

Paramodulation and Set of Support*

by

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INTRODUCTION

The applications of the set-of-support strategy in fields in which automatic theorem proving plays an important role continue to increase. Question-answering systems such as that of Green [3] and information-retrieval systems such as that of Darlington [2] rely heavily on automatic theorem proving. Since one of the principal problems in theorem proving is generation of a very large number of "irrelevant" inferences, this problem is important for any system based on a theorem-proving subprogram. The set-of-support strategy was formulated to impede the generation of irrelevant inferences and thus restrict the number of inferences to be examined during the search for a refutation. It is desirable however for the restricted system to retain refutation completeness.

Two inference systems will be considered in this paper: the system Π employing the inference rules paramodulation, resolution, and factoring, and the system Σ employing only resolution and factoring. The system Π is applicable to first-order theories with equality, while Σ is efficiently applied only to ordinary first-order theories. ΠT and ΣT are identical to Π and Σ respectively, except that only T -supported inferences are allowed. The system ΣT (Σ with T as set of support) has been shown to be refutation complete when S is finite and $S-T$ is satisfiable. (The finiteness consideration for S is not necessary for refutation-completeness, but is necessary to show that a certain procedure is a refutation procedure.)

A number of papers have been published modifying the original resolution principle. Such modifications include hyper-resolution [8], semantic resolution [9], and resolution with merging [1]. Due no doubt in part to the success

with which the set-of-support strategy has met, a common point of interest has been the question of refutation completeness of the system combining set of support with the particular modification of resolution then under consideration. For such systems the conclusion usually is, briefly speaking, that set of support is complete. More formally, for such modified systems Ω , ΩT is frequently refutation complete.

In this paper the definition of *set of support* is extended to inference systems based on paramodulation, thus extending the scope of automatic theorem-proving from ordinary first-order theories to first-order theories with equality, the former being the subject of earlier theorem-proving papers. The question which naturally comes to mind is answered in the main result of the paper: set of support is refutation complete for functionally reflexive first-order theories with equality.

DEFINITIONS AND NOTATION

In this paper A, B, C, D, E, and F will be clauses; R will be the equality predicate; S, T, and U will be (not necessarily finite) sets of (not necessarily ground) clauses; f will be a function symbol; k, l, and m will be literals; s, t, and u will be terms; x_1, x_2, \dots will be individual variables; σ and τ will be substitutions of (not necessarily ground, i.e., not necessarily variable-free) terms for variables; and Ω will be an inference system.

Definition (Paramodulation): Let A and B be clauses such that a literal Rst (or Rts) occurs in A and a term u occurs in (a particular position in) B. Further assume that s and u have a most general common instance $s' = s\sigma = u\tau$ where σ and τ are the most general substitutions such that $s\sigma = u\tau$. Where \hat{B} is obtained by replacing by $t\sigma$ the occurrence of $u\tau$ in the position in B corresponding to the particular position of the occurrence of u in B, infer the clause $C = \hat{B} \cup (A - \{Rst\})\sigma$ (or $C = \hat{B} \cup (A - \{Rts\})\sigma$). C is called a *paramodulant* of A and B (and also of B and A) and is said to be *inferred by paramodulation from A on Rst (or Rts) into B on (the occurrence in the particular position in B of) u*. The literal Rst (or Rts) is called the *literal of paramodulation* [6].

From a given pair A and B of clauses one can usually infer by paramodulation a number of clauses. Which clause is inferred depends first on the direction of paramodulation (A into B or B into A), then on the equality literal of paramodulation, then on the choice (first or second) of argument within that literal, then on the term and its occurrence within the other clause.

Notation: SPT will be $\{C \mid C \text{ can be inferred by paramodulation from } A \text{ into } B \text{ or from } B \text{ into } A \text{ where } A \in S \text{ and } B \in T\}$. " $\{A\}P\{B\}$ " will be abbreviated " APB "; " $SP\{B\}$ " by " SPB ", etc.

For example, where R is the equality predicate and, intuitively, f is product and g inverse, if A is $Rf(yg(y))e$ and B is $Rf(xx)e$, then $APB = \{\{Rf(ef(yg(y)))e\}, \{Rf(f(yg(y))e)e\}, \{Rf(xx)f(yg(y))\}, \{Rf(eg(f(xx)))e\}, \{Rf(f(xx)g(e))e\}, \{Rf(yg(y))f(xx)\}\}$. The first three elements of APB are obtained by paramodulation from A into B , and the last three from B into A .

Since the terms *resolution* and *resolvent* vary somewhat in usage throughout the literature, we give the following:

Definition: For any literal l , $|l|$ is that atom such that either $l = |l|$ or $l = -|l|$.

Definition (Resolution): If A and B are clauses with literals k and l respectively, such that k and l are opposite in sign (i.e., exactly one of them is an atom) but $|k|$ and $|l|$ have a most general common instance m , and if σ and τ are most general substitutions with $m = |k|\sigma = |l|\tau$, then infer from A and B the clause $C = (A - \{k\})\sigma \cup (B - \{l\})\tau$. C is called a *resolvent* of A and B and is inferred by *resolution* [4][7].

Definition (Factoring): If A is a clause with literals k and l such that k and l have a most general common instance m , and if σ is a most general substitution with $k\sigma = l\sigma = m$, then infer the clause $A' = (A - \{k\})\sigma$ from A . A' is called an *immediate factor* of A . The *factors* of A are given by: A is a factor of A , and an immediate factor of a factor of A is a factor of A .

As with paramodulation, resolution yields a number of possible inferences from a given pair A and B of clauses. SRT will be $\{C \mid C \text{ is a resolvent of } A \in S \text{ and } B \in T\}$, etc. FS will be $\{C \mid C \text{ is a factor of } A \in S\}$, etc.

Definition: If R is the equality predicate, a set S of clauses is *functionally reflexive* if $Rxx \in S$ and if, for each n -ary function f occurring in S , $Rf(x_1, \dots, x_n)f(x_1, \dots, x_n) \in S$.

The theories of interest are the functionally reflexive first-order theories with equality. The scope of interest would be *all* first-order theories with equality if it were not for the fact that refutation completeness for Π without functional reflexivity is an open question. If and when the corresponding theorem is proved, it would be desirable to extend the results which follow to all first-order theories with equality.

Since the inference systems which play the main role throughout are Π (the inference system consisting of paramodulation, resolution, and factoring) and ΠT (identical to Π except that only clauses with T as set of support are allowed), it becomes necessary to extend the definition of set of support [10] to include inferences made through paramodulation.

Definition: Given a set S with subset T , a clause C has *T-support* (with respect to S) if $C \in T$, or if C is a factor of a clause with T -support, or if C is a paramodulant or resolvent of clauses A and B where B has T -support and either A has T -support or is a factor of a clause in $S-T$. T is called a *set of support* for C .

The definition could be further extended to any inference system Ω by replacing paramodulation, resolution, and factoring by "some rule of Ω ".

That the definition of set of support given above is no more than an extension of the definition given in [10] can be seen by examining the alternate definition for "C having T-support" given below. In the definition below, T_S^i is extended from that given in [10].

Definition: S^0 is the set of clauses B such that B is in S or there is a clause C in S with B a factor of C. For $i > 0$, S^i is the set of clauses A such that $A \in S^{i-1}$, or there exist clauses $C \in S^{i-1}$ and $D \in S^{i-1}$ such that A is a paramodulant or a resolvent of C and D or A is a factor of a paramodulant or resolvent of C and D.

Definition: For $T \subseteq S$, T_S^0 is the set of clauses A such that $A \in T$ or such that A is a factor of some clause B with $B \in T$. For $i > 0$, T_S^i is the set of clauses A such that $A \in T_S^{i-1}$, or there exist clauses $C \in T_S^{i-1}$ and $D \in S^0 \cup T_S^{i-1}$ such that A is a paramodulant or a resolvent of C and D or A is a factor of a paramodulant or a resolvent of C and D.

Since the factors of a clause A include A itself as a trivial factor, S^0 consists of the factors of the clauses of S. When S contains only ground clauses, it is obvious that $S^0 = S$. Normally, however, S contains nonground clauses, and in many such cases $S^0 - S$ is not empty. (From the fact that A is a factor of itself it follows that some of the definitions given above can be appropriately shortened.)

Definition: The *S-level* of a clause A (relative to Ω) is the smallest i such that $A \in S^i$. The *T_S -level*¹ of A is the smallest i with $A \in T_S^i$.

Since, for all clauses A, A is a factor of itself, T_S^i for $i > 0$ can be obtained from T_S^{i-1} by adjoining to T_S^{i-1} all clauses E which are factors of some clause D where D is in turn inferrable by paramodulation or resolution from some pair B and F with B in T_S^{i-1} and F in the $S^0 \cup T_S^{i-1}$.

¹That which is now termed T_S -level was formerly termed T-level in some of our earlier papers.

Definition: Given a set S of clauses, a subset T of S , and a clause A deducible from S , A is said to have T -support if, for some $i \geq 0$, $A \in T_S^i$. T is said to be a *set of support for A* , and A is said to be *supported by T* .

Definition: A T -supported deduction D_1, D_2, \dots, D_n (relative to S and Ω) is a deduction in Ω in which every D_i has T -support in Ω or is a factor of a clause in $S-T$. If such a deduction exists we write $S \vdash_{\Omega T}^D$.

Definition: A set S of clauses is R -satisfiable if it has an R -model, i.e., a model in which the predicate R is mapped to an equality relation.

Definition: A *refutation* of S is a deduction from S of the empty clause, \square .

Definition: An inference system Ω (or ΩT) is R -refutation complete if for R -unsatisfiable S , $S \vdash_{\Omega} \square$ (or $S \vdash_{\Omega T} \square$).

Definition: If $T \subseteq S$ and $S \vdash_{\Omega} C$, then C has T -heritage (relative to S and Ω) if in Ω there is no deduction of C from $S-T$ (i.e., $S-T \not\vdash_{\Omega} C$).

The concept of T -heritage bears an interesting relation to the concept of T -support as evidenced by Lemmas 5 and 6. T -heritage is a concept which has in the past been confused with T -support; this point and related ones will be clarified in the next section. That the concept of T -heritage is distinct from the concept of T -support can be seen from the following example:

Let $A = \{-P, -Q, R\}$, $B = \{P, Q\}$, $C = \{P, -Q\}$, $S = \{A, B, C\}$,
 $T = \{C\}$. $F = \{Q, -Q, R\}$ is a (tautologous) resolvent of A and B , and
 $D = \{P, -Q, R\}$ is a resolvent of F and C . D has T -heritage,
but D is not in T_S^i for any i and, therefore does not have
 T -support.

MISCONCEPTIONS AND NON-EQUIVALENT DEFINITIONS OF SET OF SUPPORT

It is incorrect, as can be seen from the example given below, to restate casually the heart of the definition of set of support as follows: If C is inferrable by paramodulation or resolution from A and B , and if at least one of A and B has T -support and both are deducible from S , then C has T -support.

The example under consideration is that given at the end of the previous section. The clause D does not have T -support even though one of its parents, C , does. As has been said, D has T -heritage, and there exists by Lemmas 5 and 6 a subclause E of D such that E has T -support. The only element of $(CRB)RA$ will do for E (as can be seen by examining the proof of Lemma 1).

We give an additional example to show that the casual rendering of the set of support definition given above can lead to an error when both paramodulation and resolution are involved as rules of inference.

Let $A = \{Rab, -Qc\}$, $B = \{Pa, Qc\}$, $C = \{Pa, -Qc\}$, $S = \{A, B, C\}$,
 $T = \{C\}$. D , the only element of $(APB)RC$, is $\{Pa, Pb, -Qc\}$.
 Although D has T -heritage, D does not have T -support even though one of its parents does.

The proof of Lemma 3 gives the clause $E = \{Pb, -Qc\}$, which is a clause whose existence is demanded by Lemmas 5 and 6. E has T -support and is a subclause of D . E is the only element of $(CRB)PA$.

The question of T -support status for some given clause D is in general only semidecidable. If S is finite and contains only ground clauses, the question is decidable since $\bigcup_i T_S^i$ is finite. But when S contains clauses which are not ground, $\bigcup_i T_S^i$ is usually infinite. Although one can have a

decidable test for D being an element of a given T_S^j (the union of $T_S^0, T_S^1, \dots, T_S^j$ is finite for each j), all that can be said in general is that, if D has T-support, then this fact can be ascertained eventually since D will be in some T_S^i . If D does not have T-support, the situation is analogous to attempting to prove that a given non-theorem is in fact a non-theorem.

The question of T-heritage for a given clause is also in general only semidecidable. (Putting the set of support question another way, one normally cannot show that D is not in T_S^i for all i .)

For us if a clause is in some T_S^i it has T-support regardless of whether or not it is deducible from S-T.

Slagle [9] demands² that, in order for a deduction to have T-support, no resolution occurs between members of S-T (ignoring factoring for this discussion). Thus all of his T-supported deductions are for us also T-supported, but not conversely as can be seen from the following example:

S-T = {A,B,C,E}, T = {F}, A = {P,R}, B = {P,-R}, C = {Q,R},
 E = {Q,-R}, F = {-P,Q}. $D_1 = \{P,R\}$. $D_2 = \{P,-R\}$. $D_3 = \{-P,Q\}$.
 $D_4 = \{Q,R\}$, a resolvent of D_1 and D_3 . $D_5 = \{Q,-R\}$, a resolvent
 of D_3 and D_2 . $D_6 = \{Q\}$, a resolvent of D_4 and D_5 .

The deduction D_1 through D_6 has T-support for us, but not for Slagle since he does not allow the resolution of D_4 and D_5 , both of which are in

²He also assumes S-T satisfiable, which is irrelevant to what follows and is mainly done because of his intended application; we wish not to make this assumption because of the generality gained and because of other applications by other authors such as Green [3] concerning question-answering systems.

S-T. This resolution is allowable for us because D_4 and D_5 have T-support since they are elements of T_S^1 . Although Slagle does not define set of support for clauses but instead only for deductions, he would in effect exclude $\{Q\}$ from having T-support while $\{Q\}$ would have T-support for us. He would in effect generate each T_S^i , but before retaining it remove from it all elements already in S-T.

The reason for such attention to this difference in definition is twofold. First of all, one should note that his refutation completeness theorem is strictly stronger than that given in [10]. Secondly, since Slagle's definition allows fewer deductions, (smaller T_S^i), it might seem best to prove in this paper the stronger refutation completeness theorem as his approach might be more efficient.³ The proof of Lemma 5, however, breaks down immediately since, even with F in S-T one cannot conclude that the elements of CRF or CPF have T-support when C does since some or all of such elements may also be in S-T.

Even with the obvious possible modification Lemma 5 is false for Slagle. For a counter-example, let S-T consist of the three clauses $\{P,R\}$, $\{Q,-R\}$, $\{-R,S\}$, and T consist of the clause $\{-Q,S\}$. $D = \{P,S\}$ is a clause satisfying the hypothesis of Lemma 5 and, therefore, for us must have a subclass with set of support. D itself for us has T-support, but no subclass

³Slagle's definition of set of support corresponds, at least on the unit level, to that which has been programmed in PG1 through PG5. Besides the stronger completeness theorem, he has shown (unpublished) that an instance C' of a clause C in S-T can be discarded without losing refutation completeness even when C' has T-support. For unit clauses this result has been used for a number of years in the programs PG1 through PG5.

of D exists either in S - T or obtainable with a T -supported deduction in the sense of Slagle.

The question of whether or not Lemma 6 holds with Slagle's definition of T -support is at the present an open question. The example just given does not serve as a counter-example since the clause D of the example does not have T -heritage.

Lemmas 5 and 6 may give real insight into the question, intuitively speaking, of why set of support is refutation complete for ΠT (in the presence of functional reflexivity) and ΣT .

LEMMAS, COROLLARIES, AND THEOREMS

Lemmas 1 to 6 are reordering lemmas with 1 to 4 being local and 5 to 6 global. All six are proved on the ground level here, although analogous lemmas are probably provable on the non-ground level if factoring is appropriately utilized. Lemmas 7 and 8 are used to obtain a non-ground refutation from a given ground-clause refutation and are so-called "capturing lemmas" for factoring and resolution. The obvious analog to Lemma 8, but with resolution replaced by paramodulation, is frequently not true. (For a counter-example, let $A = Rab = A'$, $B = Qx$, $B' = Qg(a)$, $C' = Qg(b)$; the only factor respectively of A and B are A and B themselves, APB consists of $\{Qa\}$ and $\{Qb\}$. There is, therefore, no C in EPF with C' as an instance, see Lemma 8.) The lack of a paramodulation "capturing lemma" analogous to 8 has been *the* source of difficulty in proving refutation completeness of paramodulation-based inference systems when functional reflexivity was not assumed [6].

For the proofs of Lemmas 1 through 4, note that P and R are symmetric: $SPT = TPS$ and $SRT = TRS$ for all sets S and T .

Lemma 1. If D is a clause in $(ARB)RC$ then there exists a subclause E of D with $E \in (CRB)RA \cup (CRA)RB \cup (CRB)R(CRA)$.

Proof. Let D be in $(ARB)RC$. Then there exists a clause $F \in ARB$ such that D is a resolvent of F and C . F and C must, therefore, contain complementary literals, say q in F and $-q$ in C . Similarly, there exist literals p in B and $-p$ in A such that F is inferrable by resolution from B and A on p and $-p$. D is inferred from F and C on q and $-q$. Since $q \in F$, $q \in B$ or $q \in A$ (or both). If q is in B and not in A , then, where G is the resolvent of C and B on $-q$ and q , let E be the resolvent of G and A on p and $-p$.

$E \in (CRB)RA$ and is a subclause of D . If $q \in B$ and $q \in A$, then, where G is as above and H is the resolvent of C and A on $-q$ and q , let E be the resolvent of G and H on p and $-p$. $E \in (CRB)R(CRA)$ and is a subclause of D . The remaining cases yield a clause $E \in (CRA)RB$ or a clause $E \in (CRA)R(CRB)$ with E a subclause of D . Since R is symmetric ($SRT = TRS$ for all S and T), the proof is complete.

Lemma 2. If $D \in (APB)PC$, then there exists a subclause E of D with $E \in (CPB)PA \cup (CPA)PB \cup (CPB)P(CPA) \cup ((CPA)PB)PC \cup ((CPB)PA)PC$.

Proof. Let $D \in (APB)PC$. Then there exists $F \in APB$ with D a paramodulant of C and F .

Case 1. D is inferred by paramodulation from F into C . Let $\tilde{r}_2 \in F$ be the (equality) literal of paramodulation. Since $F \in APB$, depending on whether paramodulation was from A into B or from B into A , one of A and B contains the (equality) literal, say r_1 , of paramodulation and the other contains the literal, say p , containing the term occurrence of paramodulation. Since $\tilde{r}_2 \in F$, there exists a literal r_2 which is the ancestor of \tilde{r}_2 in A or B (or both). $r_2 \neq \tilde{r}_2$ precisely when r_2 is that literal p which is involved in inferring F in the discussion above.

Case 1a. $r_2 \in B$ and $r_2 = \tilde{r}_2$. Let G be inferred by paramodulation on $\tilde{r}_2 \in B$ into $p_2 \in C$, where p_2 contains the term occurrence in the paramodulation of C and F to get D . The literals of G are, with one possible exception, elements of D . The possible exception is the literal $(r_1$ or $p)$ from B . If $\tilde{r}_2 \notin A$, then the clause E inferable by paramodulation from G and A on r_1 and p (or p and r_1) is a subclause of D and is in $(CPB)PA$. If $\tilde{r}_2 \in A$, let H be inferred by paramodulation from C and A on p_2 and \tilde{r}_2 . Infer E from G and H on p and r_1 , and $E \in (CPB)P(CPA)$ and a subclause of D .

Case lb. $r_2 \in A$ and $r_2 = \tilde{r}_2$. There exists a subclause E of D with $E \in (CPA)PB$ or $E \in (CPA)P(CPB)$ which, by the symmetry of P equals $(CPB)P(CPA)$. The argument parallels that of la.

Case lc. $r_2 \in B$ and $r_2 \neq \tilde{r}_2$, which implies that $r_2 = p$. There exists, therefore, an argument u of r_2 such that u is replaced by \tilde{u} in inferring F. Since the literal of paramodulation of F and C is \tilde{r}_2, \tilde{u} or u_2 , the other argument of r_2 may be the argument being "matched" with a term in $p_2 \in C$. u_2 is unchanged in passing from B to F in all cases since all clauses herein are ground clauses. If u_2 is the argument for match, then let G be the paramodulant of C and B with literal of paramodulation r_2 in B, using u_2 as the match argument. $p_2 \in C$ becomes $\tilde{p}_2 \in G$. If $\tilde{r}_2 \notin A$, then let E be the paramodulant of G and A on \tilde{p}_2 and r_1 . If $\tilde{r}_2 \in A$, let H be the paramodulant of C and A on p_2 and \tilde{r}_2 , and let E be the paramodulant of G and H on \tilde{p}_2 and r_1 . $E \in (CPB)PA$ or $E \in (CPB)P(CPA)$ and is a subclause of D. On the other hand, if \tilde{u} is the match argument for F and C, then an argument of r_1 can be successfully matched with the term in p_2 . Let H be the resulting inference from A and C, and let \tilde{p}_2 be the transform of p_2 . If $\tilde{r}_2 \notin A$, let E be inferred from H and B on \tilde{p}_2 and r_2 , using $u_1 \in r_2$ as the argument for match, where $r_2 = Ru_1u_2$ or $r_2 = Ru_2u_1$. If $\tilde{r}_2 \in A$, let G be the paramodulant thus inferred by this last sequence and let E be the paramodulant of G and C on \tilde{r}_2 and p_2 . E is a subclause of D and is in $(CPA)PB$ or $((CPA)PB)PC$.

Case ld. $r_2 \in A$ and $r_2 \neq \tilde{r}_2$. With the symmetry of P, by paralleling the argument of lc, we obtain a subclause E of D with

$$E \in (CPA)PB \cup (CPB)P(CPA) \cup (CPB)PA \cup ((CPB)PA)PC.$$

Case 2. D is inferred by paramodulation from C into F. Thus there exists a literal r_2 in C of paramodulation and a literal \tilde{p}_2 in F containing the term occurrence. Let r_1 and p_1 be the literals for inferring F from A and B. Let p_2 be the ancestor of \tilde{p}_2 with $p_2 \in A$ or $p_2 \in B$.

If $p_2 = \tilde{p}_2$ and $p_2 \in B$, then as in 1a there is a subclause E of D with $E \in (CPB)PA$ unless $p_2 \in A$. But then there exists $E \in (CPB)P(CPA)$ with E a subclause of D.

If $p_2 = \tilde{p}_2$ and $p_2 \in A$, then as in 1b there exists $E \in (CPA)PB \cup (CPB)P(CPA)$ with E a subclause of D.

If $p_2 \neq \tilde{p}_2$ with $p_2 \in B$, then as in Case 1c there exists E a subclause of D in $(CPA)PB \cup ((CPA)PB)PC$. (The proof parallels that of 1c beginning with "if $\tilde{u}_2 \dots$ ", but with r_2 in C in place of p_2 in C. There is an alternate choice of obtaining E in $((CPA)PB)PC$, for one can paramodulate C and A on r_1 in A or on \tilde{p}_2 in A.)

Finally, if $p_2 \neq \tilde{p}_2$ and $p_2 \in A$, then there exists a subclause E of D in $(CPB)PA \cup ((CPB)PA)PC$.

Lemma 3. If $D \in (APB)RC$, then there exists a subclause E of D with $E \in (CRB)PA \cup (CPA)RB \cup (CRB)P(CRA) \cup ((CRA)PB)RC \cup (CRA)PB \cup (CPB)RA \cup ((CRB)PA)RC$.

Proof. Let D be a clause in $(APB)RC$. Then there exists an $F \in APB$ such that D is in FRC . Thus there exist literals \tilde{q} in F and $-\tilde{q}$ in C with $D = (F - \{\tilde{q}\}) \cup (C - \{-\tilde{q}\})$. As in the proof of Lemma 2, we can conclude that there exist literals q in A or B as ancestor of \tilde{q} , r_1 and p (one in A, the other in B) with F a paramodulant of A and B on r_1 and p and with $D \in CRF$.

If $q \in B$, $q = \tilde{q}$, and $\tilde{q} \notin A$, then resolve C and B on q to get G . Then paramodulate G and A on p and r_1 to get E . If $\tilde{q} \in A$ but the remaining conditions are as above, let H be the resolvent of C and A on \tilde{q} , and then paramodulate G and H on p and r_1 to get E . In either case E is a subclause of D . $E \in (CRB)PA$ or $E \in (CRB)P(CRA)$.

If $q = \tilde{q}$ but $q \in A$, then the desired E is in $(CRA)PB \cup (CRA)P(CRB) = (CRA)PB \cup (CRB)P(CRA)$ by the symmetry of P .

If $q \neq \tilde{q}$ and $q \in B$, then $r_1 \in A$ and r_1 has an argument which matches a term in $\tilde{q} \in C$. (One argument of r_1 in fact matches a term in q while the other argument of r_1 is the term just mentioned for the match with \tilde{q} .) If $\tilde{q} \notin A$, then paramodulate C and A on \tilde{q} and r_1 , and resolve this paramodulant with B to get E . If $\tilde{q} \in A$, then resolve C and A on \tilde{q} , paramodulate the resolvent with B on q , and finally resolve this paramodulant with C to get E . $E \in (CPA)RB \cup ((CRA)PB)RC$ and a subclause of D .

If $q \neq \tilde{q}$ and $q \in A$, one finds a subclause E of D in $(CPB)RA \cup ((CRB)PA)RC$ by paralleling that just given but interchanging the roles of A and B .

Lemma 4. If $D \in (ARB)PC$, then there exists a subclause E of D with $E \in (CPB)RA \cup (CPB)R(CPA) \cup (CPA)RB$.

Proof. Let q and $-q$ be respectively in A and B as required for $F \in ARB$ with $D \in FPC$, for arbitrary D . r_1 and p are the literals of F and C (in either order) for paramodulation. If $p \in B$ or $r_1 \in B$ (say, without loss of generality, $p \in B$), then paramodulate C and B on r_1 and p to get G . If $p \in A$, paramodulate C and A on r_1 and p , and then resolve with G on q to get E . If $p \notin A$, resolve G and A on q to get E . If $p \in A$ or $r_1 \in A$ instead of in B , then one can find E in $(CPA)RB$ or in $(CPA)R(CPB)$. The symmetry of R completes the proof.

Lemma 5. Let S and $T \subseteq S$ be given and let U be the smallest set containing $S-T$ such that U is closed both under paramodulation and resolution. (Factoring is irrelevant on the ground level.) If $F \in U$, and C has T -support, and if $D \in CPF \cup CRF$, then there exists a clause H such that H is a subclause of D and, more importantly, H has T -support.

Proof. Let $(S-T)^0 = S-T$ (since ground clauses have no non-trivial factors), and for $j \geq 0$ let $(S-T)^{j+1} = (S-T)^j \cup APB \cup ARB$ for all clauses A and B in $(S-T)^j$.

Then $U = \bigcup_j (S-T)^j$. Let F be a clause in U , C a clause with T -support and D a clause in the union of CPF and CRF . The proof proceeds by induction on the $(S-T)$ -level of F , where the $(S-T)$ -level n of F is (as given earlier) the smallest n such that $F \in (S-T)^n$. If the $(S-T)$ -level of F is 0 , then $F \in S-T$ and D by definition has T -support since C has and F is a paramodulant or a resolvent of C and a clause in $S-T$. Assume by induction that the lemma is true for clauses G with $(S-T)$ -level j with $0 \leq j \leq n$, and let F be of $(S-T)$ -level $n+1$. Then there exist clauses A and B in $(S-T)^n$ with $F \in APB \cup ARB$. D , therefore, is in the union of $(ARB)RC$, $(APB)PC$, $(APB)RC$ and $(ARB)PC$. Depending on which of the just given four sets contains D , one of Lemmas 1 through 4 applies to yield a subclause E of D . In addition one knows that E is itself contained in some union of sets dependent on C , B , and A , and on some combination of paramodulation and resolution. We shall give the argument for the case in which $E \in ((CRA)PB)RC$ and show that a subclause H of E and hence a subclause of D , exists and has T -support. The remaining cases can be proved by an argument similar to that which follows but less involved.

Since in the case under discussion E is assumed in $((CRA)PB)RC$, there exist clauses G_1 and G_2 with $G_1 \in CRA$, $G_2 \in G_1PB$, and $E \in G_2RC$. Since C has T -support and $A \in (S-T)^n$, by induction there exists the clause E_1 which is a subclause of G_1 and has T -support. E_1 is either itself a subclause of G_2 or E_1 contains the literal relevant to the paramodulation of G_1 and B . In the first case, let $E_2 = E_1$. In the second, apply the induction hypothesis to E_1 and B to show that there exists an E_2 which is a subclause of G_2 and which has T -support. Thus in either case we have a T -supported subclause E_2 of G_2 . Either E_2 is a subclause itself of E or contains the literal for resolution with C corresponding to that by which E was inferred. Since E_2 has T -support and since every resolvent of E_2 and C has T -support (for they both do), we have an E_3 which has T -support and is a subclause of E which is a subclause of D . $E_3 = E_2$ or is in E_2RC . $H = E_3$ is the desired subclause of D having T -support.

Lemmas 5 and 6 are proved with ΠT as the underlying inference system. The obvious modification of those proofs will give corresponding lemmas for ΣT .

Lemma 6. Given S and $T \subseteq S$ and a clause D with T -heritage (relative to S and Π) then there exists a subclause E of D such that E has T -support.

Proof. Let D be a clause with T -heritage, not deducible with paramodulation and resolution from $S-T$ but deducible from S with those same inference rules. The proof proceeds by induction on the S -level of D . If D has S -level 0, then $D \in T$ since D has T -heritage. So D itself has T -support. Assume by induction that all clauses with S -level less than or equal to n having T -heritage possess a subclause having T -support, and let D have S -level $n+1$. Then there exist clauses C and F of S -level less than or equal

to n such that $D \in CPF \cup CRF$. If neither C nor F have T-heritage then both are deducible in Π from S-T, and D , therefore, is deducible in Π from S-T. But D has T-heritage in Π , which would be a contradiction, so one of C and F say C has T-heritage. By the induction hypothesis, C has a subclause C_1 having T-support.

If F has T-heritage, then by the induction hypothesis there exists a subclause F_1 of F having T-support. Now C and F with paramodulation or resolution yielded D . Hence C and F each contain a literal relevant to this paramodulation or resolution. If either C_1 or F_1 (subclauses respectively of C and F) lack that particular literal, then the clause lacking the literal is a subclause of D and has T-support from the above. If both C_1 and F_1 have the literals in question then paramodulation or resolution of C_1 and F_1 on that literal pair yields a subclause of D . This subclause has T-support since C_1 and F_1 both do.

Now consider the case where F does not have T-heritage and is, therefore, deducible in Π from S-T. Paramodulation or resolution can be applied to C_1 and F on the literal pair used to infer D , unless C_1 is a subclause of D . In the latter case we are finished. In the former we infer from F and C_1 the clause G which, since C_1 is a subclause of C , is a subclause of D . We apply Lemma 5 to F , C_1 , and G to obtain a subclause C_2 of G . C_2 is, therefore, a subclause of D , and by Lemma 5 it has T-support.

It has already been shown by example (see the end of the section on definitions and notation) that the concept of T-heritage is distinct from that of T-support. It follows that, given a clause with T-heritage, Lemmas 5 and 6 may yield at best a proper subclause having T-support. In the example just cited the clause with T-heritage was $\{P, -QR\}$, and the subclause provided by Lemmas 5 or 6 is $\{-Q, R\}$.

Remark. By examining the proofs of Lemmas 5 and 6, one can prove the corresponding lemmas with Π replaced by Σ . The correspondent of 6 states that, if D has T -heritage relative to S and Σ , there is a subclause E of D which has T -support relative to S and Σ . A similar statement is the correspondent to Lemma 5. The heart of the matter is Lemma 1, which guarantees the existence of a clause (ground) inferrable by resolution with some appropriate re-ordering when presented with a clause D in $(ARB)RC$.

Definition: Ω is *R-sound* if, whenever $S \vdash_{\Omega} C$, C holds in all R -models of S in which C is defined. An *R-model* of S is a model of S in which R (the equality predicate) is mapped to an equality relation.

Corollary 1. If S is an unsatisfiable set of ground clauses with $T \subseteq S$ such that $S-T$ is satisfiable, then $S \vdash_{\Sigma T} \square$ (set of support is ground refutation complete in Σ).

Proof. Let S be an unsatisfiable set of ground clauses with $T \subseteq S$ such that $S-T$ is satisfiable. Since resolution is refutation complete, $S \vdash_{\Sigma} \square$, the empty clause is deducible from S . Since resolution is sound and $S-T$ satisfiable, the empty clause has T -heritage (relative to S and Σ). By the remark above, there exists a subclause of \square having T -support relative to S and Σ . (Contradictory units could have been the focus of attention instead.) Thus set of support is ground refutation complete in Σ .

Corollary 2. If S is an R -unsatisfiable set of ground clauses and if $T \subseteq S$ is such that $S-T$ is R -satisfiable and if S contains all clauses of the form Rtt for t in the Herbrand universe of S , then $S \vdash_{\Pi T} \square$; set of support is ground refutation complete in Π .

Proof. Let S and $T \subseteq S$ satisfy the hypothesis of the corollary. Since Π has been shown to be refutation complete for such an S [11] [5], and since paramodulation and resolution are both R-sound and S-T is R-satisfiable, the empty clause has T-heritage (relative to Π and S). Apply Lemma 6.

Paramodulation, though R-sound (i.e., sound for first order theories with equality), is of course not sound for ordinary first order theories. Qb is a paramodulant of Qa and Rab but not a logical consequence of Qa and Rab . In Corollary 2 it is not sufficient to require S-T to be satisfiable rather than R-satisfiable as can be seen from the following example.

$$S-T = \{ \{Qa\}, \{-Qb\}, \{Rab\} \}, \text{ and } T = \{Pa\}.$$

S is R-unsatisfiable. S-T is satisfiable but R-unsatisfiable. Obviously there is no T-supported refutation relative to S and Π .

Lemma 7. If A' is an instance of A , then there exists a factor B of A having the same number of literals as A' and having A' as an instance.

Lemma 8. If A' and B' are instances respectively of A and B and if C' is an element of $A'RB'$, then there exists a clause $C \in \text{ERF}$ having C' as an instance, where E is a factor of A and F is a factor of B .

Lemmas 7 and 8 are true for all instances and not just for ground instances.

For the theorem which follows, the proof is one of obtaining a deduction based on a set S of clauses from a ground clause deduction based on a set of instances of S .

Occurrences of terms in two literals are said to be in the *same position* if each is the i_1 -st argument of the i_2 -nd argument of ... of the i_n -th argument of its literal.

Lemma 9. If A' and B' are ground instances of clauses A and B respectively, and if C' is a paramodulant of A' and B' , and if B has a term in the position corresponding to the term of paramodulation in B' , then there exists a clause C in EPF where E is a factor of A and F is a factor of B [12].

Theorem 1. If S is a functionally-reflexive R -unsatisfiable set of (not necessarily ground) clauses and if $T \subseteq S$ with $S-T$ R -satisfiable, then $S \vdash_{\Pi T} \square$; set of support is R -refutation complete (in ΠT) for functionally reflexive sets.

Proof. Let H be the Herbrand universe of S , and let S' be the full instantiation of S over H . Since S is R -unsatisfiable, S' is R -unsatisfiable. S' contains all clauses of the form Rtt for all terms t in H , where R is the equality predicate. Let the full instantiation of $S-T$ over H be $(S-T)'$. $(S-T)'$ is R -satisfiable since $S-T$ is. T' , the full instantiation of T over H , is such that $S'-T'$ is R -satisfiable since $S'-T' \subseteq (S-T)'$. By Corollary 2 there exists a T' -supported refutation D'_1, D'_2, \dots, D'_n . Using the refutation D'_1, D'_2, \dots, D'_n , the following procedure yields a T -supported refutation D_1, D_2, \dots, D_h of S itself within Π .

For each clause D'_i of the ground refutation, the procedure yields a finite sequence U_i of clauses such that the last clause in U_i has D'_i as an instance and also has precisely the same number of literals as D'_i . The juxtaposition of U_1, U_2, \dots, U_n will be a T -supported refutation D_1, D_2, \dots, D_h in Π of S . In many cases h will be greater than n . This results from two causes: the need for factoring or the need for extra steps due to the lack of a capturing lemma of the desired type for paramodulation.

For each $D_i^!$ whose justification is that $D_i^!$ is in T' , there exists an A in T with $D_i^!$ as an instance. If A and $D_i^!$ have the same number of literals, let $U_i = A$. If not, let U_i be A, B as provided by Lemma 7.

For each $D_i^!$ whose justification is that $D_i^!$ is in $S'-T'$, there exists an A in $S-T$ with $D_i^!$ as an instance. If A and $D_i^!$ have the same number of literals, let $U_i = A$. If not, let U_i be A, B as provided by Lemma 7.

For each $D_i^!$ whose justification is that $D_i^!$ is a resolvent of $D_j^!$ and $D_k^!$, consider the corresponding U_j and U_k . Let A be the last element of U_j and B be the last element of U_k . Since by construction $A, B, D_j^!$, and $D_k^!$ satisfy the hypothesis of Lemma 8, there exists a C in ERF having $D_i^!$ as an instance, where E and F are respectively factors of A and B . A , however, has the same number of literals as $D_j^!$ by construction. One can deduce, therefore, that $E = A$. Similarly, it follows that $F = B$. If C and $D_i^!$ have the same number of literals, let $U_i = C$. If not, apply Lemma 7 to obtain the clause D such that D has the same number of literals as $D_i^!$ and has $D_i^!$ as an instance. Then let U_i be C, D .

For each $D_i^!$ whose justification is that $D_i^!$ is a paramodulant of $D_j^!$ and $D_k^!$, let the paramodulation be, without loss of generality, from $D_j^!$ into $D_k^!$. Let A and B be respectively the last elements of the corresponding U_j and U_k . Let u' in $D_k^!$ be the term occurrence of paramodulation relevant to the inference of $D_i^!$. If B contains a term in the corresponding position to that of u' in $D_k^!$, then by Lemma 9 there exist factors E and F of A and B respectively such that some C in EPF has $D_i^!$ as an instance. Since by construction, $D_j^!$ and A have the same number of literals and similarly $D_k^!$ and B have the same number of literals, $E = A$ and $F = B$. If C and $D_i^!$ have the same number of literals, let U_i be C . If not, let U_i be C, D , where D is obtained

by applying Lemma 7 to D'_i and C . When B does not have a term in the position corresponding to that of u' in D'_k , the property of functional reflexivity comes into play.

In the case now under discussion D'_i is inferred by paramodulation from D'_j into D'_k , A and B are respectively the last elements of U_j and U_k , and B does not have a term u in the position (within the corresponding literal) corresponding to u' (the term of paramodulation) in D'_k . Normally A and B will not have a paramodulant C having D'_i as an instance. Depending on whether or not D'_k has T -support, there are two alternatives for obtaining a pair of clauses one of whose paramodulants has D'_i as an instance.

Consider the case in which D'_k has T -support. We show that a clause B_r can be inferred by paramodulation from B and a set of functional reflexivity axioms. B_r will have D'_k as an instance and will also have a term u in the position corresponding to u' . Since D'_k is an instance of B , there is a substitution σ such that $B\sigma = D'_k$. Since B lacks a term in the position corresponding to u' , there exists a variable x in B and a non-empty set of functions f_1, f_2, \dots, f_p in D'_k such that σ contains $f_1(\dots f_2(\dots f_p(\dots u' \dots) \dots)) / x$, i.e., x is replaced in passing from B to D'_k by instantiation by $f_1(\dots f_2(\dots f_p(\dots u' \dots) \dots))$. The vector giving the position of x in B is an initial segment of that for u' , i.e., the vectors agree on the first q coordinates where q is the number of coordinates giving the position of x in B . Let G_1, G_2, \dots, G_p be the functional reflexivity axioms corresponding to f_1, f_2, \dots, f_p . Let B_1 be the paramodulant of G_1 into B on x . In general for $1 \leq m \leq p-1$, let B_{m+1} be the paramodulant of G_{m+1} into B_m on the variable occurring as the t -th argument of f_m where t is the $(q+m+1)$ -st coordinate of the position vector for u' . By construction B_1 and, therefore, all B_m ,

$1 \leq m \leq p$, have the same number of literals as D'_k . Let $B_r = B_p$. A, B_p, D'_j and D'_k satisfy the hypothesis of Lemma 9. So there exists a clause C in $E\mathcal{P}F$ having D'_i as an instance, where E and F are respectively factors of A and B_p . As earlier in the proof one can conclude that $E = A$ and $F = B_p$ since B_p and D'_k have the same number of literals. If C and D'_i have the same number of literals let U_i be $G_1, G_2, \dots, G_p, B_1, B_2, \dots, B_p, C$. If not, apply Lemma 7 to C and D'_i to obtain a clause D having D'_i as an instance and having the same number of literals as D'_i . Then let U_i be G_1, \dots, C, D .

The other alternative occurs when B'_k does not have T' -support. But then, since the ground deduction has T' -support, D'_j has T' -support. Now the procedure just given would, of course, still yield a clause D_p with most of the desired properties. But, because of the consideration of T -support desired for $D_1, D_2, \dots, D_h, B_p$ will not do. We shall show instead, therefore, that there exists a clause A_p which can be inferred from A , the last element of U_j , and the G_r , the functional reflexivity axioms of the previous paragraph such that paramodulation from A_p into B yields a clause having a subclasse of D'_i as an instance. First we re-number the G_r . For $1 \leq m \leq p$ with $r+m = p+1$, let H_m be G_r . Since the inference of D'_i was by paramodulation from D'_j into D'_k , D'_j contains an equality literal $Rs't'$. Since the term of paramodulation in D'_k relevant to this inference was u' , and since D'_j and D'_k are ground clauses, s' or $t' = u'$. Without loss of generality say that $s' = u'$. Since by construction D'_j is an instance of A (the last element of U_j), A contains a corresponding equality literal Rst . A_p will be identical to a subclasse of A except that Rst will be replaced by $Rf_1(f_2(\dots f_p(s)\dots))f_1(f_2(\dots f_p(t)\dots))$, and the last p coordinates of the

position vector of s will be identical to the last p for u' . Let $A_0 = A$, and A_t for $1 \leq t \leq p$ be the paramodulant of A_{t-1} into H_t . Each paramodulation is into the second argument of the corresponding H_t and into the r -th subargument therein where r is the $(q+p+1-t)$ -th coordinate of the position vector of u' . The literals of paramodulation are respectively, $Rst, Rf_p(s)f_p(t), \dots, Rf_2(f_3(\dots f_p(s)\dots))f_2(f_3(\dots f_p(t)\dots))$. As was seen earlier there exists the variable x occurring in the position in B whose first q position coordinates are identical to the first q of u' in D'_k .

Let B_1 be obtained by paramodulation of A_p into B on $Rf_1(f_2(\dots f_p(s)\dots))f_1(f_2(\dots f_p(t)\dots))$ into the above mentioned occurrence of x . Let x_1 denote the term occurrence resulting from the replacement of that occurrence of x . Let B_2 be the paramodulant of A on Rst into x_1 in B_1 and let x_2 denote the corresponding resulting term occurrence. Let B_3 be obtained by paramodulation of A on Rst into x_2 in B_2 . B_3 has D'_1 as an instance. If B_3 and D'_1 have the same number of literals let U_1 be $H_1, H_2, \dots, H_p, A_1, A_2, \dots, A_p, B_1, B_2, B_3$. If not, apply Lemma 7 to B_3 and D'_1 to obtain B_4 such that B_4 has D'_1 as an instance and B_4 and D'_1 have the same number of literals. Then let U_1 be H_1, \dots, B_3, B_4 .

The need for application of paramodulation to A and B_1 and then to A and B_2 is that certain literals needed to capture D'_1 as an instance of A might be lost in passing from A to A_p and then to B_1 .

Now generate in order U_1, U_2, \dots, U_n from D'_1, D'_2, \dots, D'_n by applying the procedure given above. The desired (possibly) non-ground deduction D_1, D_2, \dots, D_h is obtained by juxtaposing U_1, U_2, \dots, U_n . Since D'_n is the empty clause and, by the construction, the last element of U_n has D'_n as an instance, D_1, D_2, \dots, D_h is a refutation (that D_1, D_2, \dots, D_h is a deduction including justifications follows from the construction.)

The argument that D_1, D_2, \dots, D_h is a T-supported refutation proceeds by induction. We show that for each U_i , all elements of U_i are either factors of clauses in S-T or have T-support, and in addition we show that, for all i , if D_i' has T'-support, then the last element of U_i has T-support. D_1' is in S'-T' or in T'. So by construction the elements of U_i are either factors of clauses in S-T or factors of clauses in T. If D_1' has T'-support then by construction the last element of U_1 is a factor of a clause in T. Now assume by induction that our statement holds for U_t with $1 \leq t \leq r$, and consider U_{r+1} .

If D_{r+1}' is justified by being in S'-T', then U_{r+1} has its elements among factors of S-T. If D_{r+1}' is justified by being in T', again as above U_{r+1} consists of factors of elements of T, and, therefore, the last element of U_{r+1} has T-support.

If D_{r+1}' is a resolvent of D_j' and D_k' , at least one of D_j' and D_k' has T'-support since the ground deduction has T'-support. Say without loss of generality that D_k' has T'-support. But then by induction the last line B of U_k has T-support. Also by induction A, the last element of U_j , either has T-support or is a factor of a clause in S-T. U_{r+1} by construction consists either of C or C and D, where C is a resolvent of A and B and D (if needed) is a factor of C. In either case U_{r+1} has all of its elements with T-support.

If D_{r+1}' is justified as a paramodulant of two earlier clauses A and B, and if B has a term in a position corresponding to the term (on the ground level) of paramodulation, we can parallel the arguments just given for D_{r+1}' when D_{r+1}' is a resolvent. If D_{r+1}' is justified by paramodulation from D_j' into D_k' where D_k' has T'-support, and if B is the last element of U_k but does not have a term in the corresponding position to that of the term of

paramodulation in D'_k then the procedure gives as U_{r+1} the sequence $G_1, G_2, \dots, G_p, B_1, B_2, \dots, B_p, C$ or possibly $\dots C, D$. G_1, G_2, \dots, G_p are in S . By induction B has T-support since D'_k does. Therefore B_1, B_2, \dots, B_p have T-support. By induction the last element A of U_j either is a factor of an element of $S-T$ or has T-support, so C has T-support. If the procedure calls for an application of Lemma 7 to yield a clause D , then D has T-support. The last line of U_{r+1} , therefore, has T-support.

If D'_{r+1} has T'-support and is a paramodulant of D'_j into D'_k , but D'_k does not have T'-support, then D'_j has T'-support since the ground deduction is T'-supported. We can parallel the argument just given in the previous paragraph and conclude that U_{r+1} consists either of factors of clauses of $S-T$ or clauses with T-support and that the last element of U_{r+1} has T-support. The induction is, therefore, complete, and we have the proof of Theorem 1.

Since $S'-T'$ was shown to be R-satisfiable, the empty clause must have T'-support. D_h , therefore, must have T-support since D_h is the last element of U_n .

(It should not be concluded that a completeness theorem for paramodulation without functional reflexivity would lead directly to the completeness theorem for paramodulation with set of support in the absence of functional reflexivity. The functional reflexivity axioms were used directly in the proof of Theorem 1.)

Theorem 2. If S is an unsatisfiable set of (not necessarily ground) clauses, and if $T \subseteq S$ is such that $S-T$ is satisfiable, then $S \vdash_{\Sigma T} \square$; set of support is refutation complete within ΣT .

Proof. The proof parallels that just given for Theorem 1, omitting all references to functional reflexivity, paramodulation, and replacing R-satisfiability by satisfiability, etc.

It is perhaps interesting to note that the proof of Theorem 1 easily yields refutation completeness within ΣT , i.e., completeness of set of support for resolution. The previously given proofs for refutation completeness in ΣT [9][10], however, do not seem to generalize easily to refutation completeness in ΠT .

Corollary 3. For finite functionally reflexive sets there is a refutation procedure for Π with set of support. That is, there is a uniform procedure that will, given any T and a finite functionally reflexive R-unsatisfiable $S \supseteq T$ with S-T R-satisfiable, generate (in a finite number of steps) a refutation of S in the system ΠT .

Proof. By Theorem 1 there exists a T -supported refutation D_1, D_2, \dots, D_k . Since by definition refutations are finite in length and the D_i either are factors of $S-T$ or are in some T_S^j , there exists an n such that $S^0 \cup T_S^n$ contains all D_i . Consider the procedure which generates $(S-T)^0$, then $T_S^0, T_S^1, \dots, T_S^n$. Since S is finite, $(S-T)^0$ and T_S^j for $0 \leq j \leq n$ are all finite, the given procedure will find the T -supported refutation of S after generating only finitely many clauses.

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