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THE AMBIGUOUS MASS AND WIDTH OF THE ρ-MESON†

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ABSTRACT

The relationship between the mass and width of the ρ-meson resonance seen in colliding beam experiments and the behavior of the $\pi\pi$ scattering phase shift is discussed in detail. Two conclusions are drawn: (1) The resonance peak in e^+e^- annihilation occurs 5 or 10 MeV below where the phase shift reaches $\pi/2$. (2) The width should not be identified with the slope of the phase shift at one point; in fact, it depends critically on details of the $\pi\pi$ strong interaction over a wide energy range.

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Suppose one could do $\pi\pi$ scattering over a wide range of energies from, say, 300 to 1000 MeV center of mass energy. Presumably, one would see a wide bump in the cross section which would be associated with the ρ-meson resonance. An experimentalist might then find parameters for a Breit-Wagner (BW) resonance formula which best fits the data. These he would quote as the mass m and width Γ of the resonance. (Given sufficiently accurate data, of course, more sophisticated fitting programs would be devised.)

One would also see two pions produced in electron-positron colliding beam experiments, and measurements[1] indicate a broad peak in the invariant mass spectrum of the two pions which is also interpreted as the effect of the ρ-meson. After removing kinematical factors and performing radiative corrections, the data are given as a series of points for the absolute square of the pion form factor $F_\pi(t)$, as a function of the invariant mass^2 = t of the two pion system. A best fit of a BW resonance formula is made to the data points, and a mass m_0 and width Γ_0 are assigned to the peak.

The theoretical basis for the BW resonance formula in elastic scattering has been known[2] for some time; one derivation will be given below. The question is what, in principle, is the relation between the masses m, m_0 and widths Γ, Γ_0 of the resonance as determined by the two empirical methods outlined above?

Consider first the scattering of pions. Using the N/D representation[3], we write the I = 1, J = 1 partial wave scattering amplitude as the ratio of two analytic functions of the invariant mass^2,

$$f = \frac{N(t)}{D(t)} . \tag{1}$$

In the elastic approximation, $N = -\frac{1}{q} \operatorname{Im} D$, $(q = \sqrt{\frac{t}{4} - \mu^2})$. In this case, it is customary to write $D = |D| e^{-i\delta}$, so that*

$$qf = \frac{-1}{\frac{D_R}{D_I} + i} = \frac{1}{\cot\delta - i}$$

Clearly $|qf|^2$ is maximum for $(D_R/D_I)^2$ minimum, which occurs for $D_R = 0$ corresponding to a phase shift $\delta = \pi/2$. This defines the position $t = m^2$ of the peak, and m is called the mass of the resonance; in this case, of the ρ-meson. In a neighborhood of this point, we have

$$qf \approx \frac{-m\Gamma}{t - m^2 + im\Gamma}$$

where the width Γ is defined by $\frac{1}{m\Gamma} = \frac{D'_R(m^2)}{D_I(m^2)} = \delta'(m^2)$. $|qf|^2$ has the usual BW shape**

$$\frac{(m\Gamma)^2}{(t-m^2)^2 + (m\Gamma)^2} \tag{2}$$

The mass and width depend only upon the phase shift.

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* Here and subsequently, we denote the real and imaginary parts of D by D_R and D_I, respectively. δ is the phase shift.

** Note that, had we approximated $|f|^2$ rather than $|qf|^2$, our definitions of m and Γ would have been changed.

Now consider the pion form factor. In the elastic approximation, F_π is related to D by[4]

$$F_\pi = \frac{D(0)}{D(t)} \frac{P(t)}{P(0)} \tag{3}$$

where P is an arbitrary entire function, which is real for real t. Whereas the mass and width of the resonance in $\pi\pi$ scattering depended only upon the <u>phase shift</u> and its first derivative, the behavior of $|F_\pi|^2$ depends only upon the <u>modulus</u> of D as well as the unknown function P.

Define $\frac{1}{E} = \frac{P}{D}$ so that $|F_\pi|^2 = \frac{|E(0)|^2}{|E(t)|^2}$

$|F_\pi|^2$ is maximum where $|E|^2$ is minimum, which is certainly not, in general, at m^2, where $D_R = 0$ ($\delta = \pi/2$). If the mass shift is small, we may approximate $|E(t)|^2$ in the neighborhood of m^2 by a Taylor's series

$$|E^2(t)| \approx |E^2(m^2)| + (t-m^2)|E^2(m^2)|' + \frac{(t-m^2)^2}{2}|E^2(m^2)|''$$

$$= \frac{|E^2(m^2)|''}{2}\left((t-m_0^2)^2 + (m_0\Gamma_0)^2\right)$$

where

$$m_0^2 = m^2 - \frac{|E^2|'}{|E^2|''} = m^2 - \frac{|E|\,|E|'}{|E|\,|E|'' + (|E|')^2} \tag{4a}$$

$$(m_0\Gamma_0)^2 = \frac{2|E|^2|E^2|'' - (|E^2|')^2}{(|E^2|'')^2} = \frac{|E|^3|E|''}{((|E|')^2 + |E|\,|E|'')^2} \tag{4b}$$

Therefore*

$$|F_\pi|^2 = \frac{2E(0)^2}{|E^2|''}\left[\frac{1}{(t-m_0^2)^2 + (m_0\Gamma_0)^2}\right] \tag{5}$$

Formula (5) is the principal result. In first approximation, the peak in $|F_\pi|^2$ is also a Breit-Wigner, but it has a different mass and width from the Breit-Wigner relevant to $\pi\pi$ scattering (formula (2)). Comparing Eq. (4) with Eq. (2), it is clear why, in principal, the mass and width of the peak in $e^+e^- \to \pi^+\pi^-$ should not be expected to be the same as in $\pi\pi$ scattering. We will illustrate the result with an example.

One popular assumption[4] is that F_π has the least degree at infinity compatible with its analytic structure. The physical motivation for this assumption is obscure but its popularity depends on the fact that it eliminates the ambiguities mentioned earlier. In the elastic approximation, it implies that the correct solution[5] for F_π is Muskhelishvili's fundamental solution (sometimes called Omnès' function)

$$F_\pi = e^{\frac{t}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta(x)dx}{x(x-t)}}$$

In particular, then, $|E|$ is given by the principal value integral

$$|E| = e^{-\frac{t}{\pi}\,P\!\int_{4\mu^2}^{\infty}\frac{\delta(x)dx}{x(x-t)}}$$

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* We assume that $|E|'' > 0$, so that $(m_0\Gamma_0)^2 > 0$. If not, then one must retain higher order terms in this expansion and the shape of the curve will no longer be a Breit-Wigner but may be, for example, double-peaked and asymmetric.

Since δ corresponds to a P-wave phase shift, near threshold $(t \to 4\mu^2)$, we expect $\delta(t) \sim (t-4\mu^2)^{3/2}$. The general behavior we expect for δ is for it to remain small as t increases from threshold, rising rather rapidly through $\pi/2$ at the ρ-mass, and nearly equal to π beyond the resonance.*

In order to see what this assumption yields for the mass shift and width (formula (4)), we need $|E|'$ and $|E|''$ at $t = m^2$. One can show

$$\frac{|E|'}{|E|} = \frac{1}{\pi} P\int_{4\mu^2}^{\infty} \frac{\delta'(t)dt}{m^2-t} \equiv A$$

$$\frac{|E|''}{|E|} = \left(\frac{|E|'}{|E|}\right)^2 + \frac{1}{\pi} P\int_{4\mu^2}^{\infty} \frac{\delta''(t)dt}{m^2-t} \equiv A^2 + B$$

In this case, formulas (4) become

$$m_0\Gamma_0 = \frac{A^2+B}{2A^2+B}$$

$$\frac{m_0^2-m^2}{m_0^2} = -\frac{\Gamma_0}{m_0} \frac{A}{\sqrt{A^2+B}}$$

Other than in order of magnitude, the width Γ_0 may bear little relation to Γ, depending as it does on such different aspects of the behavior of the phase shift. Noting that

$$\frac{m_0^2-m^2}{m_0\Gamma_0} = \frac{-(\text{sign } A)}{\sqrt{1+\frac{B}{A^2}}} ,$$

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* For large t, we are ignorant of the behavior of δ, but the elastic approximation and the preceding solution are irrelevant anyway.

We must have $B >> A^2$ for the mass shift to be small compared to the width. Since we don't expect the mass to shift outside the resonance peak, $B >> A^2$ will be the usual situation, in which case, our expressions simplify to

$$m_0 \Gamma_0 \approx \frac{1}{\sqrt{B}}$$

$$\frac{m_0^2 - m^2}{m_0^2} \approx -\Gamma_0^2 A$$

Since A will be on the order of $\frac{1}{m^2}$ or less, the mass shift depends on $\left(\frac{\Gamma_0}{m_0}\right)^2$ and so will be quite small. Note that if δ' is symmetric about $t = m^2$, A will be very nearly zero. However, even a simple effective range approximation[6] suggests that δ' is skewed, especially because δ is a P-wave resonance. Since the threshold region is enhanced, A will be positive, so the mass m_0 will be less than m. For the ρ-meson, for which $\left(\frac{\Gamma_0}{m_0}\right)^2$ is 2 or 3%, we expect, therefore, the peak in $|F_\pi|^2$ to appear on the order of 5 or 10 MeV less than m.[7,8] No general statement can be made with regard to the relation between Γ and Γ_0, since the relationship between $\delta'(m^2)$ and B is extremely sensitive to the choice of a model for $\delta(t)$. The second derivative δ'' vanishes near m^2, since the rate of change of the phase shift, $\delta'(t)$, reaches a maximum near m^2. This has two important consequences for the principal value integral for B. (1) B is sensitive to the distance from m^2 to where δ'' vanishes. (2) Because this vanishes near m^2, the contributions to B from the behavior of the phase shift farther away from the resonance are enhanced, all the more so because δ'' blows up at threshold. Therefore, the relationship between Γ_0 and Γ is by no means simple; the interpretation of Γ_0 as simply the slope of the phase shift at m^2 should not be made. In fact, given more accurate data than presently available, the use of a Breit-Wigner approximation could also be questioned.

Although colliding beam experiments are regarded as the "cleanest" observation of the ρ-meson resonance, it is clear from the preceding discussion that the interpretation of the data in terms of $\pi\pi$ scattering is by no means straightforward. It is little wonder that the mass and width of the ρ-meson determined in other, more complicated reactions, such as $\pi N \rightarrow 2\pi N$ or $\gamma N \rightarrow 2\pi N$, should be slightly different. Without a theory of strong interactions, little progress can be made toward reconciling these differences.

For resonances other than the ρ, for which the ratio of the width to mass is smaller, the preceding effects will be smaller. However, a theoretical discussion is complicated seriously by inelasticity, i.e., the presence of several competing strong decay channels. In attempting to perform an analysis similar to our discussion here for colliding beam production of kaons or nucleons, one faces the problems associated with unphysical regions for the dispersion integrals. Using a method similar to that employed by Frazer and Fulco[9], one can in part remove the unphysical region, but such techniques presuppose knowledge of the pion form factor, at the very least. In a subsequent note, we shall discuss how data from colliding beam experiments may be further utilized.

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REFERENCES

1. J. E. Augustin et al., Phys. Letters 28B, 508 (1969). V. L. Auslander et al., Phys. Letters 25B, 433 (1967).

2. G. Breit and E. P. Wigner, Phys. Rev. 49, 519, 642 (1936). More modern derivations are given many places, e.g., Chapters 6 and 8 of M. L. Goldberger and K. M. Watson, Collision Theory, J. Wiley, New York, 1964.

3. G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

4. P. Federbush, M. L. Goldberger, and S. Treiman, Phys. Rev. 112, 642 (1958). R. Omnés, Nuovo Cimento 8, 316 (1958). This is the most general form for F_π consistent with its analytic properties and normalization, $F_\pi(0) = 1$. P is usually assumed to be a polynomial and, to avoid unwanted singularities of F_π, must be chosen to vanish where D vanishes. Not only is P otherwise arbitrary, but also D is ambiguously defined by f. The latter ambiguity is called the CDD ambiguity. L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. 101, 453 (1956).

5. N. I. Muskhelishvili, Singular Integral Equations, Noordhoff, Groningen, 1953.

6. This approximation is $\frac{q^3}{\sqrt{t}} \cot\delta = \left(\frac{m^2 - t}{m\Gamma}\right)\left(\frac{q_\rho^3}{m}\right)$. See, e.g., page 392 of S. Gasiorowicz, Elementary Particle Physics, J. Wiley, New York, 1966. It is interesting to compare this model to a simple Taylor's expansion $\cot\delta = \frac{m^2 - t}{m\Gamma}$, which yields nearly identical values for Γ_0 and Γ and a negligible mass shift.

7. Another example is given by Gounaris and J. J. Sakurai, Phys. Rev. Letters 21, 244 (1968). Their results would seem to indicate that higher order terms in the Taylor series for $|E|$ can significantly alter the

effective width. These authors use a different form for F_π.

8. After completing this work a CERN preprint arrived from M. Roos and J. Pišút, Pion Form Factor and Rho Meson Width, (February 1969). Several different models for the relationship between the form factor and phase shift were investigated in detail. See this paper for a number of references to related literature.

9. W. R. Frazer and J. R. Fulco, Phys. Rev. 117, 1603 (1960). Their method is a standard technique for treatment of these kinds of integral equations. See reference 5.