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POLARIZATION MEASUREMENTS AT HIGH ENERGIES*

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ABSTRACT

The problem of translating symmetry properties of the helicity amplitudes for two-body scattering into concise predictions for polarization experiments is discussed extensively. Particular attention is given to the asymptotic symmetries at high energies, which are characteristic for a Regge pole exchange model and which are associated with J-parity and G-parity exchange in a crossed channel. The idea is developed that the symmetry operator, as a matrix in the helicity space, "maps" one polarization measurement onto a superposition of other polarization measurements. Using this fact a systematic procedure is presented for separating different parts of the amplitudes which are even or odd under a symmetry. The procedure applies for any spins of the particles. To make the paper more complete and self-consistent a discussion of the density matrix formalism with particular emphasis on the choice of coordinate systems and phase conventions is included.

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1. INTRODUCTION

Recently the experimental techniques for performing polarization measurements have been greatly improved. In many laboratories equipment for polarizing targets or photon beams and for measuring the recoil polarization of nucleons is now or will soon become available. Thus it can be expected that in the very near future a deeper look into the details of strong dynamics at high energies should become possible.

Given this progress in experimental technique, questions naturally arise as to what information can be obtained from a particular type of polarization measurement or what is the relevance of a possible experiment to current high energy dynamical models. Often theoreticians, when asked questions by their experimentalist friends, find it hard to give a concise answer.

In fact there have been relatively few efforts made to answer these questions, and most attempts made so far only consider special situations. Thus F. Cooper emphasizes in a recent series of papers[1] the usefulness of linearly polarized photons in photoproduction experiments as a "parity filter" for the exchange of particles in the t-channel. These results are a generalization of a statement by P. Stichel,[2] who first observed that in single photoproduction linearly polarized γ's can be used at high energies to separate natural from unnatural parity exchange in the t-channel. Similarly, R. L. Thews[3] analyzed the information obtainable from measurements with polarized photons at high energies. For more general cases J. P. Ader et al.[4] also considered how contributions to the differential cross section from the natural and unnatural parity parts of the t-channel amplitudes can be separated. Three years earlier E. Leader and R. C. Slansky[5] discussed how measurements of spin dependent parameters in N-N scattering provide critical tests for the Regge pole theory.

The main difficulty in a general discussion of high energy polarization measurements stems from the fact that at high energies no simple systematic procedure is known for stating phenomenological properties of dynamics in a model-independent way. At low energies such a tool is given by the partial wave expansion in the direct channel, but at high energies this technique is no longer useful.[6] Thus at present a discussion of high energy experiments has to be based on our experience with some of the more successful high energy models like the Regge-pole assumption. In this way model-dependent assumptions unfortunately enter into the discussion from the very beginning. The discussion presented here, like those cited above, assumes that the dynamics at high energies are easily represented by amplitudes associated with sets of quantum numbers in the crossed t- or u-channel. Peripheral models with absorption correction or the Regge-pole assumption are the best known realizations of this idea.

In these models the leading order terms of certain amplitudes are characterized by the quantum numbers of the crossed channels, such as the J-parity. Thus under dynamical assumptions of this sort one can expect, at least in leading order, strict selection rules for polarization measurements of any kind. It is our feeling that this fact and its practical consequences deserve a more general treatment. We, therefore, consider the two theorems presented in Section 4 to be the principal result of this work. These theorems allow a general classification of experiments at high energies with respect to J-parity and G-parity exchange in the crossed channel without any restrictions on the spins of the particles. We also want to emphasize that the general properties of polarization experiments can be derived easily by using simple algebraic properties of the S-matrix (in leading order) and do not need the detailed calculations presented in some previous works. Because of its generality, it should be possible to extend our method to include any new criteria which may be proposed to dynamically distinguish different parts of the amplitudes at high energies.

A large part of this paper is concerned with the general theoretical treatment of polarization measurements, since the appropriate methods are widely scattered in the literature. Thus we present in Section 3, the density matrix formalism including a discussion of parity restrictions and the consequences of time reversal invariance. In Section 3.C we particularly draw attention to the fact that there is a simple redundancy whereby different experimental set-ups obtain the same information. Thus one can obtain the differential cross section in a very complicated way by measuring the polarization in the final state for a polarized target and beam. But this redundancy also provides in other cases a practical advantage. Section 4, the main part of this paper, contains the two theorems which allow one to classify all polarization measurements according to the dynamical principles introduced in Section 2. Finally, Section 5, which is an Appendix, presents details of the helicity formalism with particular emphasis on phase conventions, the observation of which becomes vital when one is dealing with polarization effects. Of course, for direct experimental applications, it is necessary to have more explicit results. Thus, since this work was largely motivated by the need for interpreting polarization measurements which can be made at electron machines, a paper is being prepared in which the polarization features of photoproduction will be discussed in detail.

Notations and Conventions

We end this introduction with an explanation of our basic notations. In any channel

$$A + B \rightarrow C + D \qquad (1.1)$$

of particles K with spin S_K, helicity λ_K, parity η_K, mass m_K and momentum $\vec{q}_K$, we denote the helicity amplitudes of Jacob and Wick[7] (hereafter referred to as J.W.), in the c.m. system of (1.1) by

$$f_{\lambda_C \lambda_D; \lambda_A \lambda_B}(W, \theta, \phi = 0) = f_{\Lambda_f, \Lambda_i} \tag{1.2}$$

where we use the abbreviation

$$\Lambda_f = (\lambda_C, \lambda_D), \qquad \Lambda_i = (\lambda_A, \lambda_B) \tag{1.3}$$

In (1.2) W is the total energy, θ and ϕ the scattering angles of the particle C. In this paper ϕ is in general fixed: $\phi = 0$, so that the reaction (1.1) takes place always in the (1,3) plane. In the definition of the J.W. two-body helicity states, the particle with label 2 has an additional helicity dependent phase factor (see (5.13)). To fix this phase factor we always assume that particles A and C have the lable 1 whereas B and D have the label 2. It is occasionally convenient to note that the added phase for particle "2" may be reproduced by a rotation of the single particle state through π about the particle momentum. We also introduce

$$\lambda = \lambda_A - \lambda_B, \qquad \mu = \lambda_C - \lambda_D \tag{1.4}$$

and the half-angle factor

$$\xi_{\Lambda_f \Lambda_i}(\theta) = \left(\sqrt{2} \cos \theta/2\right)^{-|\lambda+\mu|} \left(\sqrt{2} \sin \theta/2\right)^{-|\lambda-\mu|} \tag{1.5}$$

We shall denote

$$m = \text{Max}\,(|\lambda|, |\mu|) \quad . \tag{1.6}$$

Finally we make the convention that the letter v in expressions like $(-1)^{J-v}$ becomes 1/2 if J is an half integer; otherwise v is zero, so that $(-1)^{J-v}$ is always a real number.

Throughout this paper we shall use the following conventions to define an orthogonal coordinate system in the center-of-mass (c.m.) system or the rest system of a particle K:

1. c.m. system: The three-axis (polar axis) is parallel to $\vec{q}_A$, the two-axis is parallel to $(\vec{q}_A \times \vec{q}_C)$, the one-axis completes a right-handed coordinate system.

2. rest rystem of particle K: As an intermediate step we first assign to each particle in the c.m. system three orthogonal "helicity axes" (Fig. 1). The choice of these axes is naturally determined by the J.W. prescription for constructing two-body helicity states (with $\phi = 0$) from states at rest. The three-helicity axes will be pointing in the direction in which the particle moves in the c.m. system, i.e., is parallel to $(\vec{q}_K)_{c.m.}$. The two-helicity axis for particle "1" shall be identical to the two-axis of the c.m. system, while for particle "2" it shall be the reflection of the c.m. two-axis. This inversion of the two-axis for particle "2" expresses our comment above concerning the added phases for particle "2". The one-helicity axis is then chosen such that it completes a right-handed frame. Finally, we obtain corresponding axes in the rest frame of the particle K by applying a boost in the direction $\vec{q}_{c.m.}$, which brings this particle at rest. We shall refer to these axes as the "helicity frame" following common usage.

Finally, we define in the rest system of particle K a moving frame, the crossing frame, usually also called Gottfried-Jackson frame,[8] which we shall need later on in crossing from the t- or u-channel to the s-channel. Let $R_i^K(\chi_K)$ represent a rotation of the rest frame axes $\{A^K\}$ in the rest frame of particle K through the angle χ_K about the i-axis. We denote the helicity frame axes by $\{H^K\}$. Then for crossing from the t-channel the crossing frame axes are

$$\left\{J_t^K\right\} = R_2^K\left(-\chi_t^K\right)\{H^K\} \tag{1.7}$$

$$K = A, B, C, D.$$

while for crossing from the u-channel we define

$$\left\{J_u^K\right\} = R_3^{AD,K}(\pi)\, R_2^K\left(-\chi_u^K\right) R_3^{CD,K}(\pi) \left\{H^K\right\} \tag{1.8}$$

where

$$R_3^{K_1K_2,K}(\pi) = \begin{cases} R_3^K(\pi) & \text{if } K = K_1,\ K_2 \\ I & \text{if } K \neq K_1,\ K_2 \end{cases} \tag{1.9}$$

With our choice of the particle labels in Section 4, extra helicity dependent phase factors appear only in the crossing relations from the u-channel. This has as consequence the extra rotations R_3 around the 3-axis in the definition (1.8) of the axes $\left\{J_u^K\right\}$. The crossing angles χ_t^K are given by Trueman and Wick[9] and the angles χ_u^K are the corresponding angles after interchanging particles C and D. These angles are discussed in detail in Section 4A below.

2. BASIC PHENOMENOLOGICAL CONSIDERATIONS

Most phenomenological models in high energy physics describe a given reaction very simply in terms of exchanged particles or resonances like in the isobar or peripheral model. In some cases like the Regge-pole model the exchanged system is of a more complicated structure. The basis for these concepts is an identification of the dominant parts of the amplitudes with sets of quantum numbers belonging either to the reaction considered or a cross reaction. Therefore a perfection of our ability to separate parts of the amplitudes, which are characterized by sets of quantum numbers, would be the basis for an improvement of present day models.

To isolate those parts of the amplitudes, to which only certain internal quantum numbers like isospin can be ascribed, one usually has to combine the amplitudes of reactions with particles in different charge states. To isolate those parts, to which only certain total angular momenta, parities or signatures can contribute, the experiment itself has to be refined by exploiting the polarization effects.

The possibility to relate different reactions is a consequence of the substitution law, which states e.g., in photoproduction that the amplitudes for the reaction

$$N_1 + \gamma \rightarrow N_2 + \pi, \quad \text{s-channel} \tag{2.1}$$

are suitably chosen analytic continuations of the amplitudes for the crossed reactions

$$\overline{\pi} + \gamma \rightarrow N_2 + \overline{N}_1, \quad \text{t-channel} \tag{2.2}$$

or

$$\overline{N}_2 + \gamma \rightarrow \overline{N}_1 + \pi, \quad \text{u-channel} \tag{2.3}$$

It is therefore tempting to use as in the Regge-pole model the knowledge of the dynamics in the crossed t- or u-channel for a phenomenological description of the dynamics in the s-channel.

We are thus confronted with the question of finding a set of amplitudes, with which our phenomenological assumptions can be conveniently expressed. Let us first consider the usual parity conserving amplitudes[10, 11] for the reaction (1.1)

$$f^{\sigma}_{\Lambda_f\Lambda_i}(W,z) = \xi_{\Lambda_f\Lambda_i}(\theta)\, f_{\Lambda_f\Lambda_i}(W,\theta) + \sigma(-1)^{m-\mu}\, \eta_C\,\eta_D(-1)^{s_C+s_D-v}$$

$$\times\, \xi_{\Lambda_f\Lambda_i}(\pi-\theta)\, f_{\Lambda_f-\Lambda_i}(W,\theta)$$

$$= \sum_J (J+1/2)\left[e^{J+}_{\lambda,\mu}(z)\, F^{J\sigma}_{\Lambda_f,\Lambda_i}(W) + e^{J-}_{\lambda,\mu}(z)\, F^{J(-\sigma)}_{\Lambda_f\Lambda_i}(W)\right]$$

$$z = \cos\Theta \qquad (2.4)$$

The quantities $F^{J\sigma}_{\Lambda_f\Lambda_i}$ are the parity conserving partial amplitudes. The relation between σ and the parity P is given by

$$P = \sigma(-1)^{J-v} \qquad (2.5)$$

The functions $e^{J\pm}_{\lambda,\mu}(z)$ are polynomials in z with the following symmetry property

$$e^{J+}_{\lambda,\mu}(-z) = (-1)^{J-m}\, e^{J+}_{\lambda,\mu}(z), \qquad e^{J-}_{\lambda,\mu}(-z) = -\,(-1)^{J-m}\, e^{J-}_{\lambda,\mu}(z) \qquad (2.6)$$

Since we do not need the precise definition of the F's and e's, we refer to Refs. (10) or (11) for further details on these quantities. From (2.4) and (2.6) it follows that one can easily separate the parity $P = +1$ contributions from the

P = -1 contributions by forming the linear combinations

$$\bar{f}^{\sigma}_{\Lambda_f \Lambda_i}(W, z) \pm \sigma(-1)^{m-v}\, \bar{f}^{\sigma}_{\Lambda_f \Lambda_i}(W, -z) = \sum_J (J+1/2) \left[e^{J+}_{\lambda,\mu}(z)\, F^{J,\sigma}_{\Lambda_f \Lambda_i}(W) \left(1 \pm \sigma(-1)^{J-v}\right) \right.$$

$$\left. + e^{J-}_{\lambda,\mu}(z)\, F^{J,-\sigma}_{\Lambda_f \Lambda_i}(W) \left(1 \mp \sigma(-1)^{J-v}\right) \right] \qquad (2.7)$$

From the point of view of the Regge-pole model one would even like to go further and to separate the amplitudes with different signature in the combination (2.7). But in general there exist no kinematic symmetry operation, which could help to separate different signature amplitudes. Only asymptotically is such a kinematical separation possible. Also the combination (2.7) of amplitudes has the disadvantage that it relates amplitudes at different angles and is therefore not very useful. For this reason one usually works with the parity conserving amplitudes (2.4), which contain both parities P = ±1 and which have relatively good analyticity properties.[12] For our rather kinematical discussion good analyticity properties are not so relevant so that the half-angle factors $\xi(\theta)$ introduce unnecessary complications. We replace therefore (2.4) by the definition

$$f^{\sigma}_{\Lambda_f \Lambda_i}(W, z) = f_{\Lambda_f \Lambda_i}(W, \theta) + \sigma \xi_{\lambda,\mu} (-1)^{m-\mu}\, \eta_A\, \eta_B (-1)^{S_A + S_B - v}\, f_{\Lambda_f - \Lambda_i}(W, \theta) \qquad (2.8)$$

where

$$(-1)^v\, \xi_{\lambda,\mu} = \lim_{z \to \pm\infty} \frac{\xi_{\Lambda_f \Lambda_i}(\pi - \theta)}{\xi_{\Lambda_f \Lambda_i}(\theta)} = \lim_{z \to \pm\infty} \left(\frac{1+z}{1-z}\right)^{\left|\frac{\lambda+\mu}{2}\right|} \left(\frac{1-z}{1+z}\right)^{\left|\frac{\lambda-\mu}{2}\right|}$$

$$= (-1)^{\left|\frac{\lambda+\mu}{2}\right| - \left|\frac{\lambda-\mu}{2}\right|} = (-1)^{\lambda+\mu-m} \qquad (2.9)$$

We shall refer to the amplitudes $f^{\sigma}_{\Lambda_f \Lambda_i}$ as "asymptotic parity conserving amplitudes." In the limit $z \to \pm\infty$, which occupies most of our interest, they are easily related to $\bar{f}^{\sigma}_{\Lambda_f, \Lambda_i}$ because of the factor $\xi_{\lambda,\mu}$ in the definition (2.8)

$$\bar{f}^{\sigma}_{\Lambda_f \Lambda_i} \xrightarrow[z \to \pm\infty]{} \xi_{\Lambda_f, \Lambda_i}(\theta)\, f^{\sigma}_{\Lambda_f \Lambda_i} \tag{2.10}$$

Using the explicit form (2.9) of $\xi_{\lambda,\mu}$ one can write (2.8) also:

$$f^{\sigma}_{\Lambda_f \Lambda_i}(W,\theta) = f_{\Lambda_f \Lambda_i}(W,\theta) + \sigma(-1)^{-v}\, \eta_A \eta_B (-1)^{S_A+\lambda_A+S_B-\lambda_B}\, f_{\Lambda_f - \Lambda_i}(W,\theta) \tag{2.8'}$$

This form will be useful later on in Section 4.

The amplitudes $f^{\sigma}_{\Lambda_f \Lambda_i}$ are useful if the reaction proceeds via the exchange of an entity with definite quantum numbers S_R and η_R. Under this assumption the helicity amplitudes $f_{\Lambda_f \pm \Lambda_i}$ obey the relationship

$$f_{\Lambda_f \Lambda_i}(W,\theta) \approx \eta_R (-1)^{S_R - v}\, \eta_A \eta_B (-1)^{S_A+S_B-v}\, \xi_{\lambda,\mu} (-1)^{m-\mu}\, f_{\Lambda_f - \Lambda_i}(W,\theta) \tag{2.11}$$

in the limit $|z| \to \infty$ as has been discussed several times.[11, 13] This limit will be needed if one continues the amplitudes into the region of a crossed channel. The limit (2.11) holds also for the exchange of a Regge pole, provided the factor $(-1)^{S_R - v}$ is interpreted as its signature. Now, if Eq. (2.11) is true, then it follows from (2.8) or (2.8') that the leading order term in $f^{\sigma}_{\Lambda_f \Lambda_i}$ is <u>kinematically</u> suppressed for $\sigma = -\eta_R (-1)^{S_R - v}$. We note that the relationship (2.11) is a consequence of the relation

$$F^{S_R}_{\Lambda_f - \Lambda_i}(W) \approx \eta_R (-1)^{S_R - v}\, \eta_A \eta_B (-1)^{S_A+S_B-v}\, F^{S_R}_{\Lambda_f \Lambda_i}(W) \tag{2.12}$$

for the partial amplitudes, $F^J_{\Lambda_f, \Lambda_i}$ which is true if in the partial amplitudes with angular momentum $J = S_R$ only one parity η_R dominates. Finally one needs also the asymptotic relation

$$|z| \to \infty : \; d^j_{\lambda,\mu}(\theta) \approx \xi_{\lambda,\mu} (-1)^{m-\mu} \, d^j_{-\lambda,\mu}(\theta) = (-1)^{\lambda} \, d^j_{-\lambda,\mu}(\theta) \tag{2.13}$$

to derive (2.11).

There are situations where a selection rule enforces the vanishing of some couplings of an exchanged particle. Then (2.11) is fulfilled either trivially, if $f_{\Lambda_f \Lambda_i} \approx 0$, or no longer true, if other effects become important, which are not dynamically suppressed. The vanishing of certain couplings as a consequence of G-parity conservation appears e.g., for the nucleon antinucleon system $N\bar{N}$, which we want to study in Section 4. If $|p, I; J\, M \lambda_1 \lambda_2\rangle$ denotes a $N\bar{N}$ helicity state in the c.m. system with definite angular momentum J and isospin I then

$$G|p, I; J\, M \lambda_1 \lambda_2\rangle = (-1)^{J+I} |p, I; J\, M, \lambda_2 \lambda_1\rangle \tag{2.14}$$

Therefore, the state with definite parity

$$|p, I; JM, \lambda_1 \lambda_2\rangle_{\pm} = |p, I; JM, \lambda_1 \lambda_2\rangle \pm \eta_1 \eta_2 (-1)^{S_1 + S_2} |p, I; JM, -\lambda_1 -\lambda_2\rangle$$

$$= |p, I; JM, \lambda_1 \lambda_2\rangle \pm |p, I; JM, -\lambda_1 -\lambda_2\rangle \tag{2.15}$$

$\left(\eta_1 \eta_2 (-1)^{S_1 + S_2} = +1 \text{ for } N\bar{N} \text{ state}\right)$ is an eigenstate of G with $G = (-1)^{J+I} (-1)^{\frac{(1\mp 1)}{2}(\lambda_1 - \lambda_2)}$.
Since the parity P of (2.15) is $P = \pm (-1)^J$ one obtains the relationship

$$(-1)^I \, P \cdot G = \pm (-1)^{\left(\frac{1\mp 1}{2}\right)(\lambda_1 - \lambda_2)} = \left[P (-1)^J\right]^{(1 + \lambda_1 - \lambda_2)} \tag{2.16}$$

There follow some powerful selection rules from (2.16) for the coupling of the $N\bar{N}$ system to the mesons[14]

1. $PG(-1)^I = -1$: only states with $P(-1)^J = -1$ and $\lambda_1 = \lambda_2$ are possible. (Examples: π, B, η.)

2. $PG(-1)^I = +1$: (a) $P(-1)^J = -1$ only $\lambda_1 \neq \lambda_2$ is possible. (Examples: A_1)

   (b) $P(-1)^J = +1$ no restriction on λ's. (Examples: ρ, ω, ϕ, A_2, f, f', Pomeranchuk.)

3. DENSITY MATRICES AND OBSERVABLES

3A. General Results

In order to account for the many degrees of freedom introduced by the spins of the particles of any channel (1.1) an adequate formalism is necessary. Let us therefore introduce in any frame spin density matrices ρ^i and ρ^f for the initial and final states with fixed momenta $\vec{q}_K$ of the particles. The matrices ρ^f and ρ^i are related by the reaction matrix F: $\rho^f = F\rho^i F^\dagger$, which relation is known as von Neumann's formula. More explicitly one has in the c.m. system $\vec{q}_C + \vec{q}_D = \vec{q}_A + \vec{q}_B = 0$

$$\rho^f_{\Lambda_f \Lambda_f'}(\vec{q}_C) = F_{\Lambda_f \Lambda_i'}(W,\theta)\ \rho^i_{\Lambda_i',\Lambda_i}(\vec{q}_A)\ F^\dagger_{\Lambda_i',\Lambda_f'}(W,\theta) \tag{3.1}$$

The matrix elements of F are then the helicity amplitudes of J.W.

$$F_{\Lambda_f \Lambda_i}(W,\theta) = f_{\Lambda_f,\Lambda_i}(W,\theta,\phi=0), \qquad F^\dagger_{\Lambda_i \Lambda_f}(W,\theta) = f^*_{\Lambda_f,\Lambda_i}(W,\theta,\phi=0) \tag{3.2}$$

For more details we refer to the derivation of (3.1) in Section 5B.

In the initial state we assume both particles are uncorrelated. Then ρ^i is the direct product of the density matrices for each particle A and B

$$\rho^i = \rho^A \otimes \rho^B \tag{3.3}$$

As usual we shall assume in any frame a decomposition of $\rho^{A,B}$ into a polynomial of the spin matrices S_i of the rest frame. So we write for a nucleon

$$\rho^N(\vec{q}) = \frac{1}{\sqrt{2}}\left[I + p_1\sigma_1 + p_2(-\sigma_2) + p_3\sigma_3\right]$$

$$= O_0^N + p_1 O_1^N + p_2 O_2^N + p_3 O_3^N \tag{3.4}$$

where $\sigma_i = 2S_i$ are the Pauli matrices.[15] The form of this decomposition is independent of whether the nucleon is particle "1" or "2". For a spin 1 particle we write similarly

$$\rho^K(\vec{q}) = p_0 \frac{1}{\sqrt{3}} I + p_1 \frac{1}{\sqrt{2}} S_1 + p_2 \frac{(-1)}{\sqrt{2}} S_2 + p_3 \frac{1}{\sqrt{2}} S_3$$

$$+ p_{20} \frac{1}{\sqrt{6}} \left(3S_2^2 - 2I\right) + p_{23}\left(\frac{-1}{\sqrt{2}}\right)(S_2S_3 + S_3S_2) + p_{21}\left(\frac{-1}{\sqrt{2}}\right)(S_2S_1 + S_1S_2)$$

$$+ p_{22} \frac{1}{\sqrt{2}} \left(S_3^2 - S_1^2\right) + p_{31} \frac{1}{\sqrt{2}} (S_3S_1 + S_1S_3) \quad . \tag{3.5}$$

The convention for the signs and normalizations in (3.4) and (3.5) are in agreement with the rules for the general spin case given below.

It is possible to assume the decomposition (3.4) and (3.5) in any frame always with the same real, independent parameters p_r, p_{rs}, which are the components of the tensor polarizations in the rest frame with respect to the axes introduced in the Introduction. The spin matrices S_i have in this case to be understood as the representations of certain operators O_i, the representations of which in the helicity basis reduce always to the standard rest frame spin matrices S_i, with S_3, S_1 real and S_2 imaginary, S_3 being diagonal.[7] In Section 5C we shall justify this remark in detail and also write down the operators O_i explicitly. With this interpretation of the decomposition (3.4) and (3.5) it is also possible to study certain symmetry properties of their individual terms. We shall do this in our discussion of the restrictions following from parity conservation.

For a massless particle like the photons we cannot boost to a restframe. But, since only two polarizations are possible, one can assume an expansion of the spin matrix in terms of the Pauli matrices[15]

$$\rho^\gamma(\vec{q}) = \frac{1}{\sqrt{2}} (I - \ell \cos 2\Phi\sigma_1 - \ell \sin 2\Phi\sigma_2 + c\,\sigma_3)$$

$$= O_0^\gamma + \ell \cos 2\Phi\, O_1^\gamma + \ell \sin 2\Phi\, O_2^\gamma + c\, O_3^\gamma \tag{3.6}$$

The expression (3.6) refers to a standard frame, for which one chooses either the c.m. system or the laboratory system with the 1-axis in the reaction plane. The constants c and ℓ denote the linear and circular polarization and Φ is equal to the azimuthal angle of the polarization vector measured in a right-handed system with the photon momentum as polar axis. For a more detailed discussion we refer to Section 5D.

Since in the following only very general properties of the expansions of the type (3.4), (3.5) or (3.6) are used, we adopt a unified convention. In general we shall write an expansion of a single particle spin density matrix ρ^K as

$$\rho^K = \sum_\alpha p_\alpha O_\alpha \tag{3.7}$$

where p_α denotes any of the components of the tensor polarizations and O_α any corresponding spin matrix. We shall refer to O_α as the observables and assume that the matrices O_α are hermitian, linear independent and orthogonal

$$\mathrm{Tr}\ O_\gamma O_{\gamma'} = \delta_{\gamma\gamma'} \quad . \tag{3.8}$$

The task of constructing $(2S+1)^2$ such operators to describe a particle of (arbitrary) spin S has been discussed by several authors.[16, 17] Following their work we form polynomials T_{LM} in S_1, S_2, S_3 which transform under rotations as the spherical harmonics Y_L^M requiring according to Racah[18]

$$\left[S_\pm,\ T_{LM}\right] = T_{LM\pm 1}\ \left[(L\mp M)(L\pm M+1)\right]^{1/2} \tag{3.9}$$

and

$$\left[S_2,\ T_{LM}\right] = M\,T_{LM} \tag{3.10}$$

for $S_\pm = (S_3 \pm iS_1)$, $M = -L, -L+1, \ldots, L$ and $L \le 2S$. We should like to point out that our choice of the 2-axis for "quantizing" M, rather than the 3-axis as in the case of the particle spins, provides considerable simplifications in the work below where rotations around the 2-axis play an important role. We determine the overall phase of T_{LM} by requiring that T_{LL} be a positive multiple of $(S_3 + iS_1)^L$ and determine the scale by requiring $Tr(T_{LL}\, T^{\dagger}_{LL}) = 1$. Given T_{LL} one can apply Eq. (3.9) to calculate the other T_{LM}. One also has $T^{\dagger}_{LM} = (-1)^M T_{L-M}$. Finally, for each L we obtain 2L hermitian matrices

$$T^1_{LM} = \frac{1}{\sqrt{2}}\left(T_{LM} + T^{\dagger}_{LM}\right) \tag{3.11}$$

and

$$T^2_{LM} = \frac{1}{i\sqrt{2}}\left(T_{LM} - T^{\dagger}_{LM}\right) \tag{3.12}$$

for $L \geq M > 0$. Together with the 2S +1 matrices $T^1_{L0} = T_{L0}$ which are already hermitian we then have a suitable set of $(2S+1)^2$ orthogonal matrices for describing a massive particle of spin S. We will not write explicit forms for the T_{LM} here, but note that for $M \geq 0$ they can always be cast in the general form

$$T_{LM} = \left[P^M_L(S_2),\ (S_3 + iS_1)^M\right]_+ \tag{3.13}$$

where $P^M_L(S_2)$ is a unique real polynomial of degree (L-M) in S_2 and is either even or odd. For S = 1/2, 1 these T^i_{LM} just give the familiar set of observables in (3.4) and (3.5). A derivation of Eq. (3.13) is included at the end of Section 5E.

The decomposition of the two-particle spin matrix (3.3) can be written in the manner of (3.7) as

$$\rho^i = \sum_{\alpha,\beta} p^A_\alpha p^B_\beta O^A_\alpha \otimes O^B_\beta \tag{3.14}$$

For the spin density matrix of the final state ρ^f we perform again a decomposition in terms of the direct products $O^C_\gamma \otimes O^D_\delta$

$$\rho^f = \sum_{\gamma,\delta} P^{C,D}_{\gamma,\delta}\, O^C_\gamma \otimes O^D_\delta \tag{3.15}$$

But this time the coefficients $P^{C,D}_{\gamma,\delta}$ no longer factorize in general since the spins of the particles will be usually correlated.

From the orthonormality relation (3.8) one obtains

$$P^{CD}_{\gamma\delta} = \mathrm{Tr}\left[O^C_\gamma \otimes O^D_\delta \ \rho^f\right] . \tag{3.16}$$

In practice we are not so much interested in the quantities $P^{CD}_{\gamma\delta}$ which contain the dynamics of the reaction together with the information on the initial beam. To separate the dynamical information from the input data we write

$$P^{CD}_{\gamma\delta} = \sum_{\alpha\beta} M^{CD,AB}_{\gamma\delta,\alpha\beta} \ p^A_\alpha p^B_\beta \tag{3.17}$$

where we have now

$$\begin{aligned} M^{CD,AB}_{\gamma\delta,\alpha\beta} &= \mathrm{Tr}\left[O^C_\gamma \otimes O^D_\delta \ F \ O^A_\alpha \otimes O^B_\beta \ F^\dagger\right] \\ &= \frac{\partial P^{CD}_{\gamma\delta}}{\partial p^A_\alpha \, \partial p^B_\beta} \end{aligned} \tag{3.18}$$

We shall refer to M in the following as the tensor polarization matrix.

We remark that the following terms of (3.15)

$$\frac{\mathrm{Tr}\left(O^C_0\right)}{\mathrm{Tr}(\rho^f)} \sum_\delta P^{C,D}_{0,\delta} \, O^D_\delta = \sum_\delta P^D_\delta O^D_\delta \equiv \rho^D \tag{3.19}$$

define the density matrix of particle D alone in its helicity rest frame. If D is unstable and decays into two particles then the angular distribution of the decay particles in the rest frame of D as well as the angular dependence of their possible polarizations can be used to determine the coefficients P^D_δ in (3.19). The task of extracting such coefficients has been discussed in detail by Jackson.[17] We only note here the simple case of an integer spin particle D decaying into two spinless particles, which is needed for the $\rho \to 2\pi$ decay. If we use the matrices T^i_{LM} of

(3.11) and (3.12) for the observables O_δ^D then the coefficients P_{LM}^{iD} for L even are simply related to the coefficients of Re $Y_{LM}(\theta^d, \phi^d)$ and Im $Y_{LM}(\theta^d, \phi^d)$ occurring in the (normalized) angular distribution W^D of the decay products:

$$\int d\Omega^d \begin{pmatrix} \mathrm{Re}\, Y_{LM} \\ \mathrm{Im}\, Y_{LM} \end{pmatrix} W^D(\theta^d, \phi^d) = (-1)^{S_D} \frac{(2j+1)}{\sqrt{(4\pi)(2L+1)}} \langle LO | S_D S_D OO \rangle \left(\frac{1}{\sqrt{2} + \delta_{MO}(1-\sqrt{2})} \right) P_{LM}^{\binom{1}{2} D} \tag{3.20}$$

where $\langle LO | S_D S_D OO \rangle$ are standard Clebsch-Gordon coefficients.

3B. Restrictions Following From Parity Conservation.

To discuss the restrictions of parity conservation on the tensor polarization matrix M (3.18) we consider reflections Y in the (n_1, n_3)-plane, which are perpendicular to the reaction plane: $Y_{op} = e^{-i\pi \vec{n}_2 \cdot J} P$, where P is the parity operator. Under this symmetry operation the reaction matrix F is even: $F = YFY^\dagger$ or more explicitly

$$F_{\lambda_C \lambda_D, \lambda_A \lambda_B} = Y^C_{\lambda_C \lambda'_C} Y^D_{\lambda_D, \lambda'_D} F_{\lambda'_C \lambda'_D \lambda'_A \lambda'_B} Y^{A\dagger}_{\lambda'_A \lambda_A} Y^{B\dagger}_{\lambda'_B \lambda_B} \tag{3.21}$$

In the basis of the helicity states the operator Y_{op} become the matrices $Y = -i\eta_N \sigma_2$, $+ i\eta_N \sigma_2$ for a nucleon which is particle "1", "2", and $Y = -\sigma_1$ for a photon as shown in Section 5A. Therefore under the reflection Y the observables O_i of the nucleon transform as $(Y^N = -i\sigma_2)$

$$Y^N \left(O_0^N, \vec{O}^N\right) Y^{N\dagger} = \left(O_0^N, -O_1^N, O_2^N, -O_3^N\right) \tag{3.22}$$

This reflection property of the pseudovector polarization is evident from a geometrical point of view. The transformation law of the observables of the photon are $(Y = -\sigma_1)$

$$Y^\gamma \left(O_0, \vec{O}^\gamma\right) Y^{\gamma\dagger} = \left(O_0^\gamma, O_1^\gamma, -O_2^\gamma, -O_3^\gamma\right) \tag{3.23}$$

To understand the result (3.23) geometrically one has to remember that O_3^γ represents the circular polarization, which therefore changes the sign under the reflection operation Y, and that O_1^γ and O_2^γ are the two observables of the linear polarization. Now since under a reflection the azimuthal angle Φ has to be replaced by $(\pi - \Phi)$, it means that O_1^γ, the expectation value of which is proportional to $\cos 2\Phi$, does not change sign whereas O_2^γ, the expectation value of which is proportional to $\sin 2\Phi$ (see (3.6)), does change sign.

The observables T^i_{LM} introduced above for the general massive spin case have a simple symmetry under $Y = \eta e^{-i\pi S_2}$:

$$Y\, T^i_{LM}\, Y^\dagger = \pi_M\, T^i_{LM} \tag{3.24}$$

with $\pi_M = (-1)^M$, which follows from (3.10). This fact is a consequence of the particular choice for the quantization axis[7] of the tensors T_{LM}.

Also, since Y commutes with S_2, the product $T_{LM}Y$, or $Y^\dagger T_{LM}$, is again a superposition of $T_{L'M'}$ with $M' = M$:

$$T_{LM}Y = \sum_{L'=|M|}^{2S} \rho^{LM}_{L'}\, T_{L'M} \tag{3.25a}$$

In general we shall write

$$O^K_\kappa Y^K = \rho^K_{\kappa'} O^K_{\kappa'} \tag{3.25b}$$

In Section 5 we show that the constants $\rho^{LM}_{L'}$ are individually either pure real or pure imaginary and, further, that $\rho^{LM}_{L'}$ must vanish if $M = 0$ and $L - L' + 2S$ is odd or if $L' = L$ and $M + 2S$ is odd. Thus multiplication by Y maps the set T^i_{LM} essentially into itself with little or no mixing. In particular, we have for the nucleon (as particle "1") $(Y = -i\eta_N\sigma_2)$

$$\left(O^N_0, \ldots, O^N_3\right) Y^N = \eta_N\left(+iO^N_2, +O^N_3, +iO^N_0, -O^N_1\right) , \tag{3.26}$$

for the photon ($Y = -\sigma_1$)

$$\left(O_0^\gamma, \ldots, O_3^\gamma\right) Y^\gamma = \left(+O_1^\gamma, +O_0^\gamma, -iO_3^\gamma, +iO_2^\gamma\right) \tag{3.27}$$

and for a spin 1 particle $\left(Y = -\eta \frac{1}{\sqrt{3}} (O_0 + \sqrt{8}\, O_4)\right)$

$$\frac{1}{\eta}(O_0, \ldots, O_8)\, Y = -\frac{1}{3}(O_0 + \sqrt{8}\, O_4)$$

$$- O_2, iO_5, -iO_6, \frac{1}{3}(\sqrt{8}\, O_0 - O_4) \tag{3.28}$$

$$- iO_3, iO_1, -O_7, -O_8 \quad .$$

We assigned the numbering of the observables according to the order given in (3.4) - (3.6) so that e.g., for spin 1: $O_7 = \frac{1}{\sqrt{2}}\left(S_3^2 - S_1^2\right)$. These mapping relations are crucial for showing the equivalence of certain polarization measurements.

Applying now the parity conservation condition (3.27) in the expression (3.18) for the tensorpolarization matrix M to replace F and $F^\dagger$ one obtains

$$\begin{aligned} M^{CD,AB}_{\gamma\delta,\alpha\beta} &= \mathrm{Tr}\left[Y^\dagger \left(O_\gamma^C \otimes O_\delta^D\right) YFY^\dagger \left(O_\alpha^A \otimes O_\beta^B\right) YF^\dagger\right] \\ &= \pi_\gamma^C \pi_\delta^D \pi_\alpha^A \pi_\beta^B \, M^{CD,AB}_{\gamma\delta,\alpha\beta} \end{aligned} \tag{3.29}$$

by using the parity transformation properties (3.24).

From the last line in (3.29) the important result follows that only those tensor polarizations $M^{CD,AB}_{\gamma\delta,\alpha\beta}$ are nonvanishing, where the product of the parities π_r of the observables in the initial and final state is even

$$\pi_\alpha^A \pi_\beta^B \pi_\gamma^C \pi_\delta^D = +1 \tag{3.30}$$

That is the generalization of the familiar statement for the (π - N) final state that the polarization of the nucleon has to be perpendicular to the reaction plane if parity is conserved and if there is no polarization in the initial state. In this case one has

$$\pi^A_\alpha \pi^B_\beta \pi^C_\gamma \pi^D_\delta = 1 \cdot 1 \cdot \pi^N_\gamma \cdot \pi^\pi_0$$

$$= \pi^N_\gamma \qquad (3.31)$$

independent on the nature of the initial particles. $\pi^N = +1$ only for O_0 and O_2 according to (3.22). Many other similar statements can be derived from the above general rules (3.30).

3C. Mapping Relations

Applying once more the parity conservation condition (3.21) in (3.18) to replace this time only F, one obtains

$$M^{CD,AB}_{\gamma\delta,\alpha\beta} = \mathrm{Tr}\left[\left(O^C_\gamma \otimes O^D_\delta\right) YFY^\dagger \left(O^A_\alpha \otimes O^B_\beta\right) F^\dagger\right]. \qquad (3.32)$$

One can use now the mapping relations (3.26)...(3.28) or in general (3.25) to obtain the following relationships between different types of experiments

$$M^{CD,AB}_{\gamma\delta,\alpha\beta} = \sum \rho^\gamma_{\gamma'} \rho^\delta_{\delta'} \rho^{\alpha*}_{\alpha'} \rho^{\beta*}_{\beta'} \; M^{CD,AB}_{\gamma'\delta'\alpha'\beta'} \qquad (3.33)$$

where the indices $\alpha', \beta', \gamma', \delta'$ and the factors $\rho^V_{V'}$ (real or imaginary) have to be determined by inspecting the relations (3.26)...(3.28) or (3.25). As long as the particles A, B, C, D represent only pions, nucleons or photons as in pion photoproduction the sum in (3.33) extends always only over one term with our choice of observables O^K_κ. That for our set of observables, this statement remains true in most situations if particles with spin 1 or 3/2 or even high spins are involved follows from the discussion after Eq. (3.25). Since the matrices O_α are hermitian, the tensor-polarization matrix M (3.18) has also to be hermitian. Therefore, the product of the ρ's in (3.33) can only be imaginary, if $M^{CD,AB}_{\alpha\beta,\gamma\delta}$ actually vanishes. If this product is real = ± 1, the result (3.33) shows which non-vanishing elements of M are linearly dependent. For example, in photoproduction: Since under the mapping (3.27) one has

$$\left(O_0^\gamma, O_3^\gamma\right) Y = \left(O_1^\gamma, iO_2^\gamma\right) \tag{3.34}$$

a complete set of measurements can be done using either circularly polarized or plane polarized γ's. But if one has done a measurement with an unpolarized nucleon target using circularly polarized γ's, one has to use a polarized nuclear target if one now uses plane polarized γ's.

We also note that the relation (3.33) always contain the product of all four intrinsic parities in the reaction as a common factor contained in the ρ's. Hence from measurements of the left and right side in (3.33) the parity of one particle can be deduced, if the parity of the other three is known.

It is also interesting to note what happens to the mapping relations (3.33) if parity is not conserved but the definition of the reflection operator Y is still possible. Let us assume F contains a parity conserving F_+ and non-conserving part F_- so that

$$YFY^\dagger = Y(F_+ + F_-)\, Y^\dagger = F_+ - F_- \tag{3.35}$$

Then, in contrast to the preceeding discussion, measurements where the product of the parities π_κ^K of all observables is (-1) also yield non-vanishing results. Let us use for the moment a compact notation where $O^{i,f}$ and $\pi^{f,i}$ denote the product of the observables or parities, respectively, in the initial or final state. Then one obtains if $\pi^f \cdot \pi^i = +1$

$$\begin{aligned} \mathrm{Tr}\left\{O^f F\, O^i F^\dagger\right\} &= \mathrm{Tr}\left\{O^f F_+\, O^i F_+^\dagger + O^f F_-\, O^i F_-^\dagger\right\} \\ &= \mathrm{Tr}\left\{(O^f Y^f) F_+ (Y^{i\dagger} O^i)\, F_+^\dagger - (O^f Y^f)\, F_- (Y^{i\dagger} O^i)\, F_-^\dagger\right\} \end{aligned} \tag{3.36}$$

and if $\pi^f \cdot \pi^i = -1$

$$\mathrm{Tr}\left\{O^f F\, O^i F^\dagger\right\} = 2\,\mathrm{Re}\,\mathrm{Tr}\left\{O^f F_+\, O^i F_-^\dagger\right\}$$

$$= 2\,\mathrm{Re}\,\mathrm{Tr}\left\{(O^f Y^f)\, F_+ (Y^{i\dagger} O^i)\, F_-^\dagger\right\} \qquad (3.37)$$

Thus from (3.37) follow analogous mapping relations to (3.33). From (3.36) it follows that a measurement with the set of observables (O^f, O^i) and $(O^{f'} = O^f Y^f,$ $O^{i'} = Y^{i\dagger} O^i)$ serve to separate the contribution $\mathrm{Tr}\left\{O^{f'} F_\pm\, O^{i'} F_\pm^\dagger\right\}$. To these terms the relations (3.33) can again be applied apart from a change of sign for the F_- term.

3D. Consequences to Time Reversal Invariance

It is natural to ask whether a polarization measurement can be performed more simply in the time reversed reaction. Specifically if we assume time reversal invariance, how do we obtain $M^{CD,AB}_{\gamma\delta,\alpha\beta}(W, \theta, \phi = 0)$ for the reaction (1.1) $A + B \rightarrow C + D$ from measurements $N^{AB,CD}_{\alpha'\beta',\gamma'\delta'}(W, \theta_r, \phi_r = 0)$ in the reversed reaction $C + D \rightarrow A + B$? All momenta in both reactions have opposite sign. We assume that the axes in the time reversed reaction are obtained from Fig. 1 by switching the labels $(A, B) \rightarrow (C, D)$ and $\theta \leftrightarrow \theta_r$. We note that the original rest frame axes go over into the new ones by a rotation through π about the 1-axis.

The required correspondence between elements of M and N can be deduced heuristicly by noting that time reversal changes the vector polarization $\vec{S}^K$ taken relative to fixed space axes in the rest frame of particle K, to $-\vec{S}^K$. If one then adjusts for the difference between these fixed axes and the new axes by rotating $\vec{S}^K$ through π about the 1-axis, one obtains the result, which we derive formally below.

We observe that time reversal invariance implies the following relationship between the helicity amplitudes of the original and time reversed reaction

$$f_{\lambda_C\lambda_D,\lambda_A\lambda_B}(W, \theta, \phi=0) = (-1)^{\lambda_A-\lambda_B-\lambda_C+\lambda_D}\, g_{\lambda_A\lambda_B,\lambda_C\lambda_D}(W, \theta_r, \phi_r=0) \qquad (3.38)$$

where $g_{\lambda_A\lambda_B,\lambda_C\lambda_D}$ are our standard helicity amplitudes for the time reversed reaction, with the axes chosen as just mentioned. We may write Eq. (3.38) as well in matrix form

$$F = r_3^C(+\pi) \otimes r_3^D(-\pi)\ G^{tr}\ r_3^A(-\pi) \otimes r_3^B(+\pi) \tag{3.39}$$

using the matrices

$$\left\{r_3^K(\pm\pi)\right\}_{\lambda\lambda'} = (-1)^{\mp\lambda}\,\delta_{\lambda\lambda'} \tag{3.40}$$

which represent rotations through $\pm\pi$ about the 3-axis in the rest frame of each particle. Finally, using (3.18) and (3.39), we find

$$M^{CD,AB}_{\gamma\delta,\alpha\beta} = (\tau\omega)^A_\alpha\ (\tau\omega)^B_\beta\ (\tau\omega)^C_\gamma\ (\tau\omega)^D_\delta\ N^{AB,CD}_{\alpha\beta,\gamma\delta} \tag{3.41}$$

where

$$N^{AB,CD}_{\alpha\beta,\gamma\delta} = \mathrm{Tr}\left\{\left(O^A_\alpha \otimes O^B_\beta\right) G \left(O^C_\gamma \otimes O^D_\delta\right) G^\dagger\right\} ; \tag{3.42}$$

we assumed the following symmetry properties of the observables O^K_κ under transposition

$$\left(O^K_\kappa\right)^{tr} = \tau^K_\kappa O^K_\kappa\,, \quad \tau^K_\kappa = \pm 1 \tag{3.43}$$

and under rotations around the 3-axis about the angle $\pm\pi$

$$r_3^K(\pm\pi)\ O^K_\kappa\, r_3^K(\pm\pi)^\dagger = \omega^K_\kappa O^K_\kappa\,, \quad \omega^K_\kappa = \pm 1 \quad . \tag{3.44}$$

(3.43) and (3.44) are a consequence of Eqs. (3.11) - (3.13). Since the matrix S_2 is imaginary while the matrices for S_3 and S_1 are real $\left(\tau^K_\kappa\ \omega^K_\kappa\right)$ just changes the polarizations $\vec{S}$ by $\vec{S} \to -\vec{S}$ followed by a rotation through π about the 1-axis.

The result (3.41) shows that the tensor polarization matrices for channels related by time reversal are linearly dependent and are easily related with our set of observables. The result (3.41) includes the familiar statement that a measurement of the nucleon polarization of the π-N system is equivalent to measuring the cross section with a polarized target, a fact widely used at present in pion nucleon scattering.

3E. A Note on Time Reversal Invariance and Vector Dominance

To utilize some of the above results in a practical situation we derive a relationship which can be used to apply the hypothesis of vector meson dominance (VMD)[19] to photoproduction with plane-polarized γ's. To begin with let us consider a photoproduction process in which the nucleon target is unpolarized and only the total differential cross section is measured.

$$n + \gamma \longrightarrow p + \pi^- \qquad \text{(reaction I)} \qquad (3.45)$$

Then, using the photon polarization matrices of (3.6) we find for this experiment

$$\frac{1}{ph^I}\frac{d\sigma^I}{d\Omega} = M^{p\pi, n\gamma}_{00,00} + \ell \cos 2\Phi\, M^{p\pi, n\gamma}_{00,01} \qquad (3.46)$$

where ph^I is a kinematic factor (see (5.28)) and

$$M^{p\pi, n\gamma}_{00,0\beta} = \frac{1}{2}\,\mathrm{Tr}\left\{(I^p \otimes I^\pi)\, F\left(I^n \otimes O^\gamma_\beta\right) F^\dagger\right\} \qquad (3.47)$$

The photon polarizations $\beta = 2,3$ do not enter into (3.46) because of parity conservation.

Now we would like to use time reversal invariance to relate these measurements in I to measurements in the reaction

$$p + \pi^- \longrightarrow n + \gamma \qquad \text{(reaction IIa)} \qquad (3.48)$$

and then, by VDM, to measurements of the reaction

$$p + \pi^- \longrightarrow n + V \qquad \text{(reaction IIb)} \qquad (3.49)$$

where V is a vector meson (ρ, ω or ϕ). Applying Eq. (3.41) for time reversal we find that $M^{p\pi, n\gamma}_{00,00}$ and $M^{p\pi, n\gamma}_{00,01}$ can be found by measuring the usual differential cross section and the linear polarization $N^{n\gamma, p\pi}_{01,00}$ of the photon in IIa, again with the unpolarized target.

$$\left(ph^{IIa}\right)^{-1} \frac{d\sigma^{IIa}}{d\Omega} = N^{n\gamma, p\pi}_{00,00} = M^{p\pi, n\gamma}_{00,00} \tag{3.50a}$$

and

$$N^{n\gamma, p\pi}_{01,00} = M_{00,01} \tag{3.50b}$$

We have

$$\begin{aligned} N^{n\gamma, p\pi}_{0\beta,00} &= \mathrm{Tr}\left\{O^n_0 \otimes O^\gamma_\beta \; \rho^{IIa}\right\} \\ &= \mathrm{Tr}\left\{\left(O^n_0 \otimes O^\gamma_\beta\right) G^{IIa} \left(O^p_0 \times O^\pi_0\right) G^{IIa\dagger}\right\} \\ &= \mathrm{Tr}\left(O^\gamma_\beta \; \rho^\gamma\right) \frac{\mathrm{Tr}\,\rho^{IIa}}{\mathrm{Tr} O^n_0} \end{aligned} \tag{3.51}$$

In the last step of Eq. (3.51) we introduced the density matrix ρ^γ of the photons using the definition (3.19). Now according to the VDM assumption the helicity amplitudes entering into (3.51) (with $\lambda_\gamma = \pm 1$) are directly related to the same helicity amplitudes for reaction IIb, so that

$$\rho^{IIa} = \frac{\alpha\pi}{\gamma_\rho^2} \left\{\rho^{IIa}\right\}_{tr} + \ldots \tag{3.52}$$

In (3.52) the contributions of the ω and ϕ mesons is neglected (ρ-dominance). ρ^{IIb} is the density matrix in the c.m. system of the final state IIb. The subscript "tr" denotes the transversal part of ρ^{IIb}, i.e., all elements with λ_ρ, $\lambda'_\rho = 0$ are deleted. Using (3.19) to define the density matrix ρ^ρ of the ρ-mesons and the ρ-dominance assumption (3.52) one obtains

$$\begin{aligned} N^{n\gamma, p\pi}_{0\beta,00} &= \frac{\alpha\pi}{\gamma_\rho^2} \mathrm{Tr}\left(O^\gamma_\beta \left\{\rho^\rho\right\}_{tr}\right) \frac{\mathrm{Tr}\,\rho^{IIb}}{\mathrm{Tr} O^n_0} + \ldots \\ &= \frac{\alpha}{\gamma_\rho^2} \mathrm{Tr}\left(O^\gamma_\beta \left\{\rho^\rho\right\}_{tr}\right) \mathrm{Tr} O^\rho_0 \frac{1}{ph^{IIb}} \frac{d\sigma^{IIb}}{d\Omega} + \ldots \end{aligned} \tag{3.53}$$

For $\beta = 0,\ 1$ we have

$$\mathrm{Tr}\left(O_0^\gamma\{\rho^\rho\}_{tr}\right) = \frac{1}{\sqrt{2}}\ \mathrm{Tr}\{\rho^\rho\}_{tr} = \frac{1}{\sqrt{2}}\ \left(\rho_{11}^\rho + \rho_{-1-1}\right) = \sqrt{2}\,\rho_{11}^\rho \tag{3.54a}$$

$$\mathrm{Tr}\left(O_1^\gamma\{\rho^\rho\}_{tr}\right) = -\frac{1}{\sqrt{2}}\ \mathrm{Tr}\,\sigma_1\{\rho^\rho\}_{tr} = \frac{-1}{\sqrt{2}}\left(\rho_{-11}^\rho + \rho_{1-1}^\rho\right) = -\sqrt{2}\ \rho_{-11}^\rho \tag{3.54b}$$

The identities $\rho_{11} = \rho_{-1-1}$ and $\rho_{-11} = \rho_{1-1}$ used in (3.54) are a consequence of parity conservation.

Using now the result (3.50), (3.53) and (3.54) in (3.46) we can relate the photoproduction reaction I with polarized photons to the ρ-production data of IIb

$$\frac{d\sigma^I}{d\Omega} = \frac{ph^I}{ph^{II}}\sqrt{2}\ \mathrm{Tr}O_0^\rho\ \frac{\alpha\pi}{\gamma_\rho^2}\left(\rho_{11} - \ell\cos 2\Phi\,\rho_{-11}\right)\frac{d\sigma^{IIb}}{d\Omega} \tag{3.55a}$$

Using (3.46) and (5.28) for the definition of "ph" and the relation $\mathrm{Tr}O_0^\gamma = \sqrt{2}$ (3.6) one obtains $ph^I/ph^{II}\ \ 2\ \mathrm{Tr}O_0^\rho = q_n^2/q_p^2$. Thus for pure linearly polarized γ's (ℓ=1) the result (3.55a) goes over into

$$\frac{d\sigma^I}{d\Omega} = \frac{q_n^2}{q_p^2}\ \frac{\alpha\cdot\pi}{\gamma_\rho^2}\ \left(\rho_{11} - \cos 2\,\Phi\ \rho_{-11}\right)\frac{d\sigma^{IIb}}{d\Omega} \tag{3.55b}$$

According to Section 5D $\Phi = 0,\ \pi/2$ corresponds to linear polarization in the reaction plane or polarization perpendicular to it, respectively. It might be worthwhile to stress that the minus sign in front of ρ_{-11} in (3.54) and (3.55) is a consequence of the phase convention (5.9) for the helicity states of J.W. This phase convention is usually adopted, when quoting results for the density matrix ρ^ρ. In our derivation we have neglected effects of the extrapolation in the ρ-mass. For an attempt to treat these effects kinematically in a suitably chosen phase space factor see the derivation of (3.55) by Krammer and Schilling,[20] who write (3.55) also with the inclusion of the ω and ϕ meson.

4. RESULTS FROM CROSSING RELATIONS

4A. Crossing Applied to the Elements of M (3.18)

We now wish to express the elements of the tensor polarization matrix M (3.18) in the s-channel by the analytically continued t- or u-channel helicity amplitudes. This we shall perform by substituting in relation (3.18) for M the s-channel amplitudes by the crossed t- or u-channel amplitudes using the crossing relations of Trueman and Wick,[9] (hereafter referred to as T.W.). Let us first define a matrix notation in all three channels, which makes the following discussion as compact as possible, since only very general properties of the crossing relations will be used.

The s-, t- and u-channels will be denoted by

$$A + B \rightarrow C + D, \text{ s-channel} \tag{4.1}$$

$$\overline{D} + B \rightarrow C + \overline{A}, \text{ t-channel} \tag{4.2}$$

$$\overline{C} + B \rightarrow \overline{A} + D, \text{ u-channel} \tag{4.3}$$

The corresponding helicity amplitudes are $f^s_{\lambda_C\lambda_D,\lambda_A\lambda_B}(W^s, \theta^s, \phi^s = 0)$, $f^t_{\lambda_C\lambda_A,\lambda_D\lambda_B}(W^t, \theta^t, \phi^t = 0)$, $f^u_{\lambda_A\lambda_D,\lambda_C\lambda_B}(W^u, \theta^u, \phi^u = 0)$. From now on the quantities in the three channels (4.1) - (4.3) are denoted by the superscripts s, t, u. To fix the phases of the helicity amplitudes we take as always the convention (see Introduction) that the first particles in (4.1) - (4.3) are the particles with label 1, i.e., in (4.1) they are the pair (A, C), in (4.2) the pair ($\overline{D}$, C) and in (4.3) the pair ($\overline{C}$, $\overline{A}$). In photoproduction A and C represent then according to (2.1) the nucleons. Note that in this case the u-channel amplitudes are derived from the s-channel amplitudes by substituting antinucleon indices for nucleon indices. As in (3.2) we denote by F^s, F^t, F^u the three reaction matrices with elements

$$\{F^s\}_{\lambda_C\lambda_D,\lambda_A\lambda_B} = f^s_{\lambda_C\lambda_D,\lambda_A\lambda_B} ; \{F^t\}_{\lambda_C\lambda_A,\lambda_D\lambda_B} = f^t_{\lambda_C\lambda_A,\lambda_D\lambda_B} ;$$

$$\{F^u\}_{\lambda_A\lambda_D,\lambda_C\lambda_B} = f^u_{\lambda_A\lambda_D,\lambda_C\lambda_B} \tag{4.4}$$

In writing down the crossing relations it will be convenient to have introduced in the s-channel partially reflected reaction matrices $F^s_{(AD)}$ and $F^s_{(AC)}$ with elements

$$\left\{F^s_{(AD)}\right\}_{\lambda_C\lambda_A,\lambda_D\lambda_B} = f^s_{\lambda_C\lambda_D,\lambda_A\lambda_B} \tag{4.5}$$

and

$$\left\{F^s_{(AC)}\right\}_{\lambda_A\lambda_D,\lambda_C\lambda_B} = f^s_{\lambda_C\lambda_D,\lambda_A\lambda_B} . \tag{4.6}$$

Finally we need also the tensor polarization matrices of the t- and u-channel, which are defined in analogy to (3.18) again in terms of the single particle observables O^K_κ introduced in Section 3A.

$$N^{CA,DB}_{\gamma\alpha,\delta\beta}(F^t) = \mathrm{Tr}\left[\left(O^C_\gamma \otimes O^A_\alpha\right) F^t \left(O^D_\delta \otimes O^B_\beta\right) F^{t\dagger}\right] \tag{4.7}$$

and

$$L^{AD,CB}_{\alpha\delta,\gamma\beta}(F^u) = \mathrm{Tr}\left[\left(O^A_\alpha \otimes O^D_\delta\right) F^u \left(O^C_\gamma \otimes O^B_\beta\right) F^{u\dagger}\right] \tag{4.8}$$

In the s-channel we use again the definition (3.18), but use in the following the notation

$$M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^s) = M^{CD,AB}_{\gamma\delta,\alpha\beta}$$

for better clarity.

We express $M(F^S)$ also in terms of the partially reflected reaction matrices $F^s_{(AD)}$, $F^s_{(AC)}$, (4.5) and (4.6)

$$M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^S) = \mathrm{Tr}\left[\left(O^C_\gamma \otimes \left(O^A_\alpha\right)^{tr}\right) F^s_{(AD)} \left(O^D_\delta\right)^{tr} O^B_\beta F^{s\dagger}_{(AD)}\right]$$

$$= \tau^A_\alpha \tau^D_\delta N^{CA,DB}_{\gamma\alpha,\delta\beta}\left(F^s_{(AD)}\right) \tag{4.9}$$

or analogously

$$M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^S) = \tau^A_\alpha \tau^C_\gamma L^{AD,CB}_{\alpha\delta,\gamma\beta}\left(F^s_{(AC)}\right) \tag{4.10}$$

In (4.9) and (4.10) we used the fact that according to our choice the matrices O^K_κ are either symmetrical or antisymmetrical (see (3.43)). By means of the notation (4.9) and (4.10) it will be very convenient to express the tensor polarization in terms of the crossed t- and u-channel amplitudes.

After these somewhat lengthy preparations we now wish to express the elements of the tensor polarization matrix $M(F^S)$ in terms of the analytically continued elements F^t or F^u by invoking the crossing relations of T.W. These authors arrived at a result which is easy to visualize: After the helicity amplitudes, say of the t-channel, are analytically continued from the physical t-channel region to the s-channel region, the rest frame states are quantized in a direction different from that given by the helicity convention (see Section 5). By a rotation of the states in the rest frame of each particle, these have to be adjusted to the new helicity axes. The rotation is around the 2-axis through an angle χ^K. In some cases an additional rotation around the 3-axis in the helicity frame through the angle π readjusts the helicity dependent phases of the helicity states resulting from the distinction between particle "1" and "2". With our definition in (4.1) - (4.3) of the particles with the label 2, this last rotation will only appear if the u-channel amplitudes are crossed.

Using the partially reflected reaction matrices (4.5) and (4.6) the crossing relations of T.W. can be written in the compact matrix notation

$$F^{s}_{(AD)} = D^{C}\left(\chi^{C}_{t}\right)^{\dagger} \otimes D^{A}\left(\chi^{A}_{t}\right)^{\dagger} F^{T} D^{D}\left(\chi^{D}_{t}\right) \otimes D^{B}\left(\chi^{B}_{t}\right) \qquad (4.11)$$

for crossing the t-channel amplitudes and

$$F^{s}_{(AC)} = D^{D\dagger}\left(-\chi^{D}_{u}\right) \otimes \left[r^{A}_{3}(+\pi)\, D^{A\dagger}\left(-\chi^{A}_{u}\right)\right] F^{u} \left[D^{C}\left(\chi^{C}_{u}\right) r^{C}_{3}(-\pi)\right] \otimes D^{B}\left(\chi^{B}_{u}\right) \qquad (4.12)$$

for crossing the u-channel amplitudes.

The matrices

$$\left\{D^{K}\right\}_{\lambda_K \lambda'_K} = d^{S_K}_{\lambda_K \lambda'_K}\left(\chi^{K}_{t,u}\right) \qquad (4.13)$$

are the standard real rotation matrices in the (1, 3)-plane. The rotation matrices $r^{K}_{3}(\pm\pi)$ in (4.12) were defined in (3.40). For completeness we cite here also the results of T.W. for the crossing angles $\chi^{K}_{u,t}$

$$\cos\, \chi^{A}_{t}\left(2p_t \sqrt{t}\, S_1\right) = -\left(s + m^2_A - m^2_B\right)\left(t + m^2_A - m^2_C\right) - 2m^2_A \cdot L \qquad (4.14)$$

$$\cos\, \chi^{B}_{t}\left(2q_t \sqrt{t}\, S_1\right) = +\left(s + m^2_B - m^2_A\right)\left(t + m^2_B - m^2_D\right) - 2m^2_B \cdot L \qquad (4.15)$$

$$\cos\, \chi^{C}_{t}\left(2p_t \sqrt{t}\, S_2\right) = +\left(s + m^2_C - m^2_D\right)\left(t + m^2_C - m^2_A\right) - 2m^2_C \cdot L \qquad (4.16)$$

$$\cos\, \chi^{D}_{t}\left(2q_t \sqrt{t}\, S_2\right) = -\left(s + m^2_D - m^2_C\right)\left(t + m^2_D - m^2_B\right) - 2m^2_D \cdot L \qquad (4.17)$$

where

$$L = m^2_C - m^2_A + m^2_B - m^2_D \qquad (4.18)$$

and

$$p^2_t = \left(t^2 - 2t\left(m^2_A + m^2_C\right) + \left(m^2_A - m^2_C\right)^2\right)/(4t) \qquad (4.19)$$

$$q^2_t = \left(t^2 - 2t\left(m^2_B + m^2_D\right) + \left(m^2_B - m^2_D\right)^2\right)/(4t) \qquad (4.20)$$

$$S_1^2 = 4s\, p_s^2 = \left(s - (m_A - m_B)^2\right)\left(s - (m_A + m_B)^2\right) \tag{4.21}$$

$$S_2^2 = 4s\, q_s^2 = \left(s - (m_C - m_D)^2\right)\left(s - (m_C + m_D)^2\right) \tag{4.22}$$

To fix the sign of χ_t^K we cite also the result for $\sin \chi_t^K$

$$\sin \chi_t^{A,B} = \frac{m_{A,B}\, q_t}{s^{1/2}\, p_s} \sin \theta_t = \frac{m_{A,B}\, q_s}{t^{1/2}\, p_t} \sin \theta_s \tag{4.23}$$

$$\sin \chi_t^{C,D} = \frac{m_{C,D}\, q_t}{s^{1/2}\, q_s} \sin \theta_t = \frac{m_{C,D}\, p_s}{t^{1/2}\, p_t} \sin \theta_s \tag{4.24}$$

The labels "1" and "2" enter into the definition of the scattering angles θ_s, θ_u. Thus to obtain the crossing angles for the u-channel one has to replace the variable and index t in (4.14) - (4.24) by u. One also has to interchange the masses m_C and m_D and to replace θ_t in (4.23) and (4.24) by $\theta_u + \pi$ or θ by $\theta + \pi$. One should note that the crossing angle for a photon is equal to zero.

Some comments to explain the differences in the crossing formulas (4.11) and (4.12) may be useful: To derive Eq. (4.12) for crossing the u-channel amplitudes from the original result (4.11) of T.W. one first introduces in the s- and u-channel helicity amplitudes with the labels 1, 2 of the particles in the final state interchanged with respect to the definition (4.1) and (4.3).

$$f^s_{\lambda_C\lambda_D,\lambda_A\lambda_B}(W^s, \theta^s, \phi^s = 0) = (-1)^{S_D - S_C + \lambda_C - \lambda_D} f^{s'}_{\lambda_D\lambda_C,\lambda_A\lambda_B}(W^s, \theta'^s = \theta^s + \pi, \phi^{s'} = 0) \tag{4.25}$$

$$f^{u'}_{\lambda_D\lambda_A\lambda_C\lambda_B}(W^u, \theta^{u'} = \theta^u + \pi, \phi^{u'} = 0) = (-1)^{S_A - S_D + \lambda_D - \lambda_A} f^u_{\lambda_A\lambda_D,\lambda_C\lambda_B}(W^u, \theta^u, \phi^u = 0) \tag{4.26}$$

The formula (4.11) are directly applicable to cross the $f^{u'}$-amplitudes to the $f^{s'}$-amplitudes, since now the particle with label 1 in the initial state is interchanged with the particle with label 2 in the final state as in crossing from the s-channel (4.1) to the t-channel (4.2). One introduces then the rotation matrices r^K (3.40) to take care of the helicity dependent phase factors in (4.25) and (4.26). Finally one has to realize that the relations (4.23) and (4.24) apply for crossing the $f^{s'}$ and $f^{u'}$ amplitudes with the polar angles $\theta'^s = \theta^s + \pi$ and $\theta^{u'} = \theta^u + \pi$.

We now apply the relations (4.11) and (4.12) to (4.9) and (4.10), respectively, to obtain the s-channel tensor polarization matrix in terms of the t- or u-channel amplitudes: Due to this step the observables O_κ^K in (4.9) are substituted by

$$O_\kappa^{K'} = D^K\left(\chi_t^K\right) O_\kappa^K \; D^{K\dagger}\left(\chi_t^K\right) \tag{4.27}$$

and $F^s_{(AD)}$ by F^t; analogous substitutions appear in (4.10). As has been noted already by Gottfried and Jackson[8] this result can be interpreted easily if one makes use of the well known transformation properties under three-dimensional rotations of the matrices O_κ^K, which represent components of tensors made up by the spin matrices in the rest frame of each particle. The components O_κ^K of these tensors referred to the axes of the helicity frame in the rest system as explained in Section 3A. The quantities $O_\kappa^{K'}$ (4.27) are then the components of the same tensor with respect to new axes, which are obtained from the old ones by a rotation around the 2-axis of the rest frame through the crossing angle χ_t^K. Thus we have for the elements of the vector polarization O_κ, $\kappa = 1,2,3$

$$O_\kappa^{K'} = R_2^{-1}\left(\chi_t^K\right)_{\kappa,\kappa'} O_{\kappa'}^K, \qquad O_\kappa^{K'} = R_2\left(-\chi_t^K\right)_{\kappa,\kappa'} O_{\kappa'}^K \tag{4.28a}$$

$R_2(\chi)$ is the transformation matrix of cartesian coordinates for a rotation of the coordinate system around the 2-axis by the angle χ. For the u-channel due to the additional rotation around the 3-axis through the angle π mentioned earlier , Eq. (4.28a) has to be replaced by

$$O_\alpha^{A''} = D^A\left(-\chi_u^A\right) r^A(-\pi)\, O_\alpha^A\, r^A(\pi)\, D^A\left(-\chi_u^A\right)^\dagger$$

$$= \sum_{\alpha'=1}^{3} R_2^{-1}\left(\chi_u^A\right)_{\alpha\alpha'} \omega_{\alpha'}^A O_{\alpha'}^A \tag{4.28b}$$

$$O_\beta^{B''} = D^B\left(\chi_u^B\right) O_\beta^B\, D^B\left(\chi_u^B\right)^\dagger = \sum_{\beta'=1}^{3} R_2^{-1}\left(\chi_u^B\right)_{\beta\beta'} O_{\beta'}^B \tag{4.28c}$$

$$O_\gamma^{C''} = D^C\left(\chi_u^C\right) r^C(-\pi)\, O_\gamma^C\, r^C(\pi)\, D^C\left(\chi_u^C\right)^\dagger$$

$$= \omega_\gamma^C \sum_{\gamma'=1}^{3} R_2^{-1}\left(\chi_u^C\right)_{\gamma\gamma'} O_{\gamma'}^C \tag{4.28d}$$

$$O_\delta^{D''} = D^D\left(-\chi_u^D\right) O_\delta^D\, D^D\left(-\chi_u^D\right)^\dagger = \omega_\delta^D \sum_{\delta=1}^{3} R_2^{-1}\left(\chi_u^D\right)_{\delta\delta'} \omega_{\delta'}^D O_{\delta'}^D \tag{4.28e}$$

The rotation symmetry factor ω_κ^K was introduced in (3.44).

Due to the rotation (4.28) of the observables it is convenient for crossing from the t- or u-channel to introduce new sets of observables $\tilde{O}_\kappa^K$, $\approx{O}_\kappa^K$, respectively, in the crossing or Gottfried-Jackson frame as defined in the Introduction. The rotations (4.28) carry then $\tilde{O}_\kappa^K$, $\approx{O}_\kappa^K$ over into the old O_κ^K defined with respect to the helicity frame. Thus a measurement of the polarization in the respective crossing frame is expressed by the simple relations

$$\tilde{M}_{\gamma\delta,\alpha\beta}^{CD,AB}(F^s) = \tau_\alpha^A\, \tau_\delta^D\, N_{\gamma\alpha,\delta\beta}^{CA,DB}(F^t) \tag{4.29}$$

for the crossing from the t-channel and

$$\tilde{\tilde{M}}^{CD,AB}_{\gamma\delta,\alpha\beta}(F^s) = \tau^A_\alpha \tau^C_\gamma L^{AD,CB}_{\alpha\delta,\gamma\beta}(F^u) \tag{4.30}$$

for the crossing from the u-channel.

The amplitudes F^t and F^u in (4.29) and (4.30) are, of course, the analytically continued quantities. The relations (4.29) and (4.30) will be the basis of all further discussion of practical results.

4B. Consequences of the Decomposition (2.8)

In Eq. (2.8) or (2.8') we defined asymptotic parity conserving amplitudes for an arbitrary channel (1.1). One can as well write this definition in matrix form for the reaction matrix F (3.2).

$$\begin{aligned} F^\sigma &= F + \sigma(-1)^{-v} FY^A \otimes Y^B \\ &= F + \sigma(-1)^{-v} Y^C \otimes Y^D F \end{aligned} \tag{4.31}$$

by means of the reflection operator (5.11). In (4.31) $v = 1/2$, if $(S_A + S_B)$ is an half integer; otherwise it is zero. Since $(Y^K)^2 = (-1)^{2S_K}$ and $Y^{K\dagger} = (-1)^{2S_K} Y^K$, the matrices F^σ have the simple property that

$$F^\sigma Y^{A\dagger} \otimes Y^{B\dagger} = \sigma(-1)^{-v} F^\sigma, \quad \text{or} \quad Y^C \otimes Y^D F^\sigma = \sigma(-1)^{+v} F^\sigma \tag{4.32}$$

These relationships turn out to be extremely useful for the following discussion. The matrices F^σ suggest a natural decomposition of the elements of the tensor polarization matrix into four parts

$$\begin{aligned} M^{CD,AB}_{\gamma\delta,\alpha\beta}(F) &= \frac{1}{4} \sum_{\sigma,\sigma'=\pm} \mathrm{Tr}\left[\left(O^C_\gamma \otimes O^D_\delta\right) F^\sigma \left(O^A_\alpha \otimes O^B_\beta\right) F^{\sigma'\dagger}\right] \\ &= \frac{1}{4} \sum_{\sigma,\sigma'} M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^\sigma, F^{\sigma'}) \end{aligned} \tag{4.33}$$

Because of the symmetry relation (4.32) the elements $M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma}, F^{\sigma'})$ are non-vanishing only if:

$$\sigma\sigma' = \pi^{A}_{\alpha}\, \pi^{B}_{\beta} = \left(\pi^{C}_{\gamma} \cdot \pi^{D}_{\delta}, \quad \text{see (3.30)}\right), \tag{4.34}$$

where π^{K}_{κ} are the parities of the observables introduced in (3.24). Furthermore, the hermiticity of the observables O^{K}_{κ} insures the symmetry relation

$$M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma}, F^{\sigma'}) = M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma'}, F^{\sigma})^{*} \quad . \tag{4.35}$$

Thus as a consequence of (4.34) and (4.35) the sum in (4.33) reduces to one or two terms depending on the value of $\pi^{A}_{\alpha} \cdot \pi^{B}_{\beta}$

$$M^{CD,AB}_{\gamma\ \ ,\alpha\beta}(F) = \begin{cases} \frac{1}{2}\,\mathrm{Re}\, M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma=+1},\ F^{\sigma=-1}) & \text{if } \pi^{A}_{\alpha} \cdot \pi^{B}_{\beta} = -1 \\ \\ \frac{1}{4}\, M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma=+1},\ F^{\sigma'=+1}) + \frac{1}{4}\, M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma=-1},\ F^{\sigma'=-1}) & \\ & \text{if } \pi^{A}_{\alpha} \cdot \pi^{B}_{\beta} = +1 \end{cases} \tag{4.36}$$

Analogous to the result in (3.32) one can also show that certain of the non-vanishing elements of $M(F^{\sigma},\ F^{\sigma'})$ are linearly dependent as a result of the mapping relations (3.26)...(3.28) or in general (3.25) applied to the observables of particles A and B

$$M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^{\sigma},\ F^{\sigma'}) = (-1)^{-v_{\sigma}} \sum_{\gamma'\delta'} \bar{\rho}^{\,\gamma\delta}_{\gamma'\delta'}\, M^{CD,AB}_{\gamma'\delta',\alpha\beta}(F^{\sigma},\ F^{\sigma'}) \tag{4.37}$$

The $\bar{\rho}$'s are coefficients, which are analogous to the ρ's in (3.32) and are determined by the relations (3.25). In many of the interesting cases the sum in (4.37) extends only over one term, so that (4.37) becomes a simple relationship. Note that in (4.37) the right hand side flips sign under the substitution $\sigma \leftrightarrow -\sigma$. If the product of the parities $\pi_\alpha^A \cdot \pi_\beta^B = +1$, then one can exploit this fact to separate in (4.36) the $\sigma = \sigma' = \pm 1$ terms by taking the linear combination

$$M^{CD,AB}_{\gamma\delta,\alpha\beta} \pm (-1)^{-v} \sum_{\gamma'\delta'} \bar{\rho}^{\gamma\delta}_{\gamma'\delta'} M^{CD,AB}_{\gamma'\delta',\alpha\beta}$$

$$= \sum_{\sigma=\pm 1} \frac{1 \pm \sigma}{4} M^{CD,AB}_{\gamma\delta,\alpha\beta}(F^\sigma, F^\sigma) = \frac{1}{2} M^{CD,AB}_{\gamma\delta,\alpha\beta} (F^{\pm 1}, F^{\pm 1}) \tag{4.38}$$

$$\text{if } \pi_\alpha^A \cdot \pi_\beta^B = \pi_\gamma^C \cdot \pi_\delta^D = +1$$

The results (4.36) or (4.38) if $\pi_\alpha^A \cdot \pi_\beta^B = -1$ or $+1$, respectively, are the basis of very simple experimental predictions at high energies, if the amplitudes in the direct channel are well represented by the exchange of particles with definite parity type (2.5) in the crossed channels. This situation occurs in the peripheral or Regge-pole model. Thus, assume we apply the results (4.36) and (4.38) to the t- or u-channel expressions (4.7) or (4.8). They are related to the observables in the s-channel by the crossing relations (4.29) and (4.30). Hence one obtains from (4.29), (4.30) and (4.36), if the parities of the observables in the initial state of the t- or u-channel are negative,

$$\tau_\alpha^A \tau^D \tilde{M}^{CD,AB}_{\gamma\delta,\alpha\beta}(F^s) = \frac{1}{2} \text{Re } N^{CA,DB}_{\gamma\alpha,\delta\beta} (F^{t\sigma=+1}, F^{t\sigma=-1}) , \tag{4.39a}$$

if $\pi^D_\delta \cdot \pi^B_\beta = -1$ or

$$\tau^A_\alpha \tau^C_\gamma \tilde{\tilde{M}}^{CD,AB}_{\gamma\delta,\alpha\beta}(F^s) = \frac{1}{2} \text{Re } L^{AD,CB}_{\alpha\delta,\gamma\beta}(F^{u\sigma=+1}, F^{u\sigma=-1}) \tag{4.39b}$$

if $\pi^C_\gamma \cdot \pi^B_\beta = -1$. The s-channel observables refer to the Gottfried-Jackson frame as indicated by the tilde on the M's. (See the discussion in Section 4A.) Also if we try to apply the mapping relation (4.37) in the t-channel for $\pi^D_\delta \cdot \pi^B_\beta = -1$, we find $(-1)^{-v_t} \bar{\rho}^{\gamma\alpha}_{\gamma'\alpha'}$ is necessarily imaginary so that relations result between real and imaginary parts of $N(F^{t+}, F^{t-})$. Specifically, one finds

$$-i(-1)^{-v_t} \bar{\rho}^{\gamma\alpha}_{\gamma'\alpha'} \text{Re } N_{\gamma'\alpha'\delta\beta}(F^{t+}, F^{t-}) = \text{Im } N_{\gamma\alpha,\delta\beta}(F^{t+}, F^{t-}) \tag{4.39c}$$

and correspondingly for the u-channel

$$-i(-1)^{-v_t} \bar{\rho}^{\alpha\delta}_{\alpha'\delta'} \text{Re } L_{\alpha'\delta',\gamma\beta}(F^{u+}, F^{u-}) = \text{Im } L_{\alpha\delta,\gamma\beta}(F^{u+}, F^{u-}) \tag{4.39d}$$

Thus one can use Eqs. (4.39a, b, c, d) to express the imaginary parts of $N(F^{t+}, F^{t-})$ and $L(F^{u+}, F^{u-})$ (as well as their real parts) in terms of the s-channel observables $M(F^s)$.

Similarly one obtains from (4.29) and (4.38)

$$\tau^A_\alpha \tilde{M}^{CD,AB}_{\gamma\delta,\alpha\beta}(F^s) \pm (-1)^{-v_t} \sum_{\alpha',\gamma'} \left(\bar{\rho}^{\gamma\alpha}_{\gamma'\alpha'} \tau^A_{\alpha'} \tilde{M}^{CD,AB}_{\gamma'\delta,\alpha'\beta}\right)(F^s)$$

$$= \frac{1}{2} \tau^D N^{CA,DB}_{\gamma\alpha,\ \beta\delta}(F^{t,\sigma=\pm1}, F^{t,\sigma=\pm1}) \tag{4.40a}$$

if $\pi^D_\delta \cdot \pi^B_\beta = +1$. The $\bar{\rho}$'s are defined by $\sum_{\gamma'} \bar{\rho}^{\gamma}_{\gamma'} O^C_{\gamma'} = O^C_\gamma Y^C$ and $\sum_{\alpha'} \bar{\rho}^{\alpha}_{\alpha'} O^A_{\alpha'} = O^A_\alpha Y^A$, where Y^C and Y^A are now the reflection operators of the s-channel. Note that according to (5.11) $Y^{A("1")} = Y^{\bar{A}("2")}$, where A("1") is the particle "1" in the s-channel and $\bar{A}$("2") the antiparticle "2" in the t-channel. From (4.30) and (4.38) follows

$$\tau^A_\alpha \,\tilde{\tilde{M}}^{CD,AB}_{\gamma\delta,\alpha\beta}(F^s) \pm (-1)^{-v_u} \sum_{\alpha',\gamma'} \left(\bar{\rho}^{\gamma\alpha}_{\gamma'\alpha'}\, \tau^A_{\alpha'}\, \tilde{\tilde{M}}^{CD,AB}_{\gamma\delta',\alpha'\beta}\right)(F^s)$$

$$= \frac{1}{2}\, \tau^C_\gamma\, L^{AD,CB}_{\alpha\delta,\gamma\beta}(F^{u,\sigma=\pm 1},\; F^{u,\sigma=\pm 1}) \qquad (4.40b)$$

if $\pi^C_\gamma \cdot \pi^B_\beta = +1$. Here $(-1)^{2S_A} \sum_{\alpha'} \bar{\rho}^{\alpha}_{\alpha'} O^A_{\alpha'} = O^A_\alpha Y^A$ and $\sum_{\delta'} \bar{\rho}^{\delta}_{\delta'} O^D_{\delta'} = O^D_\delta Y^D$. The factor $(-1)^{2S_A}$ arises this time from the relation $Y^{A("1")} = Y^{\bar{A}("1")}(-1)^{2S_A}$.

Now for $|z_t|$ or $|z_u| \to \infty$ in the s-channel, the exchange of particles with any parity type is kinematically enhanced in $F^{t,\sigma}$ or $F^{u,\sigma}$ as discussed in Section 2. If the leading natural and unnatural trajectories α^+, α^- are not separated by more than one unit

$$|\mathrm{Re}\,\alpha^+ - \mathrm{Re}\,\alpha^-| < 1 \qquad (4.41)$$

then each $F^{t,\sigma}$ or $F^{u,\sigma}$ is in general dominated by one parity type for $|z_t|$ or $|z_u| \to \infty$. Thus one can summarize in the following theorem concerning the separation of natural and unnatural parity contributions the main result so far obtained in this subsection.

Theorem 1: Assume that the analytically continued t- or u-channel amplitudes are dominated in the s-channel by a leading natural and unnatural parity trajectory obeying the restriction (4.41). Then the observables $\tilde{M}$, $\tilde{\tilde{M}}$ in the Gottfried-Jackson frame measure, according to (4.39), the interference term between the leading t- or u-channel trajectories if $\pi^D_\delta \cdot \pi^B_\beta = -1 = \pi^A_\alpha \cdot \pi^C_\gamma$ (see (3.30)) or $\pi^C_\gamma \cdot \pi^B_\beta = -1 = \pi^A_\alpha \cdot \pi^D_\delta$, respectively. On the other hand, the linear combinations (4.40) of observables separate the leading natural and unnatural trajectories of the t- or u-channel if $\pi^D_\delta \cdot \pi^B_\beta = +1$ or $\pi^C_\gamma \cdot \pi^B_\beta = +1$, respectively.

However, one has always to keep in mind that all these order of magnitude estimates may fail if the trajectory couplings to the various parity conserving amplitudes are such that some coefficients of the leading power of s vanish as the consequence of a selection rule. For the baryon number N = 0 channel such rules follow from G parity conservation as discussed at the end of Section 2. Furthermore one has to be cautious in applying Theorem 1 to measurements, which are represented only by terms of the form $\mathrm{Im}\ f_{\Lambda_f\Lambda_i} f^*_{\Lambda_f'\Lambda_i'}$. Let us assume that only one Regge pole contributes to $f_{\Lambda_f\Lambda_i}$ as for example the ρ-Regge pole in pion nucleon charge exchange scattering. Since the phase of a single Regge-pole term is independent of the helicities any bilinear form $f_{\Lambda_f\Lambda_i} f^*_{\Lambda_f'\Lambda_i'}$ becomes real in leading order. Thus, in general, interference terms between the same Regge poles do not contribute to $\mathrm{Im}\ f_{\Lambda_f\Lambda_i} f^*_{\Lambda_f'\Lambda_i'}$. The vanishing of the leading order term in $\mathrm{Im}\ f_{\Lambda_f\Lambda_i} f^*_{\Lambda_f'\Lambda_i'}$ is even true in more general models. It is a consequence of analyticity as has been discussed for example by van Hove.[21]

4C. Constraints Due to G-Parity Conservation for the $N\bar{N}$ Systems

In the following we assume that in a t-channel with baryon number zero the final state is made up by a nucleon-antinucleon pair $N\bar{N}$ like in the photoproduction channel (2.2). Then, as discussed at the end of Section 2, G-parity conservation introduces a selection rule, if the reaction proceeds via the exchange of particles with natural or unnatural parity $P = \pm (-1)^J$. For unnatural parity exchange it was shown that the helicities λ_C and λ_A of the nucleon-antinucleon pair N_C, $\bar{N}_A$ must fulfill the restriction $\lambda_C = \pm \lambda_A$ if the quantum numbers P, G and I are related by $PG(-1)^I = \mp 1$. For natural parity exchange only particles with $PG(-1)^I = +1$ are possible with no restriction on the helicities λ_C, λ_A. This rule suggests therefore a further decomposition of $F^{t,\sigma}$ for $\sigma = -1$ in order to distinguish in the limit $|z_t| \to \infty$ the parts with $w = PG(-1)^I = \pm 1$

$$F^{t-} = {}^{+}F^{t-} + {}^{-}F^{t-} \tag{4.42}$$

or explicitly

$${}^{\pm}F^{t-}_{\lambda_N \lambda_{\overline{N}}, \Lambda_i} = F^{t-}_{\lambda_N \lambda_{\overline{N}}, \Lambda_i} \; \delta_{\lambda_N, \mp \lambda_{\overline{N}}} \tag{4.43}$$

In the limit $|z_t| \to \infty$ the parts ${}^{\pm}F^{t-}$ are dominated by the contributions with $w = \pm 1$. Now there is a matrix W such that

$$W^{\pm}F^{t-} = \pm \, {}^{\pm}F^{t-} \tag{4.44}$$

to distinguish the $w = \pm 1$ part of F^{t-} by an algebraic equation. The specific form of $W = W^N \otimes W^{\overline{N}}$ is not unique, but expressed in Pauli matrices we may take $W^K = i\sigma_3$ so that

$$W = -\sigma_3^N \otimes \sigma_3^{\overline{N}} \tag{4.45}$$

or $W'^K = i\sigma_1$ so that

$$W' = -\sigma_1^N \otimes \sigma_1^{\overline{N}} \tag{4.46}$$

This second choice with $W' = -WY$ follows from (4.32).

Analogous to the parity discussion in Section 3B we introduce a W-parity $\tilde{\pi}_\nu^N$ of the nucleon observables

$$WO_\nu W^\dagger = \tilde{\pi}_\nu O_\nu \tag{4.47a}$$

or explicitly

$$W(O_0, \vec{O}) W^\dagger = (O_0, -O_1, -O_2, +O_3) \; . \tag{4.47b}$$

Again we have also mapping relations

$$O_\nu W = \tilde{\rho}^{\nu}_{\nu'} O_{\nu'} \tag{4.48a}$$

or explicitly

$$(O_0,\vec{O})W = (-iO_3, O_2, -O_1, -iO_0) \tag{4.48b}$$

As in (4.33) we may now decompose the elements $N(F^{t-}, F^{t-})$ in terms of $N({}^{w}F^{t-}, {}^{w'}F^{t-})$ and also $N(F^{t-}, F^{t+}) = N(F^{t+}, F^{t-})^*$ in terms of $N({}^{w}F^{t-}, F^{t+})$. Using the analogous steps as in Section 4B one derives the following results, which we briefly state below

$$N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}(F^{t-}, F^{t-}) = \sum_{w,w'} N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{w}F^{t-}, {}^{w'}F^{t-}) \tag{4.49}$$

From (4.44) and (4.47) follows a restriction for the nonvanishing elements in the sum (4.49)

$$\tilde{\pi}^{N}_{\nu} \cdot \tilde{\pi}^{\bar{N}}_{\bar{\nu}} = w \cdot w' \tag{4.50}$$

The hermiticity of the observables guarantees the symmetry

$$N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{w}F^{t-}, {}^{w'}F^{t-}) = N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{w'}F^{t-}, {}^{w}F^{t-})^* \tag{4.51}$$

Thus

$$N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}(F^{t-}, F^{t-}) = \begin{cases} \sum_{w=\pm1} N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{w}F^{t-}, {}^{w}F^{t-}) & \text{for } \tilde{\pi}_{\nu} \cdot \tilde{\pi}_{\bar{\nu}} = +1 \\ \\ 2\,\mathrm{Re}\, N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{+1}F^{t-}, {}^{-1}F^{t-}) & \text{for } \tilde{\pi}_{\nu} \cdot \tilde{\pi}_{\bar{\nu}} = -1 \end{cases} \tag{4.52}$$

The mapping relations (4.48) yield again

$$N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{w}F^{t-}, {}^{w'}F^{t-}) = w\, \tilde{\rho}^{\nu}_{\nu'}\, \tilde{\rho}^{\bar{\nu}}_{\bar{\nu}'}\, N^{N\bar{N},DB}_{\nu'\bar{\nu}',\delta\beta}({}^{w}F^{t-}, {}^{w'}F^{t-}) \tag{4.53}$$

where this time we have only one term on the right-hand side. From (4.53) it follows again that the linear combinations

$$N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}(F^{t-}, F^{t-}) + w\, \tilde{\rho}^{\nu}_{\nu'}\, \tilde{\rho}^{\bar{\nu}}_{\bar{\nu}'}\, N^{N\bar{N},DB}_{\nu'\bar{\nu}',\delta\beta}(F^{t-}, F^{t-})$$

$$= 2\, N({}^{w}F^{t-}, {}^{w}F^{t-}) \qquad \text{for } \tilde{\pi}_{\nu} \cdot \tilde{\pi}_{\bar{\nu}} = +1 \qquad (4.54)$$

separate the $w = \pm 1$ contributions. (Note that the product $\tilde{\rho}^{\nu}_{\nu'} \cdot \tilde{\rho}^{\bar{\nu}}_{\bar{\nu}'}$ is real for $\tilde{\pi}_{\nu} \cdot \tilde{\pi}_{\bar{\nu}} = +1$.)

Finally for the interference terms of natural and unnatural parity terms $\left(\pi^{N}_{\nu} \cdot \pi^{\bar{N}}_{\bar{\nu}} = \pi^{D}_{\delta} \cdot \pi^{B}_{\beta} = -1\right)$ one can derive the following identities using (4.36), (4.44) and (4.32)

$$2N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}(F^{t}) = \text{Re Tr}\left(O^{N}_{\nu} \otimes O^{\bar{N}}_{\bar{\nu}}\, F^{t-}\, O^{D}_{\delta} \otimes O^{B}_{\beta}\, F^{t+\dagger}\right)$$

$$= \begin{cases} \text{Re Tr} \displaystyle\sum_{w=\pm 1} w\left(O^{N}_{\nu} \otimes O^{\bar{N}}_{\bar{\nu}}\, W^{N\bar{N}}\ {}^{w}F^{t-}\, O^{D}_{\delta} \otimes O^{B}_{\beta}\, F^{t+\dagger}\right) & \text{if } \tilde{\pi}^{N}_{\nu} \cdot \tilde{\pi}^{\bar{N}}_{\bar{\nu}} = +1 \\ \\ \text{Re Tr} \displaystyle\sum_{w=\pm 1} w\left(O^{N}_{\nu} \otimes O^{\bar{N}}_{\bar{\nu}}\, W'^{N\bar{N}}\ {}^{w}F^{t-}\, O^{D}_{\delta} \otimes O^{B}_{\beta}\, F^{t+\dagger}\right) & \text{if } \tilde{\pi}^{N}_{\nu} \cdot \tilde{\pi}^{\bar{N}}_{\bar{\nu}} = -1 \end{cases}$$

(4.55)

where $W' = -WY$. Thus in the linear combinations

$$N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}(F^{t}) + w\, \tilde{\tilde{\rho}}^{\nu}_{\nu'}\, \tilde{\tilde{\rho}}^{\bar{\nu}}_{\bar{\nu}'}\, N^{N\bar{N},DB}_{\nu'\bar{\nu}',\delta\beta}(F^{t}) = \text{Re}\, N^{N\bar{N},DB}_{\nu\bar{\nu},\delta\beta}({}^{w}F^{t-}, F^{t+}) \qquad (4.56)$$

the $w = \pm 1$ contributions interfering with the $\sigma = +1$ amplitude are separated. According to (4.55) the factors $\tilde{\tilde{\rho}}$ are just the factors $\tilde{\rho}$ of Eq. (4.48) if $\tilde{\pi}^{N}_{\nu} \cdot \tilde{\pi}^{\bar{N}}_{\bar{\nu}} = +1$, for $\tilde{\pi}^{N}_{\nu} \cdot \tilde{\pi}^{\bar{N}}_{\bar{\nu}} = -1$ one has to use the analogous relations (4.48) with W replaced by

$W' = WY$. One can show analogously to (4.39c, d) that for any value of $\tilde{\pi}^{N}_{\nu} \cdot \tilde{\pi}^{\bar{N}}_{\bar{\nu}}$

$$\mathrm{Im}\, N^{N\bar{N}, DB}_{\nu\bar{\nu}, \delta\beta}({}^{W}F^{t-},\ F^{t+}) = i\ \rho^{\nu}_{\nu'}\ \rho^{\bar{\nu}}_{\bar{\nu}'}\ N^{NN, DB}_{\nu'\bar{\nu}', \delta\beta}({}^{W}F^{t-},\ F^{t+}) \qquad (4.57)$$

where the factor $i\ \rho^{\nu}_{\nu'}\ \rho^{\bar{\nu}}_{\bar{\nu}'}$ is necessarily real.

The quantities $N^{N_1\bar{N}_2, DB}_{\nu_1\nu_2, \delta\beta}(F^{t-},\ F^{t+})$ are related to the s-channel observables $\tilde{M}^{N_2D, N_1B}_{\nu_2\delta, \nu_1\beta}(F^{s})$ in the Gottfried-Jackson frame by means of the relations (4.39a) and (4.40a). The main results of this subsection may again be summarized in a theorem in the following way:

<u>Theorem 2:</u> Assume that the final state of the t-channel (4.2) is made up by a nucleon antinucleon pair. (Analogous results apply for the $N\bar{N}$ pair in the initial state.) Assume further that the analytically continued t-channel amplitudes are dominated in the s-channel by a leading natural α^{+} (with $w = +1$) and two leading unnatural parity trajectories α^{-} (with $w = \pm 1$). The trajectories α^{+}, ${}^{w}\alpha^{-}$ shall fulfill again the restriction (4.41).

1. Case: $\pi^{D}_{\delta}\pi^{B}_{\beta} = \pi^{N_1}_{\nu_1}\pi^{N_2}_{\nu_2} = +1$. Then the linear combinations (4.40a) measure the leading $\sigma = -1$ interference terms according to Theorem 1. If furthermore the product of the W-parity of the observables $\tilde{\pi}^{N_1}_{\nu_1}\tilde{\pi}^{N_2}_{\nu_2} = -1$ then (4.40a) measures only the interference between the two leading unnatural parity trajectories ${}^{w}\alpha^{-}$ with $w = \pm 1$ (according to Eq. (4.52)). If $\pi^{N_1}_{\nu_1}\pi^{N_2}_{\nu_2} = +1$ then one has to form out of the expressions (4.40a) linear combinations according to (4.54). These then separate the two $w = \pm 1$ contributions.

2. Case: $\pi^{D}_{\delta}\pi^{B}_{\beta} = -1$. According to (4.39a) one measures the interference between $\sigma = \pm 1$ contributions. The linear combinations

$$\tau^{N_1}_{\nu_1}\tau^{D}_{\delta}\,\tilde{M}^{N_2D, N_1B}_{\nu_2\delta, \nu_1\beta}(F^{S}) + w\tau^{N_1}_{\nu_1'}\tau^{D}_{\delta}\,\tilde{\tilde{\rho}}^{\nu_1}_{\nu_1'}\,\tilde{\tilde{\rho}}^{\nu_2}_{\nu_2'}\,\tilde{M}^{N_2D, N_1B}_{\nu_2'\delta, \nu_1'\beta}(F^{S}) = \mathrm{Re}\, N^{N_2\bar{N}_1, DB}_{\nu_2\nu_1, \delta\beta}({}^{W}F^{t-}, F^{t+}) \qquad (4.58)$$

separate the $w = \pm 1$ contributions. For the factors $\tilde{\tilde{\rho}}^{\nu}_{\nu'}$, see the discussion after Eq. (4.56).

5. APPENDIX: DEFINITION AND DECOMPOSITION OF DENSITY MATRICES IN THE HELICITY BASIS

5A. Helicity States

We present here a few steps in the construction of helicity states according to J. W. to obtain some results to which we refer in the following discussion on the density matrices. Let $\vec{n}_i$ be three unit vectors in the direction of the axes of some frame, which is not necessarily the c. m. system

$$\vec{n}_i \cdot \vec{n}_j = \delta_{ij} \tag{5.1}$$

Consider then one-particle states $|q\lambda\rangle$ in the Heisenberg picture defined by

$$|q\lambda\rangle = R(\phi, \Theta, -\phi)\, U\big(L(\vec{n}_3 q)\big)\, |0\lambda\rangle \tag{5.2}$$

These states are produced from a particle in its rest frame $|0\lambda\rangle$ first by an acceleration in the direction $\vec{n}_3$ to momentum q. This Lorentz transformation is represented by the boost $L(\vec{n}_3 q)$ and

$$U\big(L(\vec{n}_3 q)\big) = e^{-i\zeta \vec{n}_3 \cdot \vec{K}}, \quad \zeta = \sinh^{-1}\frac{q}{m} \tag{5.3}$$

where $\vec{K} = (K_1, K_2, K_3)$ are the usual three generators of a pure Lorentz transformation. The subsequent rotation $R(\phi, \Theta, -\phi)$ in (5.2) brings the momentum $\vec{n}_3 q$ into the final direction $\vec{q}$ by a rotation through the angle Θ

$$R(\phi, \Theta, -\phi) = e^{-i\Theta \vec{m} \cdot \vec{J}} \tag{5.4}$$

about the axis

$$\vec{m} = -\sin\phi\, \vec{n}_1 + \cos\phi\, \vec{n}_2 \tag{5.5}$$

with $\vec{m} \cdot \vec{n}_3 = 0$. If λ, the spin component in the rest frame, is quantized in the $\vec{n}_3$-direction, into which the particle is boosted, then (5.2) is a state in the "helicity basis" chosen in accordance with the conventions of J. W. particularly with respect to the rotation $R(\phi, \Theta, -\phi)$. We remember that the relative phases of the states $|0\lambda\rangle$ are fixed by the requirement

$$(\vec{n}_1 \cdot \vec{S} \pm i\,\vec{n}_2 \cdot \vec{S})\,|0\lambda\rangle = [(s \mp \lambda)\,(s \pm \lambda + 1)]^{1/2}\,|0\lambda \pm 1\rangle. \tag{5.6}$$

where S_i are the usual spin matrices in the rest frame and s is the spin of the particle.

For a massless boson like the photon the construction prescription (5.2) for the states has to be replaced, since there is no rest frame available. In this case one starts with particles moving in the $\vec{n}_3$-direction $q_\mu = (q, \vec{n}_3 q)$

$$|\vec{q}, \lambda\rangle = R(\phi, \Theta, -\phi)\,|\vec{q}\,' = \vec{n}_3 q, \lambda\rangle \tag{5.7}$$

Under reflections in the (n_1, n_3)-plane

$$Y = e^{-i\pi\vec{m} \cdot \vec{J}}\,P, \tag{5.8}$$

where P is the parity operator, one requires (see J. W. Eq.(9))

$$Y\,|q\vec{n}_3, \lambda\rangle = \eta\,|q\vec{n}_3, -\lambda\rangle\ . \tag{5.9}$$

So, since there exists only two states $\lambda = \pm S$, the relative phases of (5.7) are fixed, once the parity η is determined.

The reflections Y turn out to be particularly useful in our work. If applied to states with momenta in the (n_1, n_3)-plane the reflections Y change only the helicity λ of the states (5.2) or (5.9) apart from a phase factor.

$$
\begin{aligned}
Y|\vec{q}(\phi=0),\lambda\rangle &= e^{-i\pi\vec{n}_2\cdot\vec{J}}\, P\, e^{-i\Theta\vec{n}_2\cdot\vec{J}}\, e^{-i\zeta\vec{n}_3\cdot\vec{K}}|0\lambda\rangle \\
&= e^{-i\Theta\vec{n}_2\cdot\vec{J}}\, e^{-i\zeta\vec{n}_3\cdot\vec{K}}\, e^{-i\pi\vec{n}_2\cdot\vec{J}}\, P|0\lambda\rangle \\
&= e^{-i\Theta\vec{n}_2\cdot\vec{J}}\, e^{-i\zeta\vec{n}_3\cdot\vec{K}}\, (-1)^{S-\lambda}\,\eta|0-\lambda\rangle \\
&= (-1)^{S-\lambda}\,\eta|q(\phi=0),-\lambda\rangle;
\end{aligned}
\tag{5.10}
$$

(for line 3 see Eq. (9') in J. W.). Therefore

$$
\langle\vec{q}(\phi=0)\lambda|Y|\vec{q}(\phi=0),\lambda'\rangle = (-1)^{S-\lambda'}\delta_{\lambda,-\lambda'} \tag{5.11a}
$$

which matrix is $(-\eta i\sigma_2)$ for a spin 1/2 particle. For a massless boson the factors $(-1)^{S-\lambda}$ in (5.10) and (5.11a) have to be omitted according to (5.9). Relation (5.11a) applies only to a particle "1"; for a particle "2" see the following Eq. (5.11b). Therefore Y corresponds always to the matrix $\eta\sigma_1$ in the helicity basis.

For a two particle state

$$
|q_1\lambda_1, q_2\lambda_2; {}^{\text{in}}_{\text{out}}\rangle = |q_1\lambda_1; {}^{\text{in}}_{\text{out}}\rangle \otimes |q_2\lambda_2; {}^{\text{in}}_{\text{out}}\rangle \tag{5.12}
$$

the construction prescription (5.2) is applied only for the particle "1", whereas for the particle "2" the rule (5.2) is modified by an extra phase factor $(-1)^{S-\lambda}$

$$
|q\lambda\rangle = (-1)^{S-\lambda} R(\phi,\Theta,-\phi)\, U\left(L(\vec{n}_3 q)\right)|0\lambda\rangle . \tag{5.13}
$$

This is done in order to conform with the relative phase convention of J.W. for the two particle states and is motivated in their paper. Consequently the reflection matrix $Y^{''2''}$ for a particle "2" differs from that of a particle "1" by the factors $(-1)^{2S}$

$$Y^{''1''} = (-1)^{2S}\, Y^{''2''} \tag{5.11b}$$

The helicity states (5.2) fulfill in the spin space of a given particle orthogonality relations which are independent of the frame

$$\langle q\lambda' | q\lambda \rangle_s = \langle 0\lambda' | U^+\big(L(q\vec{n}_3)\big)\, R^+(\phi, \Theta, -\phi) R(\phi, \Theta, -\phi)\, U\big(L(q\vec{n}_3)\big) | 0\lambda \rangle_s = \langle 0\lambda' | 0\lambda \rangle_s = \delta_{\lambda\lambda'} \tag{5.14}$$

This fact is, of course, a consequence of the unitary nature of the transformations $U\big(L(q\vec{n})\big)$ and $R(\phi, \Theta, -\phi)$ in (5.2). Similarly, the two particle states (5.12) fulfill in the spin space of the two given particles the orthogonality relations

$$\langle {}^{in}_{out}; q_B\lambda'_B, q_A\lambda'_A | q_A\lambda_A, q_B\lambda_B; {}^{in}_{out} \rangle_s = \langle {}^{in}_{out}; 0\lambda'_B | \lambda_B 0; {}^{in}_{out} \rangle_s \langle {}^{in}_{out}; 0\lambda_A | \lambda_A 0; {}^{in}_{out} \rangle_s$$

$$= \delta_{\lambda_A\lambda'_A}\, \delta_{\lambda_B\lambda'_B} \tag{5.15}$$

The relation (5.14) and (5.15) are the basis for constructing the spin density matrices in Section 5B.

Finally, we remark also that the usual Lorentz invariant inner product is defined for any single particle states in the Hilbert space by

$$\langle \psi | \phi \rangle = \int \langle \psi | \phi \rangle_s \frac{d^3p}{2p_0} \tag{5.16}$$

with an abvious extension to two particle states. In particular, we have the normalization condition

$$\langle q'\lambda' | q\lambda \rangle = \delta^3(\vec{q}\,' - \vec{q})\, 2q_0\, \delta_{\lambda'\lambda} \tag{5.17}$$

and an analogous relation for the two particle states (5.12) — Note that we distinguish the inner product in spin space (5.14) and (5.15) from the inner product in Hilbert space (5.16) by the subscript s.

5B. Von Neumann's Formula

Consider now an arbitrary state composed of the basic incoming two particle states (5.12)

$$|q_A, q_B; \text{in}\rangle = \sum_{\Lambda_i} a(\Lambda_i)|q_A\lambda_A, q_B\lambda_B; \text{in}\rangle \tag{5.18}$$

with

$$\sum_{\Lambda_i} |a(\Lambda_i)|^2 = 1 \tag{5.19}$$

The state (5.18) is expanded in terms of outgoing states $|q_1\lambda_1, q_2\lambda_2, \ldots; \text{out}\rangle$

$$|q_A, q_B; \text{in}\rangle = \int \frac{d^3q_C}{2q_0^C} \frac{d^3q_D}{2q_0^D} \delta^4(q_C + q_D - q_A - q_B) |q_C, q_D; \text{out}\rangle + \ldots \tag{5.20}$$

where

$$|q_C, q_D; \text{out}\rangle = \sum_{\Lambda_i, \Lambda_f} a(\Lambda_i) F_{\Lambda_f, \Lambda_i}(P, q_A - q_B, q_C - q_D) |q_C\lambda_C, q_D\lambda_D; \text{out}\rangle \tag{5.21}$$

and $P = (q_A + q_D) = (q_C + q_D)$. In (5.21) the reaction matrix $F_{\Lambda_f\Lambda_i}$ is defined by

$$\langle \text{out}; q_C\lambda_C, q_D\lambda_D |q_A\lambda_A, q_B\lambda_B; \text{in}\rangle = \langle \text{in}; q_C\lambda_C, q_D\lambda_D |q_A\lambda_A, q_B\lambda_B; \text{in}\rangle$$
$$- \delta^4(q_C + q_D - q_A - q_B) F_{\Lambda_f\Lambda_i}(P, q_A - q_B, q_C - q_D) \tag{5.22}$$

Note that with the definition (5.22) for F the unscattered part is left out in the first term of (5.20).

In the subspace of two particle states (5.18) and (5.21) we introduce now spin density operators in the usual way

$$\rho^{i}_{op}(q_A, q_B) = |q_A, q_B; in\rangle\ \langle in; q_B, q_A|$$

$$= \sum_{\Lambda_i \Lambda_i'} \rho^{i}_{\Lambda_i, \Lambda_i'} |q_A \lambda_A, q_B \lambda_B; in\rangle\ \langle in; q_B \lambda_B, q_A \lambda_A| \tag{5.23}$$

and

$$\rho^{f}_{op}(q_C, q_D) = |q_C, q_D; out\rangle\ \langle out; q_D, q_C|$$

$$= \sum_{\Lambda_f \Lambda_f'} \rho^{f}_{\Lambda_f \Lambda_f'} |q_C \lambda_C, q_D \lambda_D; out\rangle\ \langle out; q_D \lambda_D, q_C \lambda_C| \tag{5.24}$$

The coefficients $\rho^{i,f}_{\Lambda, \Lambda'}$, are the elements of the spin density matrix defined by

$$\rho^{i}_{\Lambda_i \Lambda_i'}(q_A, q_B) = \langle in; q_B \lambda_B, q_A \lambda_A | \rho^{i}_{op}(q_A, q_B) | q_A \lambda_A, q_B \lambda_B; in\rangle_s$$

$$= a(\Lambda_i)\, a^*(\Lambda_i') \tag{5.25}$$

and

$$\rho^{f}_{\Lambda_f \Lambda_f'}(q_C, q_D) = \langle out; q_D \lambda_D, q_C \lambda_C | \rho^{f}_{op}(q_C q_D) | q_C \lambda_C, q_D \lambda_D; out\rangle_s$$

$$= \sum_{\Lambda_i, \Lambda_i'} a(\Lambda_i)\, F_{\Lambda_f \Lambda_i}\, a^*(\Lambda_i')\, F^*_{\Lambda_f' \Lambda_i'} \tag{5.26}$$

The relation (5.26) is Von Neumann's formula, which can also be written in the compact form

$$\rho^f = F \rho^i F^\dagger \tag{5.27}$$

With our normalization the cross section in the c.m. system $\vec{q}_A + \vec{q}_B = \vec{q}_C + \vec{q}_D = 0$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{q_C}{q_A} \frac{\pi^2}{4W^2} \frac{\mathrm{Tr}\, F\rho^i F^\dagger}{\mathrm{Tr}\, \rho^i} \qquad (5.28)$$

where F is the reaction matrix Eq. (3.2) in the c.m. system, the elements of which are the helicity amplitudes. According to (3.18) and (3.14) one has

$$\mathrm{Tr}\, \rho^i = p_0^A\, p_0^B \left((2s^A+1)(2s^B+1)\right)^{-1/2} \qquad (5.29)$$

As mentioned in Section 3 we assume that the helicities of the particles in the initial state are uncorrelated, so that ρ^i is the direct product of the density matrix for each particle.

5C. Decomposition of the Single Particle Spin Density Matrix (Case M ≠ 0)

In the rest system ($\vec{q} = 0$) a single particle spin density matrix ρ is usually expanded by writing ρ as a polynomial of degree 2j in the components of the spin matrices S_i. Thus for a single particle state (5.6) in its rest frame

$$|\vec{q} = 0\rangle = \sum_\lambda c_\lambda |\vec{q} = 0, \lambda\rangle, \qquad (5.30)$$

the density matrix ρ is written

$$\rho = I + \sum_{r=1}^{3} p_r S_r + \sum_{r,s=1}^{3} p_{rs} S_r S_s + \ldots + \sum_{r,s\ldots u=1}^{3} p_{rs\ldots u} S_r S_s \ldots S_u \qquad (5.31)$$

Here the quantities $p_{rs\ldots u}$ denote the components of the polarization tensors of the particle in the rest frame with respect to certain axes, e.g. the helicities axes. The corresponding density operator $\rho_{op}(\vec{q} = 0) = |\vec{q} = 0\rangle \langle \vec{q} = 0|$ has an analgous expansion (5.31) with the S_i's replaced by the angular momentum operators J_i in the rest frame. In a general frame, where the particle is moving with momentum $\vec{q}$ the transformation law (5.2) yields

$$\rho_{op}(\vec{q}) = R(\phi,\theta,-\phi)\, U\left(L(\vec{n}_3 q)\right) \rho_{op}(\vec{q}\,' = 0)\, U^{\dagger}\left(L(\vec{n}_3 q)\right) R^{\dagger}(\phi,\theta,-\phi)$$

$$= I + \sum_{r=1}^{3} p_r \mathcal{O}_r + \sum_{r,s=1}^{3} p_{rs} \mathcal{O}_r \mathcal{O}_s + \ldots + \sum_{r,s,\ldots,u=1}^{3} p_{r,s,\ldots n} \mathcal{O}_r \mathcal{O}_s \cdots \mathcal{O}_u \tag{5.31'}$$

with (see (5.3) and (5.4))

$$\mathcal{O}_r(\vec{q}) = e^{-i\theta \vec{m}\cdot\vec{J}} e^{-i\lambda \vec{n}_3 \cdot \vec{K}} J_r e^{i\lambda \vec{n}_3\cdot\vec{K}} e^{i\theta\vec{m}\cdot\vec{J}} \tag{5.32}$$

replacing J_r in the analogous relation (5.30) for $\rho_{op}(\vec{q} = 0)$. Assuming a c omposition of $\vec{J}$ in the helicity frame with $(\vec{m}\cdot\hat{q}) = 0$

$$\vec{J} = (J_1, J_2, J_3) = (\vec{m}\times\hat{q})\cdot\vec{J},\ \vec{m}\cdot\vec{J},\ \hat{q}\cdot\vec{J}) \tag{5.33}$$

a straightforward calculation yields for $\mathcal{O}_i(q)$

$$\mathcal{O}_1(q) = -\sin\theta\, \hat{q}\cdot\vec{J} + \cos\theta\left(\frac{q_0}{M}(\vec{m}\times\hat{q})\cdot J + \frac{q}{M}\vec{m}\cdot\vec{K}\right) \tag{5.34a}$$

$$\mathcal{O}_2(q) = \frac{q_0}{M}\vec{m}\cdot\vec{J} - \frac{q}{M}(\vec{m}\times\hat{q})\cdot\vec{K} \tag{5.34b}$$

$$\mathcal{O}_3(q) = \cos\theta\, \hat{q}\cdot\vec{J} + \sin\theta\left(\frac{q_0}{M}(\vec{m}\times\hat{q})\cdot\vec{J} + \frac{q}{M}\vec{m}\cdot\vec{K}\right) \tag{5.34c}$$

The matrix elements of the operators $\mathcal{O}_i(\vec{q})$, defined for particles with arbitrary momentum $\vec{q}$, reduce by construction to the usual spin matrices S_i of the rest frame, if $\mathcal{O}_i(\vec{q})$ is calculated in the basis of the helicity states (5.2). Therefore one can assume that the expansion (5.31) is actually valid for any moving particle and that the parameters $p_{rs\ldots n}$ are always the same rest frame polarization quantities. The matrices S_i have only to be interpreted as the representation of the operators $\mathcal{O}_i(\vec{q})$ defined by (5.32) if calculated in the basis of the helicity states

(5.2). We have seen in Section 4, how from a practical point of view it is sometimes very advantageous to describe particles always in terms of properties in the rest frame.

In order to account for the different transformation law (5.13) of particles with label 2, we replace for these particles in (5.32) J_r by $(-1)^{2S} e^{-i\pi \vec{n}_3 \cdot \vec{J}} J_r e^{i\pi \vec{n}_3 \cdot \vec{J}}$. As a result of this replacement the axis $\vec{m}$ in (5.34) goes over into $(-\vec{m})$ apart from the overall factor $(-1)^{2S}$. With this convention the expansion (5.31) is true for particles with label 1 or 2.

There exists a convenient representation of the operators $\mathcal{O}_i$ in terms of the relativistic spin vector W_μ

$$W_\mu = \frac{1}{M} (\vec{q}\,\vec{J},\ q_0 \vec{J} + \vec{K} \otimes \vec{q}) \tag{5.35}$$

as one might expect. In fact the $\mathcal{O}_i$'s are projections of W_μ with respect to certain axes $(\vec{u}_i)^\mu$

$$\mathcal{O}_i(\vec{q}) = W_\mu \cdot (\vec{u}_i)^\mu \tag{5.36}$$

with

$$(\vec{u}_i)^\mu \cdot (\vec{u}_j)_\mu = -\delta_{ij},\ (\vec{u}_i)_\mu \cdot q^\mu = 0 \tag{5.37}$$

Explicitly one has for $(m \cdot \hat{q}) = 0$

$$u_1 = \left(-\frac{q}{M} \sin\theta,\ \cos\theta\ (\vec{m} \times \hat{q}) - \frac{q_0}{M}\ \hat{q} \sin\theta\right) \tag{5.38a}$$

$$u_2 = (0,\ -\vec{m}) \tag{5.38b}$$

$$u_3 \left(\frac{q}{M} \cos\theta,\ \sin\theta\ (\vec{m} \times \hat{q}) + \frac{q_0}{M}\ \hat{q} \cos\theta\right) \tag{5.38c}$$

By means of (5.36) - (5.38) it is possible to introduce a covariant notation.[22] One defines a covariant spin vector

$$S_\mu = \sum_{i=1}^{3} P_i (n_i)_\mu \tag{5.39}$$

and can then write

$$\sum_{i=1}^{3} \mathcal{O}_i(q)\, p_i = S_\mu W^\mu \tag{5.40}$$

and analogous generalizations for the higher tensors in (5.31').

Finally we note that from the representation (5.32) or (5.34) it follows easily that under the reflections Y (5.8) the operators $\mathcal{O}_i$ have the transformation property

$$Y\mathcal{O}_{1,3}(\vec{q})\, Y^+ = - \mathcal{O}_{1,3}(\vec{q}),\quad Y\mathcal{O}_2(q)\, Y^+ = + \mathcal{O}_2(q) \tag{5.41}$$

The relations (5.41) are the basis of the restrictions on the tensor polarizations, which follow from parity conservation (see Section 3.B).

5D. Density Matrix for the Photon

In the case of the photon the proceding discussion of the density matrix is not valid, since it is e.g. not possible to define the spin matrices $\vec{S}$ in a rest frame. In this case we derive the form (3.6) directly from the definition $\rho^\gamma_{op} = |\gamma\rangle\langle\gamma|$ for a general photon state in an arbitrary frame

$$|\gamma\rangle = a_+ \,|\vec{q}, \lambda = +1\rangle + a_- \,|\vec{q}, \lambda = -1\rangle,\quad |a_+|^2 + a_-|^2 = 1 \tag{5.42}$$

where $|\vec{q},\lambda\rangle$ are the states (5.7). Then

$$\begin{aligned}\rho^\gamma_{op}(\vec{q}) = |\gamma\rangle\langle\gamma| = \tfrac{1}{2}\,\{|+1\rangle\langle+1| + |-1\rangle\langle-1|\} \\ + \frac{|a_+|^2 - |a_-|^2}{2}\,\{|+1\rangle\langle+1| - |-1\rangle\langle-1|\} \\ + \{a_+a_-^* \,|+1\rangle\langle-1| + a_+^*a_-|-1\rangle\langle+1|\},\end{aligned} \tag{5.43}$$

where for notational reasons $|q, \lambda = \pm 1\rangle = |\pm 1\rangle$. The expansion (5.43) of the operator $\rho^{\gamma}_{op}(q)$ taken in the helicity basis (5.7) yields the decomposition (3.6).

In (3.6) we put

$$c = |a_+|^2 - |a_-|^2, \tag{5.44}$$

which gives the degree of circular polarization,

$$\ell = 2\,|a_+ a^*_-|, \tag{5.45}$$

which gives the degree of linear polarization, and

$$e^{-2i\Phi} = -\frac{a_+ a^*_-}{|a_+ a^*_-|} \tag{5.46}$$

which determines within $\pm\pi$ the azimuth angle Φ of the polarization vector in the plane perpendicular to the momentum of the photon. To see this consider a photon moving in the $\vec{n}_3$-direction. The relation between the helicity states $|q\vec{n}_3, \lambda\rangle$ and plane-polarized states $|\vec{e}_i\rangle$, $i = 1,2$, with the polarization vector $\vec{e}_i$ pointing in the $\vec{n}_1$ or $\vec{n}_2$ direction is as usual

$$|qn_3, \lambda\rangle = -\frac{1}{\sqrt{2}}\,(\lambda|e_1\rangle + i|e_2\rangle). \tag{5.47}$$

Then a plane polarized state with an arbitrary $\vec{e}(\phi)$ in the (n_1, n_2)-plane is given by

$$|q\vec{n}_3, \vec{e}(\phi)\rangle = \cos\phi\,|e_1\rangle + \sin\phi\,|e_2\rangle = \frac{1}{\sqrt{2}}\left\{-e^{i\phi}|q\vec{n}_3, +1\rangle + e^{+i\phi}|q\vec{n}_3, -1\rangle\right\} \tag{5.48}$$

We see under these circumstances one can put $\Phi = \phi$.

In practice we always assume that the photon is a particle with the label 2 moving in the direction $-\vec{n}_3$. Now in the case of the photon the relation

$$|-q_2\vec{n}_3, \lambda\rangle = |q_2\vec{n}_3, -\lambda\rangle \tag{5.49}$$

substitutes (5.13). We take the same decomposition (3.6) for the density matrix. If we measure the azimuthal angle ϕ' of the plane polarized state in the coordinates for particle "2" shown in Fig. 1 we can again put $\Phi = \phi'$, but we can as well take our photon coordinates to be rotated by π around the $\vec{n}_3$ direction. This ambiguity in coordinates restates the ambiguity in ϕ.

We also note that under the reflection transformation Y the parts of (5.43) connected with the coefficient 1/2 and $-\ell/2 \cos 2\Phi$ are even, whereas the other two parts c/2 and $-\ell/2 \sin 2\Phi$, are odd. This one can easily establish from the relation (5.9) introduced into (5.43) or from the matrix representation $\eta\sigma_1$ for Y and (3.6) for ρ^{γ}.

Finally we remark that the decomposition (5.42) or (5.43) is dependent on the frame and therefore not unique.

5E. Mapping of the Spherical Harmonics by Y: Further Restrictions; Eq. (3.13)

In Section (3.B) we established the mapping

$$T_{LM}Y = \sum_{L'=|M|}^{2S} \rho_{L'}^{LM} T_{L'M} \tag{5.32a}$$

We now note that via Eq. (3.13) $T^{\dagger}_{LM}$ can be linearly related to T_{LM}

$$T^{\dagger}_{LM} = (-1)^{L-M} e^{-i\pi S_3} T_{LM} e^{+i\pi S_3} \tag{5.50}$$

(see the representation (3.13) for T_{LM}). Hence one can find $(T_{LM}Y)^{\dagger}$ either directly from (3.24)

$$(T_{LM}Y)^{\dagger} = \sum_{L'=|M|}^{2S} (\rho_{L'}^{LM})^{*} T^{\dagger}_{L'M} \tag{5.51}$$

or by applying (5.50) to both sides of (3.25a). Equality of the resulting expressions then implies

$$(\rho_{L'}^{LM})^{*} = (-1)^{L'-L-M} \rho_{L'}^{LM} \tag{5.52}$$

Thus $\rho_{L'}^{LM}$ is real (imaginary) if L'-L-M is even (odd). This means that in the decomposition of $T_{LM}^{i}Y$, $T_{L'M}^{i'}$ (see Eq. (3.11) and (3.12)) appears for i' = i or for i' ≠ i but not for both i' = i and i' ≠ i.

$$(T_{LM} \pm T_{LM}^{\dagger})Y = \sum_{L'=|M|}^{2S} \rho_{L'}^{LM}\left[T_{L'M} \pm (-1)^{L'-L+2S} T_{L'M}^{\dagger}\right] \tag{5.53}$$

For M = 0 we have $T_{L0}^{\dagger} = T_{L0}$ and therefore

$$(T_{L0}Y)^{\dagger} = Y^{\dagger}T_{L0} = T_{L0}Y^{\dagger} = (-1)^{2S}T_{L0}Y \tag{5.54}$$

Consistency with (5.52) now requires

$$\rho_{L'}^{L0} = 0 \text{ if } L' - L + 2S \text{ is odd} \tag{5.55}$$

Furthermore

$$(\rho_{L'}^{LM})^{*} = \left[\mathrm{Tr}(T_{LM}Y\,T_{L'M}^{\dagger})\right]^{*} = (-1)^{2S}\,\rho_{L}^{L'M} \tag{5.56}$$

and, in particular, ρ_{L}^{LM} must be real (imaginary) if 2S is even (odd). Thus consistency with (5.52) also requires

$$\rho_{L}^{LM} = 0 \text{ if } 2S - M \text{ is odd} \tag{5.57}$$

For low values of S the restrictions (5.52), (5.55) and (5.57) are often sufficient to limit the decomposition of $T_{LM}^{i}Y$ in terms of $T_{L'M}^{i'}$ to only one term, as can be seen in Eq. (3.26) and (3.27).

We would also like to derive here Eq. (3.13) for T_{LM}, $M \geq 0$. For this purpose it is convenient to use Eq. (3.9) to generate T_{LM}, $2S \geq L > M \geq 0$, from T_{LL}, a positive multiple of $(S_3 + iS_1)^L$. Let us assume that for a given M > 0 we can write

T_{LM} in the form

$$T_{LM} = \mathscr{P}_L^M(S_2)\,(S_3 + iS_1)^M \tag{5.58}$$

where $\mathscr{P}_L^M(S_2)$ is a real polynomial in S_2 of degree L-M. Then Eq. (3.9) gives

$$T_{LM-1} = \left[(L+M)(L-M+1)\right]^{-1/2}\Big[\mathscr{P}_L^M(S_2+1)\left[S(S+1) - S_2^2 - S_2\right]$$

$$-\mathscr{P}_L^M(S_2)\left[S(S+1) - M(M-1) - S_2^2 + (2M-1)S_2\right]\Big]\,(S_3+iS_1)^{M-1} \tag{5.59}$$

Since the polynomial multiplying $(S_3 + iS_1)^{M-1}$ in Eq. (5.59) is real and has degree L-M+1, Eq. (5.58), valid for M = L, can be obtained for all $M \geq 0$ by iteration.

Now let us assume inductively that for a given positive integer ν a product of the form $\mathscr{P}^{\nu'}(S_2)(S_3 + iS_1)^M$ can be put into the form

$$\left[P^{\nu'}(S_2),\ (S_3 + iS_1)^M\right]_+ \tag{5.60}$$

if $\mathscr{P}^{\nu'}(S_2)$ is any real polynomial of degree $\nu' < \nu$ and where $P^{\nu'}(S_2)$ is also a real polynomial of degree ν'. (Clearly this can be done for $\nu = 1$.) But then for $\nu' = \nu$ we have

$$\mathscr{P}^{\nu}(S_2)(S_3 + iS_1)^M = \left[\tfrac{1}{2}\mathscr{P}^{\nu}(S_2), (S_3 + iS_1)^M\right]_+ + \tfrac{1}{2}\left[\mathscr{P}^{\nu}(S_2), (S_3 + iS_1)^M\right] \tag{5.61}$$

$$= \left[\tfrac{1}{2}\mathscr{P}^{\nu}(S_2), (S_3 + iS_1)^M\right]_+ + \tfrac{1}{2}\left[\mathscr{P}^{\nu}(S_2) - \mathscr{P}^{\nu}(S_2 - M)\right](S_3 + iS_1)^M \tag{5.62}$$

where $1/2\left[\mathscr{P}^{\nu}(S_2) - \mathscr{P}^{\nu}(S_2 - M)\right]$ is a real polynomial of degree $\nu - 1$. Hence, by induction on ν, Eq. (5.60) can be obtained for any $\nu' \geq 0$. Given Eq. (5.58) we can therefore write Eq. (3.13)

$$T_{LM} = \left[P_L^M(S_2),\ (S_3 + iS_1)^M\right]_+ \tag{3.13}$$

where P_L^M is a real polynomial of degree L-M, $M \geq 0$. Since $T^{\dagger}_{LM} = (-1)^M T_{L-M}$ we also have

$$T_{L-M} = \left[(-1)^M P_L^M(S_2), (S_3 - iS_1)^M\right]_+ \qquad (5.63)$$

The uniqueness of the L-M + 1 coefficients in P_L^M follows from considering the 2S+ 1 - M matrix elements between eigenstates of S_2, $\langle m_2 + M | T_{LM} | m_2 \rangle$, noting that $L \leq 2S$. That $P_L^M(S_2)$ must involve only even or only odd powers of S_2, depending on the sign of $(-1)^{L-M}$, follows from the observation that, as a consequence of our procedure for generating T_{LM} via commutation ($(S_3 \pm iS_1)$ are symmetrical matrices),

$$(T_{LM})^t = (-1)^{L-M} T_{LM} \qquad (5.64)$$

This behavior must be consistent with Eq. (3.13) in which the antisymmetrical matrix S_2 appears, thus restricting P_L^M to be even or odd.

It is interesting to note that the polynomial $P_L^M(S_2)$ is not, in general, a simple multiple of the corresponding associated Legrendre polynomial $\mathbf{T}^M_{L-M}\left(\frac{S_2}{\sqrt{S(S+1)}}\right)$ although the two are in many ways similar.

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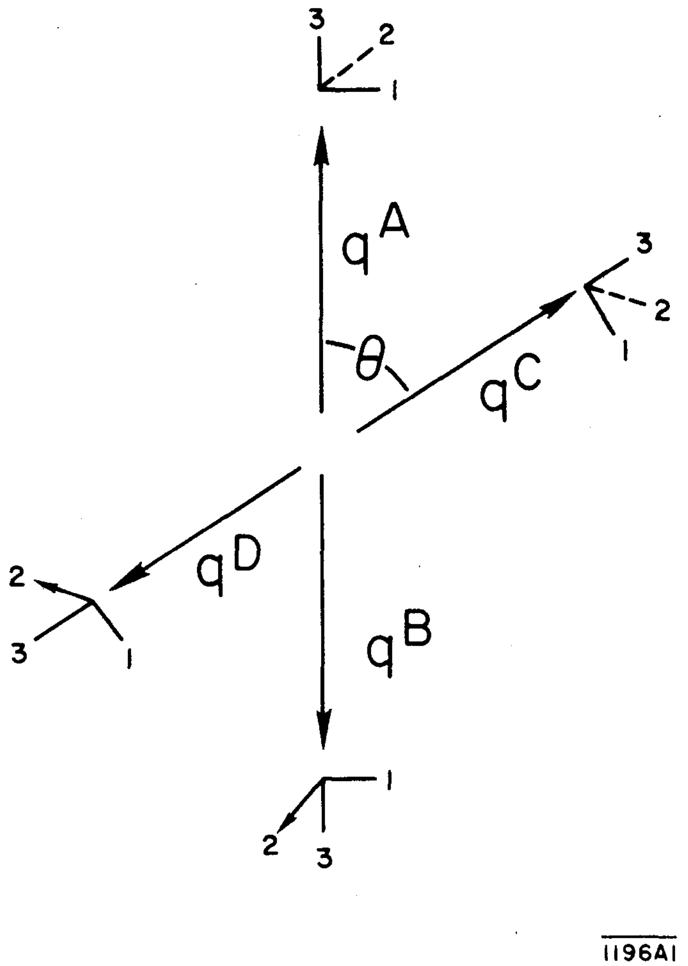

FIG. 1--Helicity coordinates for incident particles A, B and final particles C, D. The 3, 1-axes always lie in the reaction plane while the 2-axis either points into the plane (dotted line) or out of the plane (arrow).