_{i_1\ i_2}^{i_1\ i_2}$ going at least as fast as $1/\left(k_1^0\right)^3$ as $k_1^0 \to \infty$. Of course, this would be a sufficient, but not necessary, condition for these terms to be finite.

The term containing

$$\left[\partial_0\mathscr{J}_{i_1}^{e.m.}(x_1),\mathscr{J}_{e.m.}^{i_1}(y_1)\right]\delta\left(x_1^0-y_1^0\right)$$

we recognize from the second order calculation will be cancelled by an appropriate counter term added to the interaction Hamiltonian to remove the divergent part of mass renormalization corrections and of electromagnetic renormalization effects in strong interaction processes. Thus, only the second term will give a divergent contribution to the k_1 integration. The double commutator

$$\left[\mathscr{J}^{i_1}_{e.m.}(y_1), \left[\mathscr{J}^{e.m.}_{i_1}(x_1), H_{wk}(0)\right]\right]\delta\left(x_1^0\right)\delta\left(y_1^0\right) = -2\frac{G}{\sqrt{2}}\mathscr{J}^{wk(+)}_{i_1}(0)\,\mathscr{J}^{i_1(-)}_{wk}(0)\,\delta^4(x_1)\,\delta^4(y_1)$$

as in second order has the covariant generalization

$$-2\frac{G}{\sqrt{2}}\left(g^{\mu_1\nu_1} - \frac{k_1^{\mu_1}k_1^{\nu_1}}{k_1^2}\right)\mathscr{J}^{wk(+)}_{\mu_1}(0)\,\mathscr{J}^{wk(-)}_{\nu_1}(0)\,\delta^4(x_1)\,\delta^4(y_1)\ .$$

Under symmetrical k_1 integration we may replace

$$g^{\mu_1\nu_1} - \frac{k_1^{\mu_1}k_1^{\nu_1}}{k_1^2} \longrightarrow \frac{3}{4}\,g^{\mu_1\nu_1}\ ,$$

so this term effectively contributes to $T^{i_1\ i_2}_{i_1\ i_2}(k_1, k_2)$ a term

$$\left(-\frac{3}{2}\right)\frac{1}{k_1^2}\int d^4x_2\int d^4y_2\, e^{-ik_2(x_2-y_2)}\langle B\ell\nu_\ell\left|T\left\{\mathscr{J}^{e.m.}_{i_2}(x_2),\ \mathscr{J}^{i_2}_{e.m.}(y_2),\ H_{wk}(0)\right\}\right|A\rangle.$$

This matrix element now has the same structure as the second order radiative correction. To isolate the divergent contribution to the k_2 integration we therefore repeat that calculation and find that the above term's contribution to the covariant generalization of $T^{i_1\ i_2}_{i_1\ i_2}(k_1, k_2)$ is

$$\left(-\frac{3}{2}\right)^2\frac{1}{k_1^2k_2^2}\langle B\ell\nu_\ell\left|H_{wk}(0)\right|A\rangle + \left(-\frac{3}{2}\right)\frac{1}{k_1^2}\left(g^{\mu_2\nu_2} - \frac{k_2^{\mu_2}k_2^{\nu_2}}{k_2^2}\right)T^{f}_{\mu_2\nu_2}(k_2)$$

where

$$T^{f}_{\mu_2\nu_2}(k_2) = \frac{1}{k_2^2}\int d^4x_2 \int d^4y_2\, e^{-ik_2(x_2-y_2)}\Big[\langle B\ell\nu_\ell \Big| T\Big\{\partial_\beta J^{e.m.}_{\mu_2}(x_2), \partial^\beta J^{e.m.}_{\nu_2}(y_2), H_{wk}(0)\Big\}\Big| A\rangle$$

$$+ \langle B\ell\nu_\ell \Big| T\Big\{\partial_0 \mathscr{J}^{e.m.}_{\mu_2}(x_2), \Big[\mathscr{J}^{e.m.}_{\nu_2}(y_2), H_{wk}(0)\Big]\delta\big(y_2^0\big)\Big\}$$

$$+ T\Big\{\partial_0 \mathscr{J}^{e.m.}_{\nu_2}(y_2), \Big[\mathscr{J}^{e.m.}_{\mu_2}(x_2), H_{wk}(0)\Big]\delta\big(x_2^0\big)\Big\}\Big| A\rangle\Big].$$

We recognize $T^{f}_{\mu_2\nu_2}(k_2)$ as simply the finite contribution obtained in the second order result after the divergent terms were isolated. We have, of course, removed the divergent mass renormalization term in obtaining the above expression.

We now return to the three terms which gave finite contributions to the k_1 integration. We must extract the part of these terms which gives a divergent contribution to the k_2 integration. To do this we take the remaining derivatives $\frac{\partial}{\partial x_2^\beta}$ and $\frac{\partial}{\partial y_{2\beta}}$ in

$$\left(g^{\mu_1\nu_1} - \frac{k_1^{\mu_1}k_1^{\nu_1}}{k_1^2}\right) R^{\;\;\;\;\;\;i_2}_{\mu_1\nu_1 i_2}$$

inside the time-ordered product. In the algebra of fields model

$$\Big[\mathscr{J}^{e.m.}_{i_2}(y_2), \Big[\mathscr{J}^{i_2}_{e.m.}(x_2), \partial_0 \mathscr{J}^{e.m.}_{\mu_1}(x_1)\Big]\Big]\delta\big(x_2^0 - x_1^0\big)\delta\big(y_2^0 - x_1^0\big) = 0$$

and, using the Jacobi identity twice,

$$\Big[\mathscr{J}^{e.m.}_{i_2}(y_2), \Big[\mathscr{J}^{i_2}_{e.m.}(x_2), \Big[\mathscr{J}^{e.m.}_{\mu_1}(x_1), H_{wk}(0)\Big]\Big]\Big]\delta\big(x_1^0\big)\delta\big(x_2^0\big)\delta\big(y_2^0\big)$$

$$= \Big[\mathscr{J}^{e.m.}_{\mu_1}(x_1), \Big[\mathscr{J}^{e.m.}_{i_2}(y_2), \Big[\mathscr{J}^{i_2}_{e.m.}(x_2), H_{wk}(0)\Big]\Big]\Big]\ \delta\big(x_1^0\big)\delta\big(x_2^0\big)\delta\big(y_2^0\big).$$

Using these relations we see that the only terms which lead to a divergent k_2 integration, except the usual one containing $\left[\partial_0 \mathscr{J}_i^{e.m.}(x), \mathscr{J}_{e.m.}^i(y)\right] \delta(x_0 - y_0)$, which is cancelled by the mass renormalization counter term, are those involving a double commutator

$$\left[\mathscr{J}_{i_2}^{e.m.}(x_2), \left[\mathscr{J}_{e.m.}^{i_2}(y_2), H_{wk}(0)\right]\right] \delta\left(y_2^0\right)\delta\left(x_2^0\right) .$$

The remaining terms, which are too numerous to write out explicitly, will still be dependent on both k_1 and k_2 after any δ - functions have been integrated over. Thus, by assumption, they are the finite contributions to $\mathscr{M}_2$, the fourth order $\left((e^2)^2\right)$ matrix element.

Evaluating the double commutator and replacing it by the covariant form as before, we may combine the divergent pieces from the last three terms to give a contribution to $T_{i_1\ i_2}^{i_1\ i_2}(k_1, k_2)$ which is

$$\left(-\frac{3}{2}\right) \frac{1}{k_2^2} \left(g^{\mu_1\nu_1} - \frac{k_1^{\mu_1} k_1^{\nu_1}}{k_1^2}\right) T^f_{\mu_1\nu_1}(k_1)$$

where, again, $T^f_{\mu_1\nu_1}(k_1)$ is the finite part of the second order radiative correction given above.

At last, we may put everything together. We define the fourth order radiative corrected matrix element with the mass renormalization terms removed to be $\widetilde{\mathscr{M}}_2$. Also, denote by

$$L \equiv -\frac{3}{2} i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i\epsilon)^2}$$

the divergent momentum integral. Then the above calculation yields

$$\widetilde{\mathscr{M}}_2 = \frac{e^4}{2!\,2^2} L^2 \mathscr{M}_0 + \frac{e^2}{2} L \mathscr{M}_1^f + \mathscr{M}_2^f$$

where

$$\mathcal{M}_0 = \langle B\ell\nu_\ell \left| H_{wk}(0) \right| A \rangle$$

is the lowest order matrix element,

$$\mathcal{M}_1^f = \frac{ie^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 + i\epsilon} \quad T^f_{\mu\nu}(k)$$

is the finite part of the second order matrix element (with $T^f_{\mu\nu}(k)$ given above), and $\mathcal{M}_2^f$ is the finite part of the fourth order matrix element, which we have not exhibited explicitly. Hence we may write

$$\mathcal{M} \equiv \mathcal{M}_0 + \tilde{\mathcal{M}}_1 + \tilde{\mathcal{M}}_2 + \ldots = \left(1 + \frac{1}{2} e^2 L + \frac{1}{2!\,2^2} e^4 L^2\right)\left(\mathcal{M}_0 + \mathcal{M}_1^f + \mathcal{M}_2^f\right) + O(e^6).$$

We see that through order e^4 the matrix element $\mathcal{M}$ for any semileptonic or leptonic weak process can be written as a divergent constant factor times a finite part. The divergent factor can be absorbed into a rescaling of the weak coupling constant G by defining

$$G_R = G\left[1 + \frac{1}{2} e^2 L + \frac{1}{2!\,2^2} e^4 L^2 + O(e^6)\right].$$

Since the divergent term is the same for all processes, the ratio of the rates for any two processes, which is a measurable quantity, is finite. Thus, the divergences are universal and have no observable effects.

A comment concerning our technique is necessary. In order to obtain the correct covariant amplitude, note that, after extracting the divergent part of the k_1 integration, we were careful to construct the covariant generalization with respect to the indices μ_1 and ν_1 before examining the k_2 integration. If we vary only one of the momenta k_i at a time this removes any ambiguities concerning the proper covariant form.

It is now clear how to proceed in the case of radiative corrections in an arbitrary order $(e^2)^n$. Using the Landau gauge for each photon we consider successively each of the integration variables $k_1, \ldots, k_n$ and extract the part of $T_{i_1 i_2 \cdots i_n}^{i_1 i_2 \cdots i_n}(k_1, \ldots, k_n)$ which goes as $1/k_m^2$ for each k_m and thus contributes to a divergent k_m sub-integration. As before we remove all terms containing

$$\left[\mathscr{J}_{i_m}^{e.m.}(x_m), \partial_0 \mathscr{J}_{e.m.}^{i_m}(y_m)\right] \delta\left(x_m^0 - y_m^0\right)$$

since they are cancelled by the mass renormalization counter term. The only terms which will lead to a divergent k_m integral must contain

$$\left[\mathscr{J}_{i_m}^{e.m.}(x_m)\left[\mathscr{J}_{e.m.}^{i_m}(y_m), H_{wk}(0)\right]\right] \delta\left(x_m^0\right) \delta\left(y_m^0\right),$$

which we obtain either directly or by repeated use of the Jacobi identity if $H_{wk}(0)$ is contained inside some other commutator. All other terms will still be k_m-dependent after any δ - functions from equal-time commutators are integrated out, so by assumption they give only finite contributions to this integration. After isolating the divergent part in k_m and evaluating the double commutator, we construct the covariant generalization with respect to the indices μ_m, ν_m.

We do this for each k_m from m = 1 to n. Consider then a term in which the above double commutator with $H_{wk}(0)$ was taken r times. This will contain a factor

$$\left[-\frac{3}{2} i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i\epsilon)^2}\right]^r = L^r$$

from the r divergent subintegrations. The matrix element of the object remaining in the time-ordered product is proportional to

$$2^{n-r}(n-r)!\, \mathscr{M}_{n-r}^{f}$$

where $\mathcal{M}^{f}_{n-r}$ is the finite part of the amplitude to order $(e^2)^{n-r}$. Such a term can occur in

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

different ways. Thus, we have on combining all these terms

$$\tilde{\mathcal{M}}_n = \sum_{r=0}^{n} \frac{(e^2)^{n-r} L^r}{(n-r)!\,2^{n-r}} \mathcal{M}^f_{n-r} . \quad \left(\mathcal{M}^f_0 \equiv \mathcal{M}_0\right)$$

Therefore, we may write

$$\mathcal{M} = \mathcal{M}_0 + \sum_{n=1}^{\infty} \tilde{\mathcal{M}}_n = \left(1 + \frac{e^2 L}{2} + \ldots + \frac{e^{2n} L^n}{n!\,2^n}\right)\left(\mathcal{M}_0 + \mathcal{M}^f_1 + \ldots + \mathcal{M}^f_n\right) + O\left(e^{2(n+1)}\right).$$

Since this is true for any n,

$$\mathcal{M} = \exp\left(\frac{1}{2} e^2 L\right) \mathcal{M}_f$$

where

$$\mathcal{M}_f = \sum_{n=0}^{\infty} \mathcal{M}^f_n$$

is finite. Thus, the weak amplitude containing radiative corrections to all orders in e^2, here denoted by $\mathcal{M}$, is simply a divergent constant factor times a finite matrix element, which is precisely what we need to maintain finite corrections to universality of the weak coupling constant. We are, of course, working only to lowest order in the weak coupling constant.

This concludes the discussion of divergent radiative corrections in the algebra of fields model except for the question of contact terms in the electromagnetic interaction Hamiltonian. In the model of Lee and Zumino[28] in which the algebra of fields commutation relations are obtained by imposing a field-current identity, the Lagrangian density is

$$\mathcal{L} = \mathcal{L}_{e.m.} - \frac{1}{4} \hat{G}^a_{\mu\nu} \hat{G}^{\mu\nu}_a + \frac{1}{2} m_0^2 \phi^a_\mu \phi^\mu_a + \mathcal{L}_m\left(\psi, D_\mu \psi, \hat{G}^a_{\mu\nu}\right)$$

where

$$\mathscr{L}_{e.m.} = -\frac{1}{2}\,\partial_\nu \mathscr{A}_\mu \partial^\nu \mathscr{A}^\mu$$

is the free electromagnetic field Lagrangian density with $\mathscr{A}_\mu$ (x) as the electromagnetic potential, the

$$\phi_\mu^a(x) \qquad a = (1, \ldots, 8)$$

form an octet of fundamental vector fields, $\hat{G}^a_{\mu\nu}$ is given by

$$\hat{G}^a_{\mu\nu}(x) = \partial_\mu\, \hat{\phi}^a_\nu(x) - \partial_\nu\, \hat{\phi}^a_\mu(x) - g_0 f^{abc}\, \hat{\phi}^b_\mu(x)\, \hat{\phi}^c_\nu(x),$$

and $\mathscr{L}_m$ is the matter field Lagrangian density depending on the hadron and lepton fields $\psi(x)$ and their covariant derivatives

$$D_\mu \psi(x) = \partial_\mu \psi(x) - g_0\, T_a\, \hat{\phi}^a_\mu(x)\, \psi(x)$$

where T_a is the appropriate internal symmetry matrix for the $\psi(x)$. The indices a, b, c, ... use the conventional SU(3) notation. The electromagnetic interaction is introduced through

$$\hat{\phi}^a_\mu = \phi^a_\mu + \frac{e}{g_0}\left(\delta^{a3} + \frac{1}{\sqrt{3}}\,\delta^{a8}\right)\mathscr{A}_\mu(x).$$

We have for simplicity ignored axial vector field terms as these do not change the basic argument.

The interaction representation is easily obtained if we choose $\hat{\phi}^a_i$, $\mathscr{A}_\mu$, and ψ as the vasic field variables. Performing a canonical transformation on these variables

$$\hat{\phi}^a_i(x) \longrightarrow U^{-1}(t)\, \hat{\phi}^a_i(x)\, U(t)$$

$$\mathscr{A}_\mu(x) \longrightarrow U^{-1}(t)\, \mathscr{A}_\mu(x)\, U(t)$$

$$\psi(x) \longrightarrow U^{-1}(t)\, \psi(x)\, U(t)$$

and their canonically conjugate momenta, it is easily verified that the equations of motion in the interaction representation, which are those obtained from $\mathscr{L}(e=0)$, are obeyed if we choose

$$\mathscr{H}_{e.m.} \equiv i\partial_0 U(t)\, U^{-1}(t) = \int d^3x \left(e\frac{m_0^2}{g_0}\right)\left(\phi_\mu^3(x) + \frac{1}{\sqrt{3}}\,\phi_\mu^8(x)\right)\mathscr{A}^\mu(x).$$

In other words, the Hamiltonian density is

$$H_{e.m.}(x) = e\,\mathscr{J}_\mu^{e.m.}(x)\mathscr{A}^\mu(x)$$

so there are no contact terms. Note that this result is independent of the specific nature of the matter Lagrangian density $\mathscr{L}_m$.

We have just shown that universality of the divergent radiative corrections in the algebra of fields model is maintained to all orders in e^2. We now wish to consider the model which gave finite results in second order and extend the discussion to all orders. Recall that the electromagnetic current in this model is

$$\mathscr{J}_\mu^{e.m.}(x) = \bar{q}(x)\, Q_h \gamma_\mu q(x) - \bar{e}(x)\gamma_\mu\, e(x) - \bar{\mu}(x)\gamma_\mu\, \mu(x)$$

where q(x), e(x), and $\mu(x)$ and the quark, electron and muon fields, respectively, and the quark charge matrix Q_h is

$$Q_h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ or } Q_h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The weak current is, with $\mathscr{J}_\mu^{wk(-)} = \left(\mathscr{J}_\mu^{wk(+)}\right)^\dagger$,

$$\mathscr{J}_\mu^{wk(+)}(x) = \bar{q}(x)\,\lambda_c^+\,\gamma_\mu(1-\gamma_5)\, q(x) + \bar{\nu}_e(x)\,\gamma_\mu(1-\gamma_5)\, e(x) + \bar{\nu}_\mu(x)\,\gamma_\mu(1-\gamma_5)\,\mu(x)$$

where

$$\lambda_c^+ = (\lambda_1 + i\lambda_2)\cos\theta + (\lambda_4 + i\lambda_5)\sin\theta.$$

Here θ is the Cabibbo angle and the λ^a matrices follow the standard SU(3) notation. $\nu_e(x)$ and $\nu_\mu(x)$ are the electron neutrino and muon neutrino fields, respectively.

These currents satisfy the naive commutation relations

$$\left[\mathscr{J}_0^{e.m.}(x),\ \mathscr{J}_\mu^{wk(\pm)}(y)\right]\delta(x_0 - y_0) = \left[\mathscr{J}_\mu^{e.m.}(x),\ \mathscr{J}_0^{wk(\pm)}(y)\right]\delta(x_0 - y_0)$$

$$= \pm\mathscr{J}_\mu^{wk(\pm)}(x)\ \delta^4(x-y)$$

$$\left[\mathscr{J}_i^{e.m.}(x),\ \mathscr{J}_j^{wk(\pm)}(y)\right]\delta(x^0 - y^0) = \mp g_{ij}\mathscr{J}_0^{wk(\pm)}(x)\,\delta^4(x-y) + i\epsilon_{ijk}\mathscr{J}_{wk}^{k(\pm)}(x)\,\delta^4(x-y),$$

which guaranteed the finiteness of the electromagnetic radiative corrections to order e^2. In considering higher orders we shall also need the commutators

$$\left[\mathscr{J}_0^{e.m.}(x),\ \mathscr{J}_\mu^{e.m.}(y)\right]\delta(x^0 - y^0) = 0$$

$$\left[\mathscr{J}_i^{e.m.}(x),\ \mathscr{J}_j^{e.m.}(y)\right]\delta(x^0 - y^0) = -2i\epsilon_{ijk}\mathcal{a}^k(x)\ \delta^4(x-y),$$

where the axial vector current $\mathcal{a}_\mu(x)$ is

$$\mathcal{a}_\mu(x) = \bar{q}(x)\,Q_h\gamma_\mu\gamma_5\,q(x) - \bar{e}(x)\gamma_\mu\gamma_5\,e(x) - \bar{\mu}(x)\,\gamma_\mu\gamma_5\,\mu(x).$$

Of course, we are assuming, as before, that the naive commutators are correct except for possible c-number Schwinger terms which do not contribute to connected amplitudes.

For spin-1/2 fields with minimal electromagnetic interactions the interaction Hamiltonian density is simply

$$H^{e.m.}(x) = e\mathscr{J}_\mu^{e.m.}(x)\mathscr{A}^\mu(x),$$

i.e., there are no contact terms. Thus, in discussing radiative corrections to order $(e^2)^n$ we need to consider the amplitude $\mathscr{M}_n$ defined above for the algebra of fields case. To illustrate we begin with the fourth order corrections as an example. The appropriate amplitude $\mathcal{M}_2$ is

$$\mathcal{M}_2 = -\frac{e^4}{2!2^2}\int d^4k_1 \int d^4k_2\, D^{\mu_1\nu_1}(k_1)\, D^{\mu_2\nu_2}(k_2)\, T_{\mu_1\nu_1\mu_2\nu_2}(k_1, k_2).$$

In the limit as $k_1^0, k_2^0 \longrightarrow \infty$ we may write, as before,

$$k_1^2 k_2^2\, T_{\mu_1\nu_1\mu_2\nu_2}(k_1, k_2) \xrightarrow[k_1^0,\, k_2^0 \to \infty]{} R_{\mu_1\nu_1\mu_2\nu_2}(k_1, k_2)$$

where

$$R_{\mu_1\nu_1\mu_2\nu_2}(k_1,k_2) = \int d^4x_1 \int d^4y_1 \int d^4x_2 \int d^4y_2\, e^{-ik_1(x_1-y_1)-ik_2(x_2-y_2)}$$

$$\times \frac{\partial}{\partial x_1^\alpha}\frac{\partial}{\partial y_{1\alpha}}\frac{\partial}{\partial x_2^\beta}\frac{\partial}{\partial y_{2\beta}} \langle B\ell\nu_\ell \left| T\left\{\mathcal{J}_{\mu_1}^{e.m.}(x_1), \mathcal{J}_{\nu_1}^{e.m.}(y_1), \mathcal{J}_{\mu_2}^{e.m.}(x_2), \mathcal{J}_{\nu_2}^{e.m.}(y_2), H_{wk}(0)\right\}\right| A\rangle$$

Again, we use the Landau gauge for the photon propagators so we only need consider $R_{i_1\ i_2}^{\ i_1\ i_2}$. First, we take the derivatives with respect to x_1 and y_1 inside the time-ordered product. As this generates a considerable number of terms, we introduce here an abbreviated notation. We suppress the space-time dependence of the operators inside the time-ordered product and also the equal-time δ-functions multiplying the various commutators. In addition we drop the superscript (e.m.) on the electromagnetic current. One then has

$$\frac{\partial}{\partial x_1^\alpha}\frac{\partial}{\partial y_{1\alpha}}\, T\left\{\mathcal{J}_{i_1}, \mathcal{J}^{i_1}, \mathcal{J}_{i_2}, \mathcal{J}^{i_2}, H_{wk}\right\}$$

$$= T\left\{\left[\mathcal{J}^{i_1}, \partial_0\mathcal{J}_{i_1}\right], \mathcal{J}_{i_2}, \mathcal{J}^{i_2}, H_{wk}\right\} + T\left\{\mathcal{J}_{i_2}, \mathcal{J}^{i_2}, \left[\mathcal{J}^{i_1}, \left[\mathcal{J}_{i_1}, H_{wk}\right]\right]\right\}$$

$$+ 2T\left\{\left[\mathcal{J}^{i_1}, \left[\mathcal{J}_{i_1}, \mathcal{J}_{i_2}\right]\right], \mathcal{J}^{i_2}, H_{wk}\right\} + 2T\left\{\left[\mathcal{J}_{i_1}, \mathcal{J}_{i_2}\right], \left[\mathcal{J}^{i_1}, \mathcal{J}^{i_2}\right], H_{wk}\right\}$$

$$+ 4T\left\{\left[\mathcal{J}_{i_1}, \mathcal{J}_{i_2}\right], \mathcal{J}^{i_2}, \left[\mathcal{J}^{i_1}, H_{wk}\right]\right\} + 4T\left\{\partial_0\mathcal{J}^{i_1}, \left[\mathcal{J}_{i_1}, \mathcal{J}_{i_2}\right], \mathcal{J}^{i_2}, H_{wk}\right\}$$

$$+ 2T\left\{\partial_0\mathcal{J}^{i_1}, \mathcal{J}_{i_2}, \mathcal{J}^{i_2}, \left[\mathcal{J}_{i_1}, H_{wk}\right]\right\} + T\left\{\partial_\alpha\mathcal{J}_{i_1}, \partial^\alpha\mathcal{J}^{i_1}, \mathcal{J}_{i_2}, \mathcal{J}^{i_2}, H_{wk}\right\}$$

from a straightforward calculation. In obtaining this expression we have used the fact that only the part of $R_{i_1\, i_2}^{i_1\, i_2}$ even under $k_1 \longrightarrow -k_1$ or $k_2 \longrightarrow -k_2$ contributes to $\mathscr{M}_2$, so we may symmetrize with respect to $x_1 \leftrightarrow y_1$ and $x_2 \leftrightarrow y_2$.

We note that eight types of terms are found. Of these, only the first three contain equal-time commutators involving both of the currents $\mathscr{J}_{i_1}(x_1)$ and $\mathscr{J}^{i_1}(y_1)$. Since the δ-functions will cancel the factor $e^{-ik_1(x_1-y_1)}$, these terms will be independent of k_1 and hence lead to divergent k_1 subintegrations. These are the only terms which could give rise to leading divergences, i.e., ones where both the k_1 and k_2 integrations diverged, since we could continue to expand the remaining terms in powers of $1/k_1^0$ to show that they lead to a finite k_1 integration.

What follows now is a necessarily lengthy argument which shows that, in fact, all of the divergent contributions cancel in fourth order as they did in second order. We proceed by considering each of the eight terms in turn. The equal time commutator in the first term may be evaluated by using the equations of motion

$$i\gamma^{\mu}\partial_{\mu} q(x) = F(x)\, q(x)$$

$$i\gamma^{\mu}\partial_{\mu} e(x) = m_e\, e(x)$$

$$i\gamma^{\mu}\partial_{\mu} \mu(x) = m_{\mu}\, \mu(x)\, .$$

One finds

$$\left[\mathscr{J}^{i_1}_{e.m.}(y_1), \partial_0 \mathscr{J}^{e.m.}_{i_1}(x_1)\right]\delta\left(x_1^0 - y_1^0\right) = A(x_1)\,\delta^4(x_1 - y_1) + B_k(x_1, y_1)\,\partial^k \delta^4(x_1 - y_1)$$

where

$$\frac{1}{12i} A(x_1) = \frac{1}{2}\bar{q}(x_1)\left\{Q_h^2, F(x_1)\right\} q(x_1) + m_e\, \bar{e}(x_1) e(x_1) + m_{\mu}\, \bar{\mu}(x_1)\mu(x_1)$$

$$\frac{1}{4}B_k(x_1, y_1) = \bar{q}(x_1)\gamma_k q(y_1) - \bar{q}(y_1)\gamma_k q(x_1) + \bar{e}(x_1)\gamma_k e(y_1) - \bar{e}(y_1)\gamma_k e(x_1)$$
$$+ \bar{\mu}(x_1)\gamma_k \mu(y_1) - \bar{\mu}(y_1)\gamma_k \mu(x_1)$$

As in second order, $A(x_1)$ and $\partial^k B_k(x_1, y_1)$, when expressed in covariant form, are cancelled by the counter term which removes the divergent part of lepton mass renormalization and of all electromagnetic renormalization effects in strong interaction processes. However, as in that example, the factor $\partial_k \delta^4(x_1 - y_1)$ must be taken outside the time-ordered product before the covariant replacement is made. The added time derivative will produce additional equal-time commutator terms when it is taken back inside. A straightforward calculation shows that the resulting contribution to the first term above is

$$18\, \delta^4(x_1)\, \delta^4(y_1)\quad T\left\{\mathscr{J}_{i_2}^{e.m.}(x_2),\, \mathscr{J}_{e.m.}^{i_2}(y_2),\ H_{wk}(0)\right\} .$$

The commutator in the second term is

$$\left[\mathscr{J}_{e.m.}^{i_1}(y_1), \left[\mathscr{J}_{i_1}^{e.m.}(x_1),\ H_{wk}(0)\right]\right] \delta\left(x_1^0\right) \delta\left(y_1^0\right)$$

$$= \frac{G}{\sqrt{2}} \left\{- 12\, \mathscr{J}_0^{wk(+)}(0)\, \mathscr{J}_0^{wk(-)}(0) - 4\, \mathscr{J}_{i_1}^{wk(+)}(0)\, \mathscr{J}_{wk}^{i_1(-)}(0)\right\} \delta^4(x_1)\, \delta^4(y_1) .$$

We must construct the covariant generalization of this in the usual manner. One has

$$\mathscr{J}_0^{wk(+)}(x)\, \mathscr{J}_0^{wk(-)}(x) \longrightarrow \frac{k_1^\mu k_1^\nu}{k_1^2}\ \mathscr{J}_\mu^{wk(+)}(x)\, \mathscr{J}_\nu^{wk(-)}(x)$$

$$\mathscr{J}_{i_1}^{wk(+)}(x)\, \mathscr{J}_{wk}^{i_1(-)}(x) \longrightarrow \left(g^{\mu\nu} - \frac{k_1^\mu k_1^\nu}{k_1^2}\right) \mathscr{J}_\mu^{wk(+)}(x)\, \mathscr{J}_\nu^{wk(-)}(x)$$

and, under symmetrical k-integration,

$$\frac{k_1^\mu k_1^\nu}{k_1^2} \longrightarrow \frac{1}{4}\, g^{\mu\nu}, \qquad g^{\mu\nu} - \frac{k_1^\mu k_1^\nu}{k_1^2} \longrightarrow \frac{3}{4}\, g^{\mu\nu} .$$

Thus, the generalization of the right-hand side of the above commutator is

$$- 6 \frac{G}{\sqrt{2}} \mathscr{J}_{\mu}^{wk(+)}(0) \mathscr{J}_{wk}^{\mu(-)}(0) \delta^4(x_1) \delta^4(y_1) = - 6 H_{wk}(0) \delta^4(x_1) \delta^4(y_1) .$$

The second term therefore contributes

$$- 6 \delta^4(x_1) \delta^4(y_1) T \left\{ \mathscr{J}_{i_2}^{e.m.}(x_2), \mathscr{J}_{e.m.}^{i_2}(y_2), H_{wk}(0) \right\} .$$

The third term requires some care. Since we have not yet let $k_2^0 \longrightarrow \infty$, we should properly consider

$$2 \left(g^{\mu_2 \nu_2} - \frac{k_2^{\mu_2} k_2^{\nu_2}}{k_2^2} \right) T \left\{ \left[\mathscr{J}_{e.m.}^{i_1}(y_1), \left[\mathscr{J}_{i_1}^{e.m.}(x_1), \mathscr{J}_{\mu_2}^{e.m.}(x_2) \right] \right] \delta\left(x_1^0 - x_2^0\right) \delta\left(y_1^0 - x_2^0\right), \mathscr{J}_{\nu_2}^{e.m.}(y_2), H_{wk}(0) \right\} .$$

Now the double commutator is easily verified to be

$$\left[\mathscr{J}_{e.m.}^{i_1}(y_1), \left[\mathscr{J}_{i_1}^{e.m.}(x_1), \mathscr{J}_{\mu_2}^{e.m.}(x_2) \right] \right] \delta\left(x_1^0 - x_2^0\right) \delta\left(y_1^0 - x_2^0\right)$$

$$= - 8 g_{\mu_2 k} \mathscr{J}_{e.m.}^{k}(x_2) \delta^4(x_1 - x_2) \delta^4(y_1 - x_2) .$$

The covariant replacement yields

$$g_{\mu_2 k} \mathscr{J}_{e.m.}^{k}(x_2) \longrightarrow \frac{3}{4} \mathscr{J}_{\mu_2}^{e.m.}(x_2) .$$

Thus, the third term's contribution to $R_{i_1 \; i_2}^{\; i_1 \; i_2}$ is

$$- 12 \delta^4(x_1 - x_2) \delta^4(y_1 - y_2) T \left\{ \mathscr{J}_{i_2}^{e.m.}(x_2), \mathscr{J}_{e.m.}^{i_2}(y_2), H_{wk}(0) \right\} .$$

We now see that the first three terms, after integration over d^4x_1 and d^4y_1, exactly cancel one another so that there are no leading divergences in fourth order. We now proceed to the non-leading divergences contained in the remaining five terms.

The fourth term is easily handled. Using the commutator

$$\left[\mathscr{J}_{i_1}^{e.m.}(x_1), \mathscr{J}_{i_2}^{e.m.}(x_2)\right]\delta\left(x_1^0 - x_2^0\right) = -2\, i\epsilon_{i_1 i_2 k}\, \mathcal{a}^k(x_2)\, \delta^4(x_1 - x_2),$$

this term may be written as

$$16\, T\left\{\mathcal{a}_k(x_2),\ \mathcal{a}^k(y_2),\ H_{wk}(0)\right\}\delta^4(x_1 - x_2)\,\delta^4(y_1 - y_2)\,.$$

We note that this time-ordered product has the same structure as the expression for the second order radiative correction except that electromagnetic current $\mathscr{J}_k^{e.m.}$ is replaced by an axial current of the same form. It is easily verified that this makes no difference in extracting the divergent part of the k_2-integration. This is apparent from the V-A nature of the weak Hamiltonian. Thus, the divergent contributions in this expression cancel.

The fifth and sixth terms are best considered together. To extract the contribution which leads to a divergent momentum subintegration we must now take the derivatives with respect to x_2 and y_2 in the original expression for $R_{i_1\ i_2}^{\ i_1\ i_2}$ inside the time-ordered product. Thus, we must consider

$$\frac{\partial}{\partial x_2^\beta}\frac{\partial}{\partial y_{2\beta}}\left[T\left\{\left[\mathscr{J}_{i_1}(x_1), \mathscr{J}_{i_2}(x_2)\right]\delta\left(x_1^0 - x_2^0\right), \mathscr{J}^{i_2}(y_2), \left[\mathscr{J}^{i_1}(y_1), H_{wk}(0)\right]\delta\left(y_1^0\right)\right\}\right.$$
$$\left. + T\left\{\partial_0\mathscr{J}_{i_1}(x_1), \left[\mathscr{J}^{i_1}(y_1), \mathscr{J}^{i_2}(y_2)\right]\delta\left(y_1^0 - y_2^0\right), \mathscr{J}_{i_2}(y_2), H_{wk}(0)\right\}\right].$$

Taking these derivatives leads to numerous terms, eight of which contribute to divergent momentum integrations. Using again the abbreviated notation, these terms are

$$\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \left[\mathscr{J}^{i_2}, \left[\mathscr{J}^{i_1}, H_{wk}\right]\right]\right] + T\left\{\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \partial_0 \mathscr{J}^{i_2}\right], \left[\mathscr{J}^{i_1}, H_{wk}\right]\right\}$$

$$+ T\left\{\partial_0 \mathscr{J}^{i_2}, \left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \left[\mathscr{J}^{i_1}, H_{wk}\right]\right]\right\} + T\left\{\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \partial_0 \mathscr{J}^{i_1}\right], \left[\mathscr{J}^{i_2}, H_{wk}\right]\right\}$$

$$+ T\left\{\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \partial_0 \mathscr{J}^{i_1}\right], \partial_0 \mathscr{J}^{i_2}, H_{wk}\right\} + T\left\{\partial_0 \mathscr{J}^{i_1}, \left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \left[\mathscr{J}^{i_2}, H_{wk}\right]\right]\right\}$$

$$+ T\left\{\partial_0 \mathscr{J}^{i_1}, \left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \partial_0 \mathscr{J}^{i_2}\right], H_{wk}\right\} + T\left\{\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \left[\mathscr{J}^{i_2}, \partial_0 \mathscr{J}^{i_1}\right]\right], H_{wk}\right\}.$$

It should perhaps be pointed out that in taking the first derivative $\partial/\partial y_{2\beta}$ inside the time-ordered product, the terms involving the equal-time commutator

$$\left[\mathscr{J}^{i_2}(y_2), \left[\mathscr{J}_{i_1}(x_1), \mathscr{J}_{i_2}(x_2)\right]\right] \delta\left(x_1^0 - x_2^0\right) \delta\left(y_2^0 - x_2^0\right)$$

give a contribution to $R_{i_1\ i_2}^{i_1\ i_2}$ which goes as a constant times k_2^0 as $k_2^0 \longrightarrow \infty$. Such a term vanishes under symmetrical integration and hence may be dropped. By noting that the commutator

$$\left[\mathscr{J}_{i_1}^{e.m.}(x_1), \mathscr{J}_{i_2}^{e.m.}(x_2)\right] \delta\left(x_1^0 - x_2^0\right)$$

is antisymmetric under $i_1 \longleftrightarrow i_2$ we observe that all of the eight terms cancel in pairs except the first and the last. These two may be rewritten as

$$-\frac{1}{2}\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \left[\left[\mathscr{J}^{i_1}, \mathscr{J}^{i_2}\right], H_{wk}\right]\right] - \frac{1}{2} T\left\{\left[\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right], \partial_0 \left[\mathscr{J}_{i_1}, \mathscr{J}^{i_2}\right]\right], H_{wk}\right\}$$

by using the above mentioned antisymmetry and the Jacobi identity. By evaluating the commutator $\left[\mathscr{J}_{i_1}, \mathscr{J}_{i_2}\right]$ we obtain the form

$$- 4\left[\mathcal{a}_k(y_2), \left[\mathcal{a}^k(x_2), H_{wk}(0)\right]\right] \delta\left(x_2^0\right) \delta\left(y_2^0\right)$$

$$- 4\, T\left\{\left[\mathcal{a}^k(y_2), \partial_0\, \mathcal{a}_k(x_2)\right] \delta\left(x_2^0 - y_2^0\right), H_{wk}(0)\right\}.$$

Once again we note that, except for the axial vector nature of the current, these two terms have the same structure as the divergent contribution found in second order. Hence, after removal of the mass renormalization contribution, it is easily verified that they cancel.

Next, consider the seventh term, By noting that

$$\left[\mathscr{J}_{i_1}^{e.m.}(x_1),\ H_{wk}(0)\right]\delta\left(x_1^0\right) = -2\,\frac{G}{\sqrt{2}}\left[\mathscr{J}_0^{wk(+)}(0)\mathscr{J}_{i_1}^{wk(-)}(0) - \mathscr{J}_{i_1}^{wk(+)}(0)\mathscr{J}_0^{wk(-)}(0)\right]\ \delta^4(x),$$

we see that the covariant generalization of this term is proportional to

$$T\left\{\partial_\mu \mathscr{J}_\nu^{e.m.}(x_1),\ \mathscr{J}_{i_2}^{e.m.}(x_2),\ \mathscr{J}_{e.m.}^{i_2}(y_2),\ \mathscr{J}_{wk}^{\mu(+)}(0)\mathscr{J}_{wk}^{\nu(-)}(0) - \mathscr{J}_{wk}^{\nu(+)}(0)\mathscr{J}_{wk}^{\mu(-)}(0)\right\} .$$

As usual, we must now take the derivatives with respect to x_2 and y_2 in $R_{i_1\, i_2}^{i_1\, i_2}$ inside this time-ordered product. The terms which lead to divergent k_2 integrations are

$$T\left\{\left[\mathscr{J}^{i_2},\ \left[\mathscr{J}_{i_2},\ \partial_\mu \mathscr{J}_\nu\right]\right],\ \mathscr{J}_+^\mu\ \ \mathscr{J}_-^\nu - \mathscr{J}_+^\nu \mathscr{J}_-^\mu\right\}$$

$$+\,T\left\{\partial_\mu \mathscr{J}_\nu,\ \left[\mathscr{J}^{i_2},\ \left[\mathscr{J}_{i_2},\ \mathscr{J}_+^\mu \mathscr{J}_-^\nu - \mathscr{J}_+^\nu \mathscr{J}_-^\mu\right]\right]\right\}$$

$$+\,T\left\{\partial_\mu\ \mathscr{J}_\nu,\ \left[\mathscr{J}^{i_2},\ \partial_0\ \ \mathscr{J}_{i_2}\right],\ \mathscr{J}_+^\mu\ \mathscr{J}_-^\nu - \mathscr{J}_+^\nu\ \mathscr{J}_-^\mu\right\} .$$

The appropriate equal-time commutators in the first two terms, and their respective covariant generalizations, are easily verified to be (using the abbreviated notation)

$$\left[\mathcal{J}^{i_2}, \left[\mathcal{J}_{i_2}, \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu\right]\right] = -8\left[\left(\partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu\right) - g_{\nu 0}\left(\partial_\mu \mathcal{J}_0 - \partial_0 \mathcal{J}_\mu\right)\right]$$

$$\longrightarrow -6\left(\partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu\right)$$

$$\left[\mathcal{J}^{i_2}, \left[\mathcal{J}_{i_2}, \mathcal{J}_+^\mu \mathcal{J}_-^\nu - \mathcal{J}_+^\nu \mathcal{J}_-^\mu\right]\right] = -4\left(\mathcal{J}_+^\mu \mathcal{J}_-^\nu - \mathcal{J}_+^\nu \mathcal{J}_-^\mu\right)$$

$$-4 g^{\mu 0}\left(\mathcal{J}_+^0 \mathcal{J}_-^\nu - \mathcal{J}_+^\nu \mathcal{J}_-^0\right) - 4 g^{\nu 0}\left(\mathcal{J}_+^\mu \mathcal{J}_-^0 - \mathcal{J}_+^0 \mathcal{J}_-^\mu\right)$$

$$\longrightarrow -6\left(\mathcal{J}_+^\mu \mathcal{J}_-^\nu - \mathcal{J}_+^\nu \mathcal{J}_-^\mu\right)$$

where the arrows indicate the covariant generalization. The third term contains the commutator

$$\left[\mathcal{J}^{i_2}, \partial_0 \mathcal{J}_{i_2}\right]$$

which we handle in the usual fashion. After removing the covariant mass renormalization term we recognize that the remaining contribution cancels that of the first two terms.

Finally, we have the eighth term, whose covariant form with respect to i_1 is

$$\frac{3}{4} T\left\{\partial_\alpha \mathcal{J}_\mu, \partial^\alpha \mathcal{J}^\mu, \mathcal{J}_{i_2}, \mathcal{J}^{i_2}, H_{wk}\right\}.$$

Extracting once again the divergent contribution to the k_2 integration, we obtain commutators of the same form as those considered above. It is straightforward to show that the resulting terms cancel after removal of mass renormalization contributions.

This concludes the demonstration of the finiteness of the fourth order radiative corrections in this model. Note that this was done by showing that, after all the commutators were evaluated, the various terms cancelled precisely as they did in the second order calculation. Thus, it becomes obvious that the result must hold to all orders in e^2. The commutators occuring in any higher order have already been encountered in fourth order. By suitable manipulations we can reduce them to terms which cancel as in the second order case.

This result is hardly surprising. It is a well-known[13] fact that for μ- decay the radiative corrections may be shown to be finite to all orders in e^2 by performing a Fierz transformation on the current-current Hamiltonian. What we have shown is that, with the stated assumptions, the divergent part of the radiative corrections depends only on the commutation relations of the currents, and not on the detailed nature of the particles in the initial and final states or their strong interactions, if any. Thus, the μ-decay result must hold for any leptonic or semileptonic (or non-leptonic, for that matter) weak process in this model.

IV. CONCLUSION

The purpose of this work has been to attempt to find models for the hadron and lepton weak and electromagnetic currents which yield a consistent theory of radiative corrections to lowest order weak processes. We mean, by consistent, that divergent momentum integrals do not enter in the calculation of physically measurable quantities. In the previous section we exhibited two models which were consistent in this sense, to all orders in e^2. One, a quark model with integrally charged quarks,[6,7] gave finite radiative corrections and another, the algebra of fields model, contained only an unobservable, universally divergent factor. We also observed that in theories with q-number Schwinger terms in the current commutators this consistency condition could not be satisfied. Since currents constructed from spin-1/2 fields and the algebra of fields model are the only two simple cases we know of where the Schwinger terms are c-numbers, the two models mentioned above thus seem to have a special position.

As we mentioned previously, we have not completely solved the consistency problem with these two examples. We removed divergent contributions to electromagnetic mass shifts and to strong coupling constant renormalization by adding a counter term to the Hamiltonian. It remains to be shown that these terms do not lead to divergences in the calculation of mass ratios, strong coupling constant ratios, and, ultimately, the ratio of the strong and electromagnetic coupling constants to the weak interaction constant G. The simplest way for this to happen would be for the commutator

$$\left[\mathcal{J}^{i}_{e.m.}(y),\ \partial_0 \mathcal{J}_i^{e.m.}(x)\right] \delta(x^0 - y^0)$$

to be a c-number so that it would not contribute to connected amplitudes, although this is clearly not a necessary condition. It has been noted[34] that in the algebra of fields model the above commutator can be made a c-number by taking an appropriate limit on the underlying Yang-Mills field theory. In general, however, it is far from obvious that the commutator could not give rise to unwanted divergences. Hence, this problem, which is outside the scope of our paper, remains an open question.

To get as far as we did, it was necessary to make several strong assumptions, which we must now discuss. We made three assumptions which are all closely related. These are that the Bjorken expansion was justified, that naive commutators could be used, and that any divergences in closed loops in the hadron or lepton "blobs" could be ignored. Recent investigations[17-19] have shown that if the last assumption is unjustified, then the first two break down also. Adler and Tung[17] considered the "gluon" model of strong interactions, a renormalizable model with an SU(3) triplet of spin-1/2 particles bound by the exchange of an SU(3)-singlet massive vector particle. They considered the current-fermion scattering amplitude to second order in the gluon-fermion coupling constant g. They showed that as the current momentum $k_0 \longrightarrow \infty$, the coefficient of $1/k_0$ was not that obtained by calculating the naive commutator in the Bjorken expansion, but that it contained a correction term of order g^2. Furthermore the amplitude contained a term going as $(\ln k_0^2)/k_0^2$, so that the expansion would not be valid to order $1/k_0^2$.

Both of these effects have their origin in primitively divergent subgraphs which appear in the perturbation expansion of the interaction, as was emphazised more recently by Tung.[19] The divergent integrals are made finite by the usual regulator technique, but as the regulator masses are let to approach infinity, additional contributions are picked up which imply that naive use of equations of motion and canonical commutators is no longer justified. Also, whenever there are logarithmic divergences, $\ln k_0^2$ terms will always occur. Since there are divergent graphs in almost any non-trivial strong interaction model, it appears that all three of the above assumptions break down in perturbation theory.

Thus, if we are to maintain the assumptions, we must suppose that the perturbation calculations are misleading and that the exact theory must somehow be more convergent than individual terms in the series expansion. It turns out in all of these examples that the commutators of the time components of the currents are unchanged simply because of current conservation. This suggests that what is needed to make the naive commutation relations also hold for the space-space commutators is some analogous smoothness condition involving time derivatives of the spatial components of currents.

One might speculate that perhaps the Bjorken expansion is justified but the use of naive commutators is not. However, if this were the case, it is very difficult to see how universality of the divergent radiative corrections could be maintained because we do not expect the commutators of the lepton currents to be modified, except perhaps by higher order weak interactions, which is a whole new subject.

Most of the above comments have been made with reference to the current model with spin-1/2 fields. However, for the algebra of fields model things are no better. Behind the facade of simple current commutation relations lurks a non-renormalizable theory of matter. We do not know how to interpret a perturbation expansion of such a theory so the question of the validity of the naive commutators is completely open in this case.

This brings us naturally to a discussion of the use of the Bjorken expansion in analyzing divergences in non-renormalizable field theories. A concrete example would be higher order weak corrections. Having seen our partial success in handling divergences in electromagnetic radiative corrections, one might suppose that at least the leading divergences to each order in the weak coupling constant could be shown to contribute only to an overall coupling constant renormalization plus mass shift terms. Unfortunately, there is a serious question of principle in applying the non-covariant expansion procedure. In order to avoid possible ambiguities in making the covariant generalization, we must let only one loop momentum approach infinity at a time, holding all others fixed. For this to be justified, free translations of the origin of the momentum integrals must be permitted. However, it is well known[35] that this is true only if the integrals are no worse than logarithmically divergent.

The attendant difficulties may be illustrated by considering second order weak corrections to the electromagnetic vertex of a particle in a W-boson theory of weak interactions. A naive application of the technique used in this paper seems to give quadratically divergent corrections. However, a Feynman diagram calculation shows that the quadratically divergent terms cancel, as they must because of the Ward identity.[36] The discrepancy is due to an unjustified translation of the momentum loop integration variable. Thus, greater care must be exercised in using the Bjorken expansion when higher than logarithmic divergences occur, to insure that the Ward identity holds. This problem is as yet unresolved. Hence, the divergences in higher order weak interactions are another open question. Of course, this is hardly a new state of affairs.

Our main concern here has been with the divergences occurring in the calculation of radiative corrections. Ultimately, having shown that divergences are either universal or are not present at all, we would like to calculate the finite parts of the corrections to test quantitatively the hypothesis of universality[1] of the weak interaction coupling. Unfortunately, in the absence of a more detailed theory of strong interactions than we now have, only crude estimates can be made. The fact that such estimates[5,20] of finite contributions are in rough agreement with experiment is the main motivation for attempting to show that divergences are absent order by order in e^2. Of course, it is also possible that the divergences are only a property of the perturbation expansion and do not occur in the exact theory. Another way out is to assume the existence of negative metric states.[37] We have shown here that a more conventional solution to the problem of divergences can be found consistent with the restrictions of current algebra. However, the justification of our various assumptions awaits, on the one hand, further developments in the theory of strong interactions, and, on the other, a satisfactory theory of higher order weak interactions. These two problems may well be related.

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APPENDIX

In the main text we considered various matrix elements between particle states of a time-ordered product of the weak Hamiltonian density and several electromagnetic currents. We wish here to give a discussion of external line wave function renormalization contributions to divergent radiative corrections. To do this we must start with the usual perturbation expansion[24] obtained from the reduction formula. In the weak process $A \to B\ell\nu_\ell$ let all particles be spinless, for simplicity, since this does not affect the argument. The matrix element $\mathcal{M}_0$ for this process, to lowest order in G and zeroth order in e^2, is

$$\mathcal{M}_0 = \left(Z^A Z^B Z^\ell Z^{\nu_\ell}\right)^{-1/2} \prod_{i=1}^{4} \int d^4u_i e^{-ip_A u_1 + ip_B u_2 + ip_\ell u_3 + ip_{\nu_\ell} u_4}$$

$$\times D_{u_1} D_{u_2} D_{u_3} D_{u_4} \mathcal{G}_0(u_1, u_2, u_3, u_4)$$

where

$$D_{u_1} \equiv \frac{\partial^2}{\partial u_1^\mu \, \partial u_{1\mu}} + m_A^2, \text{ etc.,}$$

and

$$\mathcal{G}_0(u_1, u_2, u_3, u_4) = \langle 0 | T\left\{\phi^A(u_1), \phi^B(u_2), \phi^\ell(u_3), \phi^{\nu_\ell}(u_4), H_{wk}(0)\right\} | 0 \rangle .$$

Here the ϕ's are the fields of the respective particles, which for the case of hadrons obey the exact strong interaction equations of motion. The Z's are the appropriate wave function renormalization constants. The matrix element $\mathcal{M}_n$ which gives the order $(e^2)^n$ radiative corrections to $\mathcal{M}_0$ is given by a similar expression with $\mathcal{G}_0$ replaced by

$$\mathcal{G}_n(u_1, u_2, u_3, u_4) = \frac{(-ie)^{2n}}{(2n)!} \prod_{i=1}^{2n} \int d^4x_i \; \langle 0 | T\left\{H_{e.m.}(x_1), \ldots, H_{e.m.}(x_{2n}),\right.$$

$$\left. \phi^A(u_1), \phi^B(u_2), \phi^\ell(u_3), \phi^{\nu_\ell}(u_4), H_{wk}(0)\right\} | 0 \rangle$$

where

$$H_{e.m.}(x) = e \mathscr{J}_\mu^{e.m.}(x) \mathscr{A}^\mu(x)$$

since, again for simplicity, we ignore possible contact terms in $H_{e.m.}(x)$.

The wave function renormalization constants Z are determined from the exact particle propagators by going to the mass shell. For example, if $S_F^{'A}(p_A)$ is the exact propagator for particle A, then

$$S_F^{'A}(p_A) \xrightarrow[p_A^2 \to m_A^2]{} \frac{Z^A}{p_A^2 - m_A^2 + i\epsilon} .$$

$S_F^{'A}(p_A)$ has the perturbation expansion

$$S_F^{'A}(p_A) = \sum_{n=0}^{\infty} \frac{(-ie)^{2n}}{(2n)!} \prod_{i=1}^{2n} \int d^4x_i \int d^4u_1 e^{-ip_A u_1}$$

$$\times \langle 0 | T \left\{ H_{e.m.}(x_1), \ldots, H_{e.m.}(x_{2n}), \phi^A(u_1), \phi^A(0) \right\} | 0 \rangle .$$

To calculate the divergent contributions to Z^A we first contract the $\mathscr{A}^\mu(x)$'s in $H_{e.m.}(x)$ to form n photon propagators. We then extract the part of the remaining time-ordered product which goes as $\left(1/k_i^0\right)^2$ as $k_i^0 \to \infty$ for each loop momentum k_i. The divergent contributions come from double commutators of the form

$$\left[\mathscr{J}_\mu^{e.m.}(y), \left[\mathscr{J}^\mu_{e.m.}(x), \phi^A(u)\right]\right] \delta(x^0 - u^0)\, \delta(y^0 - u^0),$$

or in constructing the covariant generalization of

$$\left[\mathscr{J}^i_{e.m.}(y), \partial_0 \mathscr{J}_i^{e.m.}(x)\right] \delta(x^0 - y^0)$$

if it contains operator Schwinger terms.

Since for the propagator we are interested only in the divergent contributions to Z^A, we may assume that the double commutator is proportional to $\phi^A(u)$, i.e.,

$$\left[\mathscr{J}_\mu^{e.m.}(y), \left[\mathscr{J}^\mu_{e.m.}(x), \phi^A(u)\right]\right] \delta(x^0 - u^0)\, \delta(y^0 - u^0) = \alpha\, \phi^A(u)\, \delta^4(x-u)\, \delta^4(y-u)$$

for some constant α. Furthermore, in calculating $\mathcal{M}_n$, any other terms in the double commutator will not give a pole in the external line momentum and hence will not contribute to the on-mass-shell amplitude. It is also easily verified that, in any of the models we have considered, any additional equal-time commutators which arise in making the mass shift term covariant are also proportional to ϕ^A.

The divergent contribution to Z^A we denote by $Z^A_{div.}$ It may now be expressed as a power series in e^2 with divergent coefficients. These terms come from equal-time commutators with both $\phi^A(u_1)$ and $\phi^A(0)$ in the time-ordered product. By considering all possible combinations of such commutators it is easily seen that we may write $Z^A_{div.}$ as $\left(\sqrt{Z^A_{div.}}\right)^2$, where the power series for $\sqrt{Z^A_{div.}}$ has coefficients which are obtained by including the commutators with only one of the two ϕ^A, say $\phi^A(u_1)$.

A similar argument holds when we consider $\mathcal{M}_n$. There will now be additional divergent contributions to the various photon loop momenta involving equal-time commutators with the fields for the external particles. As in the single particle propagator example, when we sum over all orders these additional terms will factor out to give a contribution of $\sqrt{Z_{div.}}$ for each particle. These terms are then cancelled by the divergent part of the $1/\sqrt{Z}$ appearing in the reduction formula.

Having removed these divergent contributions, it is now presumably safe to reverse the procedure for obtaining the reduction formula and put the external particles into in- and out-states. We thus obtain an expression which is the same as that used in the main text except for certain finite electromagnetic corrections to the wave function renormalization constants. These clearly do not affect any of our arguments since we are not interested in the details of the finite parts.

FOOTNOTES

(1) The use of the Bjorken expansion for time-ordered products of an arbitrary number of currents has been studied in a different context by P. Olesen (Ref. 16).

(2) This example was discussed briefly by A. Sirlin (Ref. 11).

(3) An implicit assumption of the model is that F(x) is free of operator derivatives.

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