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THE ELECTROMAGNETIC INTERACTIONS OF COMPOSITE SYSTEMS*

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ABSTRACT

Starting from Lagrangian field theory, we derive the interaction Hamiltonian of a composite system with an external electromagnetic field. Upon this basis, we develop the theoretical foundations of the atomic Zeeman effect, with specific reference to the fine structure and Lamb shift measurements. We also explicitly verify the Drell-Hearn-Gerasimov sum rule and the low energy theorem for Compton scattering for composite systems. An essential result of our investigation is that the interaction of a loosely bound composite system with an external electromagnetic field can be well approximated by the sum of the relativistic interaction Hamiltonians appropriate to the free constituents, but that in general the non-relativistic reduction of this Hamiltonian does not yield the sum of the corresponding free reduced (e. g. Foldy-Wouthuysen) Hamiltonians. New features of this work include an extended Salpeter equation which includes interactions with an external electromagnetic field, explicit wave packet solutions to a two-body relativistic equation, and a calculational approach to perturbation theory with composite systems in which sums over intermediate or final states preceed non-relativistic approximations. Our explicit calculation of the DHG sum rule illustrates its superconvergent nature.

CONTENTS

Section 1
Introduction

The most familiar objects of physics - atoms, nuclei, and perhaps even elementary particles - are composite systems. Yet one searches the literature in vain for an adequate practical treatment of the electromagnetic interactions of such systems, taking into account the essential complications of relativity and spin. In this work[1] we derive from Lagrangian field theory an expression for the interaction Hamiltonian of a composite system with an external electromagnetic field. For simplicity, we consider two-body systems, and, in particular, systems in which the particles can be regarded as interacting through an instantaneous Bethe-Salpeter kernel - which is to say, interacting through a potential - and we investigate the corrections to this approximation. We find that for such systems the external electromagnetic interaction Hamiltonian can be well approximated by the sum of the relativistic interaction Hamiltonians appropriate to the free constituents, but that the non-relativistic reduction of this Hamiltonian does not yield the sum of the corresponding reduced (e.g. Foldy-Wouthyusen (F-W)) Hamiltonians if the constituents have spin. The crucial error in deriving F-W additivity is in neglecting the spin transformation of the composite state wavefunction associated with the center of mass (CM) motion.

The correct non-relativistic reduction of the interaction Hamiltonian for a composite system of two spin $\frac{1}{2}$ particles in an external electromagnetic field takes the following form[2]

$$H_{NR}^{em} = \sum_{s=a,b} \left[\frac{-\vec{p}_s \cdot e_s \vec{A}_s}{m_s} + \frac{e_s^2 \vec{A}_s^2}{2m_s} + e_s A_s^o - \mu_s \vec{\sigma}_s \cdot \vec{B}_s - (2\mu_s - \frac{e_s}{2m_s}) \vec{\sigma}_s \cdot \vec{E}_s \times \frac{(\vec{p}_s - e_s \vec{A}_s)}{2m_s} \right]$$

$$+ \frac{1}{4M_T} \left(\frac{\vec{\sigma}_a}{m_a} - \frac{\vec{\sigma}_b}{m_b} \right) \cdot \left(e_b \vec{E}_b \times (\vec{p}_a - e_a \vec{A}_a) - e_a \vec{E}_a \times (\vec{p}_b - e_b \vec{A}_b) \right) + 0(1/m^3). \tag{1.1}$$

The terms proportional to $(M_T m_a)^{-1}$ or $(M_T m_b)^{-1}$ are correction terms to F-W additivity.

We were motivated to undertake this investigation in attempting to understand certain difficulties in connection with the Drell-Hearn (1) - Gerasimov (2) (DHG) sum rule and the low energy theorem for Compton scattering (3). The low energy theorem for Compton scattering is a fundamental statement for electrodynamics: given the total mass, spin, charge, and magnetic moment of any discrete system, the photon scattering amplitude is determined to first order in frequency. As a consequence of the low energy theorem, the DHG sum rule for the photoabsorption cross section is obtained by assuming an unsubtracted dispersion relation for the spin-flip forward amplitude. Again, the result depends only on the spin and static properties of the target system - no distinction is made between elementary and composite systems.

The calculation of Barton and Dombey (4), which purported to show that if the DHG sum rule holds for nucleons, it must fail for bound states containing a nucleon, thus seemed very suspicious. If this calculation had been correct, the only way to reconcile the DHG sum rule with the low energy theorem would have been to assume that there is an additive constant, sometimes called a "subtraction at ∞", present in the dispersion

relation for the spin flip forward Compton amplitude f_2 for a composite system, even if it is not present for the constituents. Such a state of affairs would be physically most unreasonable, since a "subtraction at ∞" is associated with the asymptotic behavior of $f_2(\omega)$ for $|\omega| \to \infty$, and the asymptotic behavior of the Compton amplitude for the composite system should be no worse than that of the sum of the amplitudes of the constituents (cf. Section 7D).

In this paper we show that the calculations in ref. (4) are incorrect because of the unjustified assumption, which pervades the literature (5), that the electromagnetic interaction of the composite system can be calculated from the sum of the Foldy-Wouthuysen Hamiltonians of the constituents. In fact, such a Hamiltonian does not yield the correct low energy limit of the Compton scattering amplitude[3] for the bound system, nor does it have the correct Thomas term appropriate to the system's momentum and spin[4]. We shall show that when the spin transformations are properly included the new non-relativistic interaction Hamiltonian (1.1) emerges, the low energy theorem is not violated, and the DHG sum rule for composite systems is verified. After this work was completed, we learned that some of these results have also been obtained using quite different methods by H. Osborn (6).

The outline of the paper is as follows. We begin in Section 2 with a derivation of the relativistic electromagnetic interactions of a two-particle composite system. In order to do this, we return to the definition of the electromagnetic current as given in Lagrangian field theory. The matrix elements of the current are then readily expressible in terms of Bethe-Salpeter (1) (BS) amplitudes. In the approximation in which the BS interaction kernel is replaced by a neutral instantaneous kernel (i.e. potential) in ladder approximation a relativistic interaction Hamiltonian emerges. For

Salpeter's (81) equation which includes interactions with a static external field. In fact, it turns out that a "Breit" Hamiltonian extended to include external electromagnetic interactions leads to the same results as the BS approach up to terms of relative order $< U^2/m_a m_b >$, where U is the interaction potential.

Using these procedures, the corrections to the impulse approximation can then be readily traced. In Section 3 we apply these results to the analysis of the Zeeman spectrum in hydrogen-like atoms, in order to obtain estimates of radiative and reduced mass corrections not already included in standard calculations. We emphasize that the comparison of theory with experimental measurements of the Lamb shift and fine structure intervals in H and D requires a precise theoretical extrapolation of the experimental results to zero magnetic field. Thus care in the calculation of the Zeeman effect is as essential as it is in the calculation of the zero field energy levels themselves.

In general, we require matrix elements of the current connecting composite states of different total momentum. The Lorentz transformation properties of the Bethe-Salpeter and time-independent Salpeter wavefunctions are derived in Section 4. In addition, a very convenient relativistic two-body bound state wavefunction, correct to first order in the binding potential, is given. The effect of the spin transformations, resulting from boosting the CM wavefunctions to other Lorentz frames, is shown explicitly.

The foundations are thus prepared for the calculation in Section 5 of the photo-absorption matrix elements required for the DHG sum rule. Two complimentary derivations of the sum rule for composite systems are

given. The first derivation is in the spirit of numerous atomic and nuclear physics calculations and shows how such treatments - including that of ref. (4) - must be modified when relativistic effects are included properly. The second method of derivation is based on closure of relativistic states and is more elegant and instructive. As is discussed in Section 7C, this technique expresses the DHG sum rule in the form of a superconvergence relation. Our calculations are given for realistic models of bound states of two spin $\frac{1}{2}$ particles or of a spin $\frac{1}{2}$, spin 0 combination. The second method of derivation, however, shows that the sum rule only requires the correct external electromagnetic interaction of the CM motion and total spin; hence this method of proof may be readily generalized to other bound systems.

In Section 6, we explicitly prove the low energy theorem for a composite system. The correct spin terms in the non-relativistic reduction are again essential to the calculations. We also obtain the effective Hamiltonian (1.1) (which includes the negative energy state contributions which enter through second order perturbation theory), to be used as the non-relativistic "large component" reduction of the electromagnetic interaction of the composite system instead of the usually assumed, but incorrect, F-W form. We discuss in Section 7A how the correct interaction can be derived in an alternative way through a modification of the F-W procedure.

We recommend that the reader interested chiefly in calculational details, as presented in the derivation of the DHG sum rule or the calculation of the low energy limit of the Compton amplitude for composite systems, should proceed directly to Sections 5 through 8.

Section 2

The Electromagnetic Interactions of a Composite System

One of our purposes in this work is to derive the electromagnetic interaction Hamiltonian for the two-body system as it is prescribed by the axioms of Lagrangian field theory. For example, starting from the definition of the electromagnetic current for two fermion fields[5]

$$j_\mu(x) = e_a \bar{\psi}_{(a)} \gamma_\mu^{(a)} \psi_{(a)} + e_b \bar{\psi}_{(b)} \gamma_\mu^{(b)} \psi_{(b)} , \tag{2.1}$$

we shall show that in an approximation which is often physically reasonable, the interaction Hamiltonian is the sum of interaction Hamiltonians appropriate for free particles:

$$H^{em} = e_a \gamma_0^{(a)} \gamma_\mu^{(a)} A^\mu(x_a) + e_b \gamma_0^{(b)} \gamma_\mu^{(b)} A^\mu(x_b) + \left(\begin{array}{l} \text{anomalous} \\ \text{moment} \\ \text{contributions} \end{array} \right) \tag{2.2}$$

where $A^\mu(x)$ is the external electromagnetic field and H^{em} is an operator in the two-fermion Hilbert space which can be time dependent ($x_a^o = x_b^o = t$). Eq. (2.2) is usually assumed without proof in quantum-mechanical treatments of a two-fermion system. In general, however, expression (2.2) is not exact, and we shall discuss the source of the additional terms for H^{em} below (after Eq. (2.33)).

A. The Bethe-Salpeter Equation

Before we proceed to a derivation of the interaction Hamiltonian we shall review the application of the Bethe-Salpeter (BS) equation (9) and S-matrix perturbation methods to the two-fermion system. The wavefunction of each

two-fermion state $|n>$ (corresponding to a bound or scattering system) satisfies the BS equation[6]:

$$(i\partial_\mu^{(a)}\gamma_{(a)}^\mu - m_a)(i\partial_\mu^{(b)}\gamma_{(b)}^\mu - m_b)\chi_n(x_a, x_b) = (G\chi_n)(x_a, x_b) \tag{2.3}$$

where

$$\chi_n(x_a, x_b) \equiv < 0 \mid T(\psi^{(a)}(x_a), \psi^{(b)}(x_b)) \mid n > . \tag{2.4}$$

The center of mass coordinate dependence in the eigensolutions can be exhibited since the states $|n>$ are eigenstates of total four-momentum:

$$\chi_n(x_a, x_b) = e^{-iP_n \cdot X} \chi_n(x) \tag{2.5}$$

with

$$\begin{aligned} X &= \tau_a x_a + \tau_b x_b \\ x &= x_a - x_b \\ \tau_a &= m_a/(m_a+m_b), \quad \tau_b = m_b/(m_a+m_b) \\ P_n^2 &= E_n^2 - \vec{P}_n^2 = M_n^2 . \end{aligned} \tag{2.6}$$

In "ladder" approximation one takes

$$G\chi_n = G(x_a - x_b)\chi_n(x_a - x_b) \tag{2.7}$$

although in general the BS equation (2.3) includes (and we will consider) self-energy corrections and irreducible kernels in addition to the kernel $G(x_a - x_b)$ from single boson exchange.

The most practical application of the BS equation has been the calculation of the energy levels of the hydrogen atom. In lowest, non-relativistic approximation - the exchange of Coulomb photons in ladder approximation with $m_e/M_p \to 0$ - the Schrödinger equation for an electron in a Coulomb potential emerges. Figure (1) illustrates the contribution of other irreducible kernels. Using the techniques of Feynman, Salpeter (8), Erickson and Yennie (10), and others, the energy levels can, in principle, be calculated to any degree of precision. In practice, one uses as expansion parameters m_e/M_p, R_p/a_0 (ratio of proton rms radius to Bohr radius), α (from vacuum polarization and self-energy kernels), as well as $Z\alpha$ and $Z\alpha \log Z\alpha$ (from the binding interaction).

B. The Composite System in an External Field

The effect of an electromagnetic field $A_\mu(x)$ on the two-fermion system may be calculated in the usual way from the reduction of the S-matrix in field theory, using as a perturbation the Heisenberg interaction density

$$\mathcal{H}_I(x) = j_\mu(x)A^\mu(x) \tag{2.8}$$

with $j_\mu(x)$ defined by (2.1). Examples are

(1) Scattering of a composite system in a static external field in

lowest order: The linear term in the S-matrix is

$$S^{(1)} = -2\pi i\, \delta(E_n - E_m) \int < n \,|\mathscr{H}_I(0,\vec{x})\,|\, m > d^3x. \tag{2.9}$$

(2) Compton scattering on a two fermion composite system, to lowest order in α.

$$S^{(2)} = \frac{(-i)^2}{2!} \int d^4x_1 \int d^4x_2 \; < f, \vec{k}'\vec{\epsilon}' |\; T(\mathscr{H}_I(x_1), \mathscr{H}_I(x_2)) \,|\, i, \vec{k}\;\vec{\epsilon}\; >$$

$$= -\frac{1}{\sqrt{4\omega\omega'}} \int d^4x_1 \int d^4x_2 \; e^{i(k'\cdot x_2 - k\cdot x_1)} \tag{2.10}$$

$$< f \,|\, T(\vec{\epsilon}\cdot\vec{j}(x_1), \vec{\epsilon}'\cdot\vec{j}(x_2)) \,|\, i > .$$

The radiation gauge is used for the photons. We have dropped an equal time term in reducing out the photons from the states since this gives no contribution when the exchanged boson is chargeless. Inserting a complete set of states and integrating over x_{10}, x_{20}, one obtains a Lippman-Schwinger form

$$S^{(2)} = -2\pi i\, \delta(E_i + \omega - E_f - \omega') \frac{1}{\sqrt{4\omega\omega'}} \int d^3x_1 \int d^3x_2 \; e^{i\vec{k}\cdot\vec{x}_1 \,-\, i\vec{k}'\cdot\vec{x}_2}$$

(2.11)

$$\sum_j \left(\frac{< f \,|\, \vec{\epsilon}'\cdot\vec{j}(\vec{x}_2) \,|\, j >< j \,|\, \vec{\epsilon}\cdot\vec{j}(\vec{x}_1) \,|\, i >}{E_i + \omega - E_j + i\epsilon} + \frac{< f \,|\, \vec{\epsilon}\cdot\vec{j}(\vec{x}_1) \,|\, j >< j \,|\, \vec{\epsilon}'\cdot\vec{j}(\vec{x}_2) \,|\, i >}{E_i - \omega' - E_j + i\epsilon} \right) .$$

Here $j^{\mu}(\vec{x}) = j^{\mu}(0,\vec{x})$, $|\, i >$ and $|\, f >$ represent the initial and final bound system in different four-momentum states, and $|\, j >$ represents all possible states connected to $|\, i >$ or $|\, f >$ through one photon emission or absorption (conserving three-momentum) and includes disintegration channels. The method of Low (3) can be applied at this point to establish the low energy theorem for Compton scattering. The direct derivation we give in Section 6 is based on an explicit form for the currents for the two-body system.

As is apparent from our examples, explicit calculations in perturbation theory will require matrix elements of the electromagnetic current between composite states. The general method of calculating such matrix elements has been discussed by Mandelstam (11). A simple result can be obtained for the two-fermion problem in ladder approximation [Eqs. (2.3), (2.7)] :

$$< n\,|\, j_{\mu}(x)\,|\, m > = -ie_a\int d^4x_b\, \bar{\chi}_n(x,x_b)\gamma_{\mu}^{(a)}(i\partial^{(b)}\cdot\gamma_{(b)} - m_b)\chi_m(x,x_b)$$

$$- ie_b\int d^4x_a\, \bar{\chi}_n(x_a,x)\gamma_{\mu}^{(b)}(i\partial^{(a)}\cdot\gamma_{(a)} - m_a)\chi_m(x_a,x) \qquad (2.12)$$

where the conjugate wavefunction is

$$\bar{\chi}_n(x_a,x_b) = < n\,|\, T(\bar{\psi}(x_a)\,\bar{\psi}(x_b))\,|\, 0 > . \qquad (2.13)$$

In terms of Mandelstam's graphical analysis, this result for $j_{\mu}(x)$ in ladder approximation corresponds to Figure 2. In general, $j_{\mu}(x)$ is modified by additional terms when other irreducible kernels or self-energy corrections are

included in the BS equation. (See Figure 2b) Such terms correspond to anomalous moment contributions, form factor corrections, exchange currents, etc. The amplitudes χ_n are assumed to be normalized (8, 11) to give the correct total charge:

$$\int < n \mid j_0(x) \mid m > d^3x = (e_a + e_b)\delta_{mn}. \tag{2.14}$$

C. The Instantaneous Ladder Approximation

We will now discuss a further simplification of the electromagnetic current of the two-fermion system made possible if the BS (ladder approximation) kernel is instantaneous: $G(x_a - x_b) = ig(\vec{x}_a - \vec{x}_b)\delta(x_a^o - x_b^o)$. This is, of course, a good approximation in the low energy or weak binding region and corresponds to a description of the two-body interaction in terms of potentials[7]. For the instantaneous kernel the x^o integration in (2.12) can be performed and, as we shall see, yields an effective interaction Hamiltonian.

We must first recover some results of Salpeter (8) for the case of the instantaneous kernel, but without specializing to the CM frame. Introducing total and relative momentum variables,

$$\begin{aligned} P &= p_a + p_b \\ p &= \tau_b p_a - \tau_a p_b \end{aligned} \tag{2.15}$$

the BS equation in momentum space is[8]

$$[\gamma^{(a)}\cdot(\tau_a P+p)-m_a][\gamma^{(b)}(\tau_b P-p)-m_b]\psi(p_a,p_b)=\frac{-1}{2\pi i}\int d^4p'\,g(\vec{p}-\vec{p}')\,\psi(p'_a,p'_b) \qquad (2.16)$$

with

$$\psi(p'_a,p'_b)=\int d^4x\,d^4x_b\,e^{ip'_a\cdot x_a+ip'_b\cdot x_b}\,\chi(x_a,x_b)=\psi(\tau_a P+p',\tau_b P-p')$$

$$g(\vec{p})=\frac{1}{(2\pi)^3}\int d^3x\,e^{-i\vec{p}\cdot\vec{x}}\,g(\vec{x}).$$

We define free particle projection operators

$$\Lambda^a_{\pm}(\vec{p}_a)=[E_a(\vec{p}_a)\pm H_a(\vec{p}_a)]/2E_a(\vec{p}_a) \qquad (2.17)$$

$$E_a(\vec{p}_a)=\sqrt{\vec{p}_a^{\,2}+m_a^2},\quad H_a(\vec{p}_a)=\vec{\alpha}_a\cdot\vec{p}_a+\beta_a m_a$$

and obtain the four equations

$$[\tau_a P_0-s_a E_a(\vec{p}_a)+p_0][\tau_b P_0-s_b E_b(\vec{p}_b)-p_0]\psi_{s_a s_b}(p_a,p_b)= \qquad (2.18)$$

$$=\frac{-1}{2\pi i}\,\Lambda^a_{s_a}(\vec{p}_a)\Lambda^b_{s_b}(\vec{p}_b)\gamma_0^{(a)}\gamma_0^{(b)}\int d^4p'\,g(\vec{p}-\vec{p}')\,\psi(p'_a,p'_b)\equiv\Gamma_{s_a s_b}$$

for

$$\psi_{s_a s_b} \equiv \Lambda^a_{s_a} \Lambda^b_{s_a} \psi, \qquad s_a, s_b = \pm 1.$$

One is then able to integrate over p_0 [9]

$$\varphi^{P_0}_{s_a, s_b}(\vec{p}, \vec{P}) \equiv \int_{-\infty}^{\infty} dp_0 \, \psi_{s_a s_b}(p_a, p_b) \tag{2.19}$$

$$= \int_{-\infty}^{\infty} dp_0 \, \frac{\Gamma_{s_a s_b}}{[\tau_a P_0 - s_a E_a(\vec{p}_a) + p_0 + i\delta s_a][\tau_b P_0 - s_b E_b(\vec{p}_b) - p_0 + i\delta s_b]}$$

$$= -\left[\delta_{s_a,+1}\,\delta_{s_b,+1} - \delta_{s_a,-1}\,\delta_{s_b,-1}\right] \frac{2\pi i}{P_0 - s_a E_a(\vec{p}_a) - s_b E_b(\vec{p}_b)} \Gamma_{s_a s_b}.$$

Adding these equations together gives Salpeter's equation (8)

$$[P_0 - H_a(\vec{p}_a) - H_b(\vec{p}_b)]\,\varphi^{P_0}(\vec{p}, \vec{P}) = \tag{2.20}$$

$$(\Lambda^a_+ \Lambda^b_+ - \Lambda^a_- \Lambda^b_-)\gamma_0^{(a)}\gamma_0^{(b)} \int d^3p' g(\vec{p} - \vec{p}')\varphi^{P_0}(\vec{p}', \vec{P}),$$

a consequence of the instantaneous kernel. Combining (2.18) and (2.19), the p_0 dependence of $\psi(p_a, p_b)$ is completely determined. We define an auxiliary wavefunction $\eta(\vec{p}, \vec{P})$ where

$$[P_o - H_a(\vec{p_a}) - H_b(\vec{p_b})]\,\eta^{P_o}(\vec{p}, \vec{P}) = \gamma_o^{(a)}\gamma_o^{(b)} \int d^3p'\, g(\vec{p} - \vec{p}')\varphi^{P_o}(\vec{p}', \vec{p}) \,. \tag{2.21a}$$

Then

$$\psi_{s_a s_b}(p_a, p_b) = \tag{2.21b}$$

$$-\frac{1}{2\pi i}\ \frac{P_0 - s_a E_a(\vec{p}_a) - s_b E_b(\vec{p}_b)}{[\tau_a P_0 - s_a E_a(\vec{p}_a) + p_0 + i\delta s_a][\tau_b P_0 - s_b E_b(\vec{p}_b) - p_0 + i\delta s_b]}\ \eta^{P_0}_{s_a s_b}(\vec{p}, \vec{P}).$$

D. The Interaction Hamiltonian

- Assuming ladder approximation and an instantaneous kernel for the BS equation we will now be able to eliminate the relative time dependence in the matrix element of the current (2.12). We shall express the result in terms of the matrix element of the interaction Hamiltonian between two composite-state wavefunctions.

$$\int d^3x \langle n | \mathcal{H}_I(\vec{x}, 0) | m \rangle$$

$$= -ie_a \int d^3x \int d^4x_b\ \bar{\chi}_n(x, x_b)\, \gamma^{(a)} \cdot A(\vec{x})(i\partial^{(b)} \cdot \gamma^{(b)} - m_b)\chi_m(x, x_b)\Big|_{x_o = 0} \tag{2.22}$$

$$= -2\pi i e_a \int d^3p'_a \int d^3p_a \int d^4p_b\ \bar{\psi}_n(p'_a, p_b)\, \gamma^{(a)} \cdot A(\vec{p}'_a - \vec{p}_a)(\gamma^{(b)} \cdot p_b - m_b)\psi_m(p_a, p_b)$$

where, for simplicity, we only display the e_a contribution.

Concentrating on the p_b^o integration,[10]

$$- 2\pi i \int_{-\infty}^{\infty} dp_b^o \, \bar{\psi}_n (p'_a, p_b) \, \gamma_\mu^{(a)} (\gamma^{(b)} \cdot p_b - m_b) \, \psi_m (p_a, p_b)$$

$$= \sum_{s_a s_b s'_a} \eta^{n\dagger}_{s'_a, s_b} (\vec{p}', \vec{P}') \, \gamma_o^{(a)} \, \gamma_\mu^{(a)} \, \eta^{m}_{s_a s_b} (\vec{p}, \vec{P}) \tag{2.23}$$

$$\cdot \int_{-\infty}^{\infty} dp_o \, \frac{(P_o^n - s'_a E'_a - s_b E_b)(P_o^m - s_a E_a - s_b E_b)(-1/2\pi i)}{(\tau_a P_o^n - s'_a E'_a + p'_o + i\delta s'_a)(\tau_b P_o^n - s_b E_b - p'_o + i\delta s_b)(\tau_a P_o^m - s_a E_a + p_o + i\delta s_a)}$$

where

$$p'_o - p_o = \tau_b (p'_{ao} - p_{ao}) = \tau_b (P_o^n - P_o^m) \, .$$

The integral in (2.23) equals s_a if $s_a = s'_a = s_b$, equals 0 if $s_a = s'_a = - s_b$, and leads to terms proportional to the binding potential $g(\vec{x})$ for $s_a \neq s'_a$. The complete result for the B S equation in ladder approximation with an instantaneous potential is

$$\int d^3x \, < n | \mathcal{H}_I (\vec{x}, o) | m > \equiv \int d^3x_a d^3x_b \varphi^\dagger_n (\vec{x_a}, \vec{x_b}) \, H^{em}_{BS} \, \varphi_m (\vec{x_a}, \vec{x_b}) \tag{2.24a}$$

where

$$H_{BS}^{em} = \Lambda_+^{(a)} e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A(\vec{x}_a) \Lambda_+^{(a)} - \Lambda_-^{(a)} e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A(\vec{x}_a) \Lambda_-^{(a)}$$

$$+ \left[\Lambda_+^{(a)} e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A(\vec{x}_a) \Lambda_-^{(a)} \frac{1}{P_o^m - P_o^n + E_a + E'_a} \gamma_o^{(a)} \gamma_o^{(b)} g(\vec{x}) + (\text{h.c.}, n \longleftrightarrow m) \right] \quad (2.24b)$$

$$- \left[\Lambda_-^{(a)} e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A(\vec{x}_a) \Lambda_+^{(a)} \frac{1}{P_o^n - P_o^m + E_a + E'_a} \gamma_o^{(a)} \gamma_o^{(b)} g(\vec{x}) + (\text{h.c.}, n \longleftrightarrow m) \right]$$

$$+ (a \rightarrow b).$$

The terms including the binding potential in H_{BS}^{em} insure that the correct external electromagnetic interaction is obtained in the static limit, where the results of the Dirac equation must apply.[11] Note that for $m_b \rightarrow \infty$, the ladder approximation Salpeter equation with no external field reduces to

$$[E - (\vec{\alpha}_a \cdot \vec{p}_a + \beta_a m_a)] \varphi_S = \Lambda_+^a V(x_a) \varphi_S \quad (2.25a)$$

where

$$\gamma_o^{(a)} \gamma_o^{(b)} g(\vec{x}) \rightarrow V(\vec{x}_a)$$

instead of the Dirac equation

$$[E - (\vec{\alpha}_a \cdot \vec{p}_a + \beta_a m_a)] \varphi_D = V \varphi_D. \quad (2.25b)$$

It is easy to check that the terms in the second line of (2.24b) insure that to first order in the external field the Salpeter formalism in the static limit gives the same result as the Dirac theory

$$(\varphi_S, H_{BS}^{em} \varphi_S) = (\varphi_D, e_a \gamma_o^{(a)} \gamma_a \cdot A_a \varphi_D) \quad (2.26)$$

through contributions linear in $g(\vec{x})$. The addition of the cross-graph kernels in the Bethe-Salpeter equation are required to establish (2.26) to all orders in the binding potential. The relationship of (2.24b) to results obtained from a Breit equation approach is discussed in the next subsection.

E. The Extended Bethe-Salpeter Equation

An alternative way of introducing electromagnetic interactions in the treatment of composite systems is to use the minimal substitution

$$\begin{aligned} i\partial_\mu^{(a)} &\to i\partial_\mu^{(a)} - e_a A_\mu(x_a) \\ i\partial_\mu^{(b)} &\to i\partial_\mu^{(b)} - e_b A_\mu(x_b) \end{aligned} \tag{2.27}$$

in the BS equation. The resulting "extended" BS equation has in fact been derived by Schwinger (9) from Lagrangian field theory for the case of a time-independent (or adiabatic) external field $A_\mu(\vec{x})$. Although this approach is not directly applicable to Compton scattering, it does have a natural application to the study of the Zeeman effect in Section 3, and it allows us to make comparisons with the simpler Breit equation approach.

The extended BS equation in ladder approximation is

$$\left(\not{p}_a - e_a \not{A}_a - m_a\right)\left(\not{p}_b - e_b \not{A}_b - m_b\right)\psi\left(p_a, p_b\right) = G\psi\left(p_a, p_b\right) \tag{2.28}$$

where

$$\begin{aligned} \not{A}_a \psi\left(p_a, p_b\right) &= 2\pi \int d^4 p'_a \; \delta\left(p_a^o - p_a^{o'}\right) \gamma^{(a)} \cdot A\left(\vec{p}_a - \vec{p}'_a\right) \psi\left(p'_a, p_b\right) \\ G\psi\left(p_a, p_b\right) &= \int d^4 p' \, d^4 P' \delta^4\left(P - P'\right) G\left(p - p'\right) \psi\left(p'_a, p'_b\right) \end{aligned} \tag{2.29}$$

One can verify that perturbation theory (8) for (2.28) and the result (2.12) both give the same current and interaction energy to first order in e_a and e_b. If

self-energy corrections and other irreducible kernels are included in the BS equation, then the extended equation (2.28) will be correspondingly modified by additional terms as a consequence of the substitution (2.27) and gauge invariance. In particular, anomalous moment and form factor contributions arise when self-energy corrections to the fermion lines are included in the BS equation, in analogy to our discussion after (2.13).

In the presence of an external field $A^{\mu}(\vec{x})$, we can derive, for an instantaneous kernel, an extended Salpeter equation. We proceed exactly as in Eqs. (2.16) through (2.21), except that we use the following "Furry" projection operators

$$\Lambda^{a}_{\pm}(\pi_a) = \frac{1}{2} \pm \frac{H_a(\pi_a)}{2E_a(\pi_a)} \tag{2.30a}$$

where

$$\pi^{\mu}_a \equiv \left(p_a - e_a A_a\right)^{\mu}, \quad A^{\mu}_a \equiv A^{\mu}\left(\vec{x}_a\right)$$

$$H_a(\pi_a) \equiv \vec{\alpha}_a \cdot \vec{\pi}_a + e_a A^{o}_a + \beta_a m_a \tag{2.30b}$$

$$\frac{1}{E_a(\pi_a)} \equiv \frac{1}{+\sqrt{H_a(\pi_a)^2}} \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dx}{x^2 + H_a(\pi_a)^2} \ .$$

The last identity is quite useful; it enables one to expand the "Furry" projection operator to arbitrary order in the external potential.

The resulting "extended Salpeter equation" has a form analogous to (2.20):

$$\left[P_o - H_a(\pi_a) - H_b(\pi_b)\right]\varphi^{P_o} = \left[\Lambda^{a}_{+}(\pi_a)\Lambda^{b}_{+}(\pi_b) - \Lambda^{a}_{-}(\pi_a)\Lambda^{b}_{-}(\pi_b)\right]\tilde{g}\varphi^{P_o} \tag{2.31}$$

where $\tilde{g} \equiv \gamma_o^{(a)} \gamma_o^{(b)} g$. If we expand the "Furry" projection operators through order e_a, e_b, and $e_a e_b$, we obtain

$$\left[P_o - H_a\left(\vec{p}_a\right) - H_b\left(\vec{p}_b\right)\right]\varphi^{P_o} = \left[\Lambda_+^a\left(\vec{p}_a\right)\Lambda_+^b\left(\vec{p}_b\right) - \Lambda_-^a\left(\vec{p}_a\right)\Lambda_-^b\left(\vec{p}_b\right)\right] \tilde{g}\varphi^{P_o}$$

$$+ \left\{ e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A_a \varphi^{P^o} + \Lambda_-^a\left(\vec{p}_a\right) \frac{e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A_a}{E_a + E'_a} \Lambda_+^a\left(\vec{p}_a\right)\tilde{g}\varphi^{P_o} \right.$$

$$\left. + \Lambda_+^a\left(\vec{p}_a\right)\frac{e_a \gamma_o^{(a)} \gamma^{(a)} \cdot A_a}{E_a + E'_a} \Lambda_-^a\left(\vec{p}_a\right)\tilde{g}\varphi^{P_o} \right\} + \left\{ a \rightarrow b \right\} \qquad (2.32)$$

where $E_a \equiv \sqrt{\vec{p}_a^{\,2} + m_a^2}$ acts to the right and E'_a acts to the left of A_a. Equation (2.32) can also be obtained without operator manipulation from (2.28) by separating terms containing A^μ at the start and using the ordinary projection operators $\Lambda_\pm^s\left(\vec{p}_s\right)$ in the Salpeter reduction. In this method, we just repeat Eq. (2.19), except that Γ' now contains the A^μ terms:

$$\Gamma \rightarrow \Gamma' \equiv \left[\tilde{g} + e_a \not{A}_a\left(\not{p}_b - m_b\right) + e_b \not{A}_b\left(\not{p}_a - m_a\right) - e_a \not{A}_a e_b \not{A}_b\right]\psi \ . \qquad (2.33)$$

It is interesting to compare these two methods of derivation of Eq. (2.32) when contributions of order $e_a e_b$ are considered. In the method based on (2.33), $e_a e_b$ terms arise from the $e_a \not{A}_a\left(\not{p}_b - m_b\right)\psi$ contribution with ψ expanded to first order in e_b, and vice-versa, $\left(a \leftrightarrow b\right)$, as well as from the explicit $e_a e_b \not{A}_a \not{A}_b \psi$ term. If ψ is expanded using the <u>free</u> two body propagator $\left[\left(\not{p}_a - m_a\right)\left(\not{p}_b - m_b\right)\right]^{-1}$, then the $e_a e_b$ contributions cancel. Thus, as in the method based on (2.31), there is no explicit $e_a e_b$ contribution required in (2.32). Indeed, the $e_a e_b$ contribution which is zeroth order in g <u>must</u> vanish because there is no

$e_a e_b$ contribution to the total energy shift of two free particles in an external field. This rule emerges directly from (2.32) as a consequence of omitting the initial state from the sum over intermediate states in second order perturbation theory.

The energy shift due to the external field may be calculated from (2.32) by means of perturbation theory and compared with the Mandlestam result (2.24). We mention two unconventional aspects of the perturbation calculation: (1) The normalization condition of the unperturbed wavefunctions is

$$\int |\varphi_{++}|^2 - |\varphi_{--}|^2 \, d^3p_a \, d^3p_b = 1.$$

(2) The binding potential in (2.32) is non-hermitian. Condition (1) guarantees agreement with the change in sign of the φ_{--} contribution in (2.24b)[11]; condition (2) gives rise to extra contributions to the perturbation theory energy shift, which corresponds to the (h.c.) terms in (2.24b).

Thus the two derivations of the energy shift do agree, but both calculations become awkward and difficult to extend to higher order in e_a and e_b, and--because of the omission of crossed-graph kernels in the ladder approximation treatment--also difficult to extend to higher order in the binding potential. This should be contrasted with the Breit equation (7) extended via (2.25):

$$\left[E - H_a(\vec{\pi}_a) - H_b(\vec{\pi}_b)\right]\varphi_B = \tilde{g}\varphi_B \qquad (2.34)$$

which has none of the above difficulties. Although the Breit equation is an approximate formalism we note the following result: <u>The Bethe-Salpeter energy shift (2.24) is the same as that obtained from the Breit equation (2.34) to first order in the binding.</u> This is readily verified by expanding the equation for φ^B_{++} to first order in $A_\mu(x)$ and $\tilde{g}$. The result is the same equation which follows for

φ_{++} from (2.32). In fact, since Eq. (2.34) is exact in the static limit, the energy shift due to an external field A_μ as calculated from the Breit equation is correct except for terms of order $\langle e\gamma \cdot A \frac{\tilde{g}^2}{m_a m_b} \rangle$.

We should, however, remind the reader that the Breit equation does not have correct charge conjugation properties. The sign of the charges must be changed by hand to describe the bound states of antifermions ($\varphi \sim \varphi_{--}$). Of course, in the case of weak binding of fermions the φ_{--} amplitudes (Breit or Salpeter) are of little consequence. These components correspond to the amplitude for finding both fermions in free negative energy states and are suppressed by two powers of the binding energy divided by the total mass.[11]

Thus we finally obtain the "impulse" approximation result (2.2), but only after

(1) adopting the Breit equation description --with errors in the interaction energy of relative order $\langle \tilde{g}^2 / m_a m_b \rangle$;

(2) neglecting non-instantaneous terms in the kernel, in particular the self-energy graphs; and

(3) neglecting charged boson "exchange" currents.

All of these approximations can be reasonable in practice. Corrections can be made for the neglect of contributions from (1), (2), or (3). In particular, inclusion of self-energy graphs in the BS kernel can be partially taken into account by the usual form factor modifications of (2.24). The practical effect of contributions from (1) and (2) in the theory of the atomic Zeeman effect is discussed in Section 5. Corrections analogous to (2) and (3) in the relativistic treatment of the electromagnetic interaction of the deuteron have been discussed by Gross (12).

Section 3

The Exact Zeeman Effect in Hydrogen-Like Atoms

In this section we discuss the sources and importance of radiative and higher order reduced mass corrections to the atomic Zeeman spectrum. We concern ourselves here with determining the magnitude of the contributions not included in standard treatments (13-15) of the Zeeman effect in hydrogen, which are based on an additive Dirac Hamiltonian, Eq. (2.2). Our aim is to understand the Zeeman spectrum to an intrinsic accuracy of 1 ppm.

This study is of more than historical or academic interest. For example, recent level crossing measurements (16) of the hydrogen fine structure have determined the fine structure constant to a few ppm. The analysis depends on the extrapolation of the experimental results from rather large magnetic fields to zero magnetic field. Similarly, the results quoted for the Lamb shift (17) require precise understanding of the Zeeman effect. It is especially worthwhile to be critical of the usual analyses in view of the serious disagreement of the Lamb shift measurements and the theoretical predictions.

Let us now analyze what would constitute an exact treatment of the Zeeman spectrum of hydrogen. As stated in Section 2, the energy levels of the unperturbed H atom are determined to arbitrary accuracy by the BS equation if one includes the prescribed irreducible kernels G_i (see Fig. 1). The energy levels of the H atom in a given static (or adiabatic) external field are in turn defined by the extended BS equation (2.26). In general, with the application of the field $A_\mu(x)$, the kernels are modified

$$G_i \rightarrow G_i^A$$

through (2.27), corresponding to insertion of photons on each internal charged line. Thus, extra contributions to the Zeeman interaction are obtained from all kernels G_i except the one-photon-exchange kernel.

Our reduction sequence is as follows: We shall first itemize the contribution of the higher order kernels and demonstrate explicitly the origin of anomalous moment interactions. We next check the corrections due to the reduction of (2.28) to the extended Salpeter equation (2.31) and then to the two component Hamiltonian (1.1). Finally, we compare our results with the work of ref. (13) and (15). Throughout we make use of the available small expansion parameters, especially α and m_e/M_p. Our purpose is to determine the response of the atomic system to an external magnetic field. We do not discuss the line shape.

A. The Contribution of Higher Order Kernels to the Zeeman Spectrum

It is convenient to examine the contribution of currents induced from the non-ladder kernels by first taking the limit $m_e/M_p \to 0$. In this limit (8) the ladder and crossed photon graphs give the Dirac equation for an electron in the static field of the nucleus plus the external field. The self-energy and vacuum polarization kernels then correspond to the electrodynamic corrections to the bound state equation in this limit. The formalism in ref. (10) is particularly useful here in determining the dependence on the external field of the radiative corrections to the energy levels of the electron, since the level shifts are calculated in (10) as an expansion in the total field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The result may be written in the form

$$\Delta E_n = \Delta E_n(L) + \Delta E_n(M) + \Delta E_n(R)$$

where

$$\Delta E_n(L) = \frac{-2\alpha}{3\pi m^2} < n | \vec{p} \cdot (\ell n \frac{m}{2(H_{NR} - E_n)} + \frac{11}{24}) \gamma_0 [\vec{p}, eA] | n >$$

$$\Delta E_n(M) = \frac{\alpha}{2\pi} (\frac{-e}{2m}) < n | \tfrac{1}{2} \gamma_0 \sigma_{\mu\nu} F^{\mu\nu} | n >$$

and $\Delta E_n(R)$ contains terms explicitly quadratic in $F_{\mu\nu}$ as well as terms which modify the operators in M and L at small distances. Here H_{NR} is the non-relativistic Hamiltonian for the electron in the total field. For corrections to the Zeeman spectrum, we are interested in the dependence of ΔE_n on the external magnetic field. When the part of $F^{\mu\nu}$ corresponding to $\vec{H}$ is inserted in $\Delta E_n(M)$, we obtain the contribution of the anamolous moment of the electron to order α. The remainder of the dependence of ΔE_n (as reflected in changes in the binding energy and wavefunction of $| n >$) for a static field $\vec{H}$ is readily found (15) to be of order $\alpha(Z\alpha)^2 \mu_e H$. The vacuum polarization loop level shift is unchanged to first order in $\mu_e H$, because of Furry's theorem.

We can now relax the requirement $m_e/M_p = 0$ and thus obtain reduced mass corrections to the radiative level shifts. If we use the reduced mass dependence of the Schrödinger equation for $| n >$ in ΔE_n, the resulting corrections to the Zeeman spectrum are of order

$$\alpha(Z\alpha)^2 (m_e/M_p) \mu_e H ,$$

the $\vec{\sigma} \cdot \vec{H}$ operator in $\Delta E_n(M)$ having no direct reduced mass correction.

We also note that the extra contribution of the crossed graph kernel will be of order

$$(Z\alpha)^4 (m_e/M_p)\mu_e H$$

since its effects of order $(Z\alpha)^4 \mu_e H$ have already been included in the Dirac equation when $m_e/M_p \to 0$. (8)

We next consider the correction to the Zeeman spectrum accompanying the replacement of the one photon kernel by an instantaneous potential. It is readily seen to be of order $(Z\alpha)^4 (m_e/M_p)\mu_e H$, since this correction does not appear in first Born approximation or for infinite proton mass. Also, if the atom is moving with velocity $\vec{V}$ with respect to the external magnetic field, we can have a binding correction of order $\mu_e H(Z\alpha)^2 \vec{V}^2$. [8]

B. Other Corrections

We now have examined all the corrections to the Zeeman spectrum due to the assumption of an instantaneous potential in ladder approximation. The extended Salpeter equation is now applicable and the resulting interaction form expressed in (2.24), augmented by anamolous moment terms, is justified.

Finally we replace the extended Salpeter equation by the extended Breit equation. As shown in Section 2, the error made in using the Breit formalism for the electromagnetic interactions of hydrogenic atoms is only of order $\frac{W^2 \mu_e H}{m_a m_b} \sim (Z\alpha)^4 (m_e/M_p)\mu_e H$. The φ_{--} amplitudes may be discarded since they are suppressed by a factor $(Z^2\alpha^2 m_e)^2/M_p^2$. The Hamiltonian (2.2), which is in fact the interaction Hamiltonian of the Breit equation, may thus be used for an analysis of the Zeeman spectrum which is to be accurate to 1 ppm. The approximations

are summarized in Table I. We also note that in accordance with the rule expressed in the last part of Section 2, cross terms in the electron and proton interaction with the external field are to be neglected in the zero binding limit.

In fact the Hamiltonian (2.2) and the Breit formalism were used in the analyses (13-15) of the Zeeman spectrum necessary for the critical $n = 2$ measurements. In particular, Dirac wavefunctions with reduced mass corrections are sufficiently accurate; and quadratic terms in the Pauli reduction, as well as $\Delta n \neq 0$ contribution, can be ignored. The magnetic field dependence of the energy levels (line centers) of a stationary hydrogen atom in a uniform magnetic field may thus be determined to an intrinsic accuracy of 1 ppm.

Finally, we must consider the experimental effects in atomic beam measurements due to the revised spin orbit terms in the reduction of the interaction of external fields to Pauli form as obtained in the next sections. [See especially Eq. (1.1).] The spin-independent terms are unchanged, hence the external field interactions (13) of the orbital motion and the non-spin effects of the motional electric field (motional Stark effect) will not be modified. On the other hand, measurements correlating the atomic spin with an external field (static, oscillatory, or motional), such as spin-dependent Stark effects, electric perturbation of hyperfine levels, polarized target measurements of electron scattering, or polarized atoms in a strong electric field, will be sensitive to the corrected spin-orbit terms. These electric field effects are negligible in the experiments of ref. (13) and (14), since they are reduced by a factor of m_e/M_p compared to the usual non-spin Stark contribution.

Section 4

An Approximate Wavefunction and its Relativistic Transformation

In order to calculate matrix elements needed in Sections 5 and 6, we require a wavefunction which is valid in the "loose binding" approximation - i.e., accurate to first order in the binding potential U and in the squared relative momentum $\vec{p}^2 << m_a^2, m_b^2$. We will construct such a wavefunction by solving a relativistic equation (2.20) to the required accuracy in the CM frame, and then transforming the resulting wavefunction to any desired frame[13]. As in Section 2, we will discuss explicitly only the case of two spin $\frac{1}{2}$ particles.

It will be adequate[14] for our purposes to drop the projection operators in (2.20). The resulting "Breit" equation takes the following form in the CM frame

$$(\vec{\alpha}^a \cdot \vec{p} + \beta^a m_a - \vec{\alpha}^b \cdot \vec{p} + \beta^b m_b + U - \mathcal{M}) \varphi_m = 0 \tag{4.1}$$

where, in momentum space, U is the integral operator

$$(U \varphi_m)(\vec{p}) = \int d^3p' \, \tilde{g}(\vec{p} - \vec{p}') \, \varphi_m(\vec{p}') \tag{4.2}$$

and we assume for simplicity that $\tilde{g}$ contains no Dirac matrices[15]. Since φ_m will reduce to a product of free positive-energy Dirac spinors in the limit of zero binding, we will attempt to find a solution which is of the form

$$\varphi_m(\vec{p}) = \begin{pmatrix} 1 \\ \omega_a \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \omega_b \end{pmatrix} \phi_m(\vec{p}) \chi_{SM}, \tag{4.3}$$

where $\omega_{a,b}$ is a 2×2 matrix and a function of $\vec{p}$ and $\vec{\sigma}_{a,b}$, $\phi_{\mathcal{M}}(\vec{p})$ is a one-component function, and $\chi_{SM}(S=1, 0)$ is a constant spinor[16]. It is useful to define

$$U - \mathcal{M} \equiv U - (m_a + m_b - W) = - (m_a + k_a) - (m_b + k_b),$$

$$k_{a,b} \equiv - \tau_{b,a}(U + W), \qquad \tau_{a,b} \equiv m_{a,b}/(m_a + m_b). \tag{4.4}$$

W is the binding energy and $k_{a,b}$ is a kinetic energy operator (for example, in the limit of zero binding, $k_{a,b} = \vec{p}^2/2m_{a,b} + 0(\vec{p}^4/m^3)$).

In terms of these quantities, Eq. (4.1) becomes

$$\left[\begin{pmatrix} \vec{\sigma}_a \cdot \vec{p}\,\omega_a - k_a \\ \vec{\sigma}_a \cdot \vec{p} - (2m_a + k_a)\omega_a \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \omega_b \end{pmatrix} + \begin{pmatrix} 1 \\ \omega_a \end{pmatrix} \otimes \begin{pmatrix} -\vec{\sigma}_b \cdot \vec{p}\,\omega_b - k_b \\ -\vec{\sigma}_b \cdot \vec{p} - (2m_b + k_b)\omega_b \end{pmatrix} \right] \phi_{\mathcal{M}}(\vec{p}) = 0. \tag{4.5}$$

This equation is satisfied if we take

$$\omega_a = \frac{1}{2m_a + k_a}\, \vec{\sigma}_a \cdot \vec{p}\,, \qquad \omega_b = - \frac{1}{2m_b + k_b}\, \vec{\sigma}_b \cdot \vec{p}\,, \tag{4.6}$$

and if $\phi_{\mathcal{M}}$ satisfies the following equation:

$$\left[\vec{\sigma}_a \cdot \vec{p}\, \frac{1}{2m_a + k_a}\, \vec{\sigma}_a \cdot \vec{p} + \vec{\sigma}_b \cdot \vec{p}\, \frac{1}{2m_b + k_b}\, \vec{\sigma}_b \cdot \vec{p} + U + W \right] \phi_{\mathcal{M}} = 0. \tag{4.7}$$

If we drop spin-orbit and other relativistic terms, Eq. (4.7) reduces to the two-body Schrödinger equation in the CM frame:

$$(\vec{p}^{\,2}/2m_r + U + W)\phi_{\mathcal{M}} = 0, \tag{4.7'}$$

where $m_r \equiv m_a m_b/(m_a + m_b)$.

The CM solution for total spin S and projection M takes the following form in position space:

$$\varphi_{\mathcal{M}}(\vec{x}_a, \vec{x}_b, X^o)_{SM} = \tag{4.8}$$

$$\int \frac{d^3p}{(2\pi)^{3/2}} \left(\frac{p_a^o + m_a}{2p_a^o} \; \frac{p_b^o + m_b}{2p_b^o}\right)^{\frac{1}{2}} \left(\frac{1}{\vec{\sigma}_a \cdot \vec{p}}{2m_a + k_a}\right) \otimes \left(\frac{1}{-\frac{\vec{\sigma}_b \cdot \vec{p}}{2m_b + k_b}}\right) \phi_{\mathcal{M}}(\vec{p}) \chi_{SM} e^{i\vec{p}\cdot\vec{x} - i\mathcal{M}X^o}$$

where $x \equiv x_a - x_b$, $X \equiv \tau_a x_a + \tau_b x_b$, and $p^o_{a,b} \equiv \sqrt{\vec{p}^{\,2} + m^2_{a,b}}$. Eq. (4.8) is written so that the normalization condition[17] compatible with Eq. (2.24), namely

$$\int d^3x_a \, d^3x_b \; \varphi_{\mathcal{M}}(\vec{x}_a, \vec{x}_b)^{\dagger} (\Lambda_{++} - \Lambda_{--}) \varphi_{\mathcal{M}}(\vec{x}_a, \vec{x}_b) = 1, \tag{4.9}$$

is satisfied if

$$\int d^3p \; | \; \phi_{\mathcal{M}}(\vec{p}) \; |^2 = 1.$$

In the matrix element of the interaction with an external field, the initial and final states will in general have different total momenta; it will consequently be necessary to know how to transform the CM wavefunction to an

arbitrary reference frame. In order to learn the transformation properties of the BS wavefunction, we return to the definition

$$\chi^{\alpha\beta}_{m}(x_a, x_b)_{SM} = < 0 \mid T(\psi^{\alpha}_{a}(x_a)\psi^{\beta}_{b}(x_b)) \mid \vec{0}mSM > .$$

Since a Lorentz transformation leaves the vacuum invariant, $U(\Lambda) \mid 0 > = \mid 0 >$,

$$\chi^{\alpha\beta}_{m}(x_a, x_b) = < 0 \mid U(\Lambda)T(\psi^{\alpha}_{a}(x_a), \psi^{\beta}_{b}(x_b)U(\Lambda^{-1})U(\Lambda) \mid \vec{0}mSM >$$

$$= \sum_{M'\alpha'\beta'} S^{-1}_{a}(\Lambda)^{\alpha\alpha'} S^{-1}_{b}(\Lambda)^{\beta\beta'} < 0 \mid T(\psi^{\alpha'}_{a}(x'_a), \psi^{\beta'}(x'_b)) \mid \vec{P}ESM' > \mathcal{D}^{S}_{M'M}(R_W) \tag{4.10}$$

where $x' = \Lambda x$, $(E, \vec{P}) = \Lambda(m, \vec{0})$, $S(\Lambda)$ is the usual spinor transformation matrix, and $\mathcal{D}^{S}_{M'M}(R_W)$ is the Wigner rotation matrix, which equals $\delta_{M'M}$ here since the initial state is at rest. Inverting (4.10) gives the required transformation law

$$\chi^{\alpha'\beta'}_{E\vec{P}}(x'_a, x'_b)_{SM} = S^{\alpha'\alpha}_{a}(\Lambda)\, S^{\beta'\beta}_{b}(\Lambda)\, \chi^{\alpha\beta}_{m}(x_a, x_b)_{SM}. \tag{4.11}$$

An explicit form for $S(\Lambda)$ is

$$S_a(\Lambda) = \exp\left(\tfrac{1}{2}\vec{\alpha}_a \cdot \vec{V} \tanh^{-1} \mid \vec{V} \mid\right) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{\vec{\alpha}_a \cdot \vec{P}}{m+E}\right), \tag{4.12}$$

where $\vec{V} = \vec{P}/E$. With $\gamma = (1-V^2)^{-\frac{1}{2}}$, the Lorentz transformation on x_a is

$$\vec{x}_a = \vec{x}'_a + (\gamma - 1)\vec{x}'_a \cdot \hat{V}\hat{V} - \gamma \vec{V} x_a^{o'} \equiv \vec{\tilde{x}}_a - \gamma \vec{V} x_a^{o'}, \quad x_a^o = \gamma(x_a^{o'} - \vec{V}\cdot\vec{x}'_a). \tag{4.13}$$

Now Eq. (2.19), which we apply in the transformed reference frame as well as the CM frane, implies the following transformation law for φ :

$$\begin{aligned} \varphi_{E\vec{P}}(\vec{x}', \vec{X}', X^{o'}) &= \chi_{E\vec{P}}(x^{o'} = 0, \vec{x}'; X^{o'}, \vec{X}') \\ &= S_a(\Lambda)\, S_b(\Lambda)\, \chi_m(-\gamma \vec{V}\cdot\vec{x}', \vec{\tilde{x}}'; \gamma X^{o'} - \gamma \vec{V}\cdot\vec{X}', \vec{\tilde{X}}' - \gamma \vec{V} X^{o'}) . \end{aligned} \tag{4.14}$$

As might be expected, the equal-time ($x^{o'} = 0$) wavefunction in the new reference frame corresponds to an unequal-time wavefunction in the CM frame. However, the dependence of χ_m on the CM relative time $x^o = -\gamma \vec{V}\cdot\vec{x}'$ is completely determined by Eq. (2.21); for small $\vec{V}$ we are justified in neglecting x^o. [18] Then (omitting primes)

$$\varphi_{E\vec{P}}(\vec{x}_a, \vec{x}_b, X^o)_{SM} = \frac{E + m}{2m} \int \frac{d^3p}{(2\pi)^{3/2}} \left(\frac{p_a^o + m_a}{2p_a^o} \, \frac{p_b^o + m_b}{2p_b^o} \right)^{\frac{1}{2}} \tag{4.15}$$

$$\begin{pmatrix} 1 + \dfrac{\vec{\sigma}_a \cdot \vec{P}}{m + E} \, \dfrac{\vec{\sigma}_a \cdot \vec{p}}{2m_a + k_a} \\ \vec{\sigma}_a \cdot \left(\dfrac{\vec{P}}{m + E} + \dfrac{\vec{p}}{2m_a + k_a} \right) \end{pmatrix} \otimes \begin{pmatrix} 1 - \dfrac{\vec{\sigma}_b \cdot \vec{P}}{m + E} \, \dfrac{\vec{\sigma}_b \cdot \vec{p}}{2m_b + k_b} \\ \vec{\sigma}_b \cdot \left(\dfrac{\vec{P}}{m + E} - \dfrac{\vec{p}}{2m_b + k_b} \right) \end{pmatrix}$$

$$\phi_m(\vec{p}) \chi_{SM} e^{i\vec{p}\cdot\vec{\tilde{x}} + i\vec{P}\cdot\vec{X}} \, e^{-iEX^o}.$$

Here $\vec{\tilde{x}} = \vec{x} + (\gamma - 1)\hat{V}\hat{V}\cdot\vec{x}$ includes the Lorentz-Fitzgerald contraction of the wave function. Again, $p^o_{a,b} = \sqrt{\vec{p}^2 + m^2_{a,b}}$.

We will have occasion in Sections 5 and 6 to evaluate matrix elements of commutators like $[X^i, P^j] = i\delta_{ij}$. In order to avoid confusion, we can imagine the physical state to be a wave packet, constructed by superposition of an arbitrarily small range of momentum eigenstates:

$$\varphi(\vec{x}_a, \vec{x}_b, X^o) = \int \frac{d^3P}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}}\, \Phi(\vec{P})\, \varphi_{E\vec{P}}(\vec{x}_a, \vec{x}_b, X^o). \tag{4.16}$$

This is done in Eqs. (5.8), (5.29). This wavefunction will be properly normalized if

$$\int d^3P\, |\Phi(\vec{P})|^2 = 1.$$

Section 5

The Drell-Hearn-Gerasimov Integral

In this section we will show in some detail how the formalism of Section 2 provides a framework for a correct calculation of the electromagnetic interactions of composite particles. In Section 5A, we present an explicit derivation of the DHG sum rule for a spin $\frac{1}{2}$ bound state composed of a spin $\frac{1}{2}$ and a spin 0 particle; in Section 5B, the same system is treated more elegantly. The former derivation is in the spirit of numerous atomic and nuclear physics calculations, and it shows how such treatments must be modified when relativistic effects are included properly. The second derivation is simpler and explicitly relativistic, and in addition it exhibits especially clearly the superconvergent nature of the DHG sum rule. In Section 5C, the same techniques are applied to a spin 1 model consisting of two fermions.

A. A Spin $\frac{1}{2}$ Composite System

Consider any system of total angular momentum $\frac{1}{2}$, charge $Z_T e$, mass m, and magnetic moment μ. The DHG sum rule (1,2) reads

$$\int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega) - \sigma_A(\omega)}{\omega}\, d\omega = 8\pi^2 \left(\mu - \frac{Z_T e}{2m}\right)^2 \tag{5.1}$$

Here $\sigma(\omega)$ is the total cross section for photoabsorption of circularly polarized light on a polarized target. The subscript P refers to the configuration where the photon helicity ρ and target spin are parallel, $(\rho, S_z) = (+1, +\frac{1}{2})$ or $(-1, -\frac{1}{2})$,

and A refers to the anti-parallel configuration $(\rho, S_z) = (-1, +\frac{1}{2})$ or $(+1, -\frac{1}{2})$, where z is the incident photon direction. The integral begins at the laboratory photon energy threshold ω_{th} for first-order electromagnetic processes. For an elementary particle, this is the threshold energy for photoproduction ω_{pp}; but for bound systems, ω_{th} is ordinarily the photoeffect threshold, equal to the energy difference ΔE_{12} between the ground and the first excited state plus the recoil energy:

$$\omega_{th} = \Delta E_{12} + \frac{(\Delta E_{12})^2}{2m} << \omega_{pp}.$$

The object of this subsection will be to show by explicit calculation that the DHG sum rule is satisfied for a loosely bound composite system consisting of a spin 0 particle and a spin $\frac{1}{2}$ Dirac particle in an S state.

We propose to calculate the DHG integral

$$I_{P,A} = \int_{\omega_{th}}^{\infty} \frac{\sigma_{P,A}(\omega)}{\omega} d\omega \tag{5.2}$$

by first order perturbation theory. Suppressing the polarization index, we must evaluate

$$I = 8\pi^2 \sum_{f > i} \frac{| < f | H^{em} | i > |^2}{\omega}, \tag{5.3}$$

where $| i >$ represents the spin $\frac{1}{2}$ ground state and the sum ranges over all discrete and continuum excited states $E_f - E_i \geq \omega_{th} > 0$. For the first order

electromagnetic interaction Hamiltonian we use

$$-H^{em} = Ze\,\vec{\alpha}\cdot\vec{A}(\vec{r}_p) + ze\,\vec{v}_\pi\cdot\vec{A}(\vec{r}_\pi), \tag{5.4}$$

with $\vec{v}_\pi = \dfrac{\vec{p}_\pi}{\sqrt{p_\pi^2+m^2}}$, $\vec{A}(\vec{r}) = \dfrac{\hat{\epsilon}}{\sqrt{2\omega}}\, e^{i\vec{k}\cdot\vec{r}}$, $|\hat{\epsilon}| = 1$, $\omega = E_f - E_i$. The rest of the notation is defined in Table II. Note that we have included no "exchange current" interaction in (5.4); throughout this paper we assume that the composite systems considered are bound by the exchange of neutral particles.

The unperturbed Hamiltonian for the two-body system is taken to be

$$H_0 = \vec{\alpha}\cdot\vec{p}_p + \beta M + \sqrt{\vec{p}_\pi^{\,2}+m^2} + U(\vec{r}_p - \vec{r}_\pi) \tag{5.5}$$

and we note the identity

$$\vec{\alpha} = i[H_0, \vec{r}_p]\ . \tag{5.6}$$

Then

$$\langle f|\,\vec{\alpha}\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_p}\,|i\rangle = \langle f|\, i[H_0, \vec{r}_p\cdot\hat{\epsilon}]\, e^{i\vec{k}\cdot\vec{r}_p}\,|i\rangle$$

$$= i\langle f|\,[H_0, \vec{r}_p\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_p}] - \vec{r}_p\cdot\hat{\epsilon}[H_0, e^{i\vec{k}\cdot\vec{r}_p}]\,|i\rangle \tag{5.7}$$

$$= i\omega\langle f|\,\vec{r}_p\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_p}(1-\vec{\alpha}\cdot\hat{k})\,|i\rangle$$

where we made essential use of energy conservation, $\omega = |\vec{k}| = E_f - E_i$, in the last step. This identity would be convenient for making a multipole expansion of the radiation interaction with the proton.

In order to parallel the usual atomic physics calculations[4], we will at this point reduce the matrix element to an approximate non-relativistic form. In doing this, we use the analogues of the equal-time BS wavefunctions derived in Section 4. We introduce the following notations for the relativistic (four-component) and "non-relativistic" (two-component) wavefunctions in position representation:

$$\varphi_i(\vec{r},\vec{R}) = \langle \vec{r},\vec{R} \mid i \rangle$$

$$\equiv \int \frac{d^3p d^3P}{(2\pi)^3} N_{p,P} \begin{pmatrix} 1 + \vec{\sigma}\cdot\vec{\underline{P}}\,\vec{\sigma}\cdot\vec{\underline{p}} \\ \vec{\sigma}\cdot(\vec{\underline{p}}+\vec{\underline{P}}) \end{pmatrix} \phi_i(\vec{p})\Phi_i(\vec{P})\, e^{i(\vec{\tilde{p}}\cdot\vec{r}+\vec{P}\cdot\vec{R})}\chi_i \tag{5.8a}$$

$$\langle \vec{r},\vec{R} \mid i \rangle \equiv \int \frac{d^3p d^3P}{(2\pi)^3} \phi_i(\vec{p})\Phi_i(\vec{P})\, e^{i(\vec{\tilde{p}}\cdot\vec{r}+\vec{P}\cdot\vec{R})}\chi_i \tag{5.8b}$$

where $\vec{\underline{p}} \equiv \frac{1}{2M+k_i}\vec{p}$, $\vec{\underline{P}} \equiv \frac{1}{2E_i+K_i}\vec{P}$, and $\vec{\tilde{p}} \equiv \vec{p} + (\gamma-1)\hat{V}\hat{V}\cdot\vec{p} \simeq \vec{p} + \frac{1}{2}\frac{\vec{P}}{M_T}\frac{\vec{P}}{M_T}\cdot\vec{p}$

appears because of the Lorentz-Fitzgerald contraction of the wavefunction. (For our purposes we can approximate $\vec{\tilde{p}}$ by $\vec{p}$.) The normalization factor is

$$N_{p,P} = 1 - \frac{\vec{p}^2}{8M^2} + \frac{\vec{P}^2}{8M_T^2} + 0\,(\text{momentum}^4).$$

See Section 4 for further details[19].

The reduction of the matrix element of $\vec{\alpha}\cdot\vec{A}(\vec{r}_p)$ proceeds as follows:

$$< f | \vec{\alpha}\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_p} | i > = i\omega \left(f | \vec{r}_p\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_p} \right\{ 1 + (\vec{p}+\vec{p}')\cdot(\vec{P}+\vec{P}') + \vec{p}\cdot\vec{p}' + \vec{P}\cdot\vec{P}' - \frac{\vec{p}^2+\vec{p}'^2}{8M^2} + \frac{\vec{P}^2+\vec{P}'^2}{8M_T^2}$$

$$+ i\vec{\sigma}\cdot\left[(\vec{p}'-\vec{p}) \times \frac{\vec{p}'+\vec{p}}{2} + (\vec{P}'-\vec{P}) \times \frac{\vec{P}'+\vec{P}}{2} + (\vec{p}'-\vec{p}) \times (\vec{P}'+\vec{P}) \right] - \hat{k}\cdot(\vec{p}+\vec{p}' + \vec{P}+\vec{P}')$$

$$+ i\,\vec{\sigma}\cdot\hat{k} \times \left[(\vec{p}'-\vec{p}) + (\vec{P}'-\vec{P}) \right] \Big\} | i) \qquad (5.9)$$

$$= i\omega \left(f | e^{i\vec{k}\cdot\vec{r}_p} \left[\vec{r}_p\cdot\hat{\epsilon} \left(1 - \hat{k}\cdot\frac{\vec{p}_p}{M} + \frac{\vec{p}_p^{\,2}}{2M^2} - \frac{\vec{p}_{av}^{\,2}}{2M^2} \right) + \frac{1}{2M}\,\vec{\sigma}\cdot\hat{k}\times\hat{\epsilon} \right.\right.$$

$$\left.\left. + \frac{1}{2M}\left(\frac{1}{2M} - \frac{1}{2M_T}\right)\vec{\sigma}\cdot\hat{\epsilon}\times\vec{p}_{av} + \frac{1}{2M_T}\left(\frac{2}{2M} - \frac{1}{2M_T}\right)\vec{\sigma}\cdot\hat{\epsilon}\times\vec{P}_{av} \right] | i \right).$$

In the second-to-last step we have dropped the higher-order $\vec{\sigma}\cdot\hat{k}\ 0(\text{momenta}^3)$ terms and in the last step we have integrated $(\vec{p}'-\vec{p})$ and $(\vec{P}'-\vec{P})$ by parts. Note that $\vec{r}_p = \vec{R} + (1 - \frac{M}{M_T})\vec{r}$ and that higher order terms in ω are omitted in the last step. These approximations are justified in the discussion following Eq. (5.13). For the average of the initial and final momenta we use the notation $\vec{p}_{av} = \frac{1}{2}\ (\vec{p}' + \vec{p})$, $\vec{P}_{av} = \frac{1}{2}\ (\vec{P}' + \vec{P})$, and we define $\vec{p}_p/M \equiv \vec{P}_{av}/M_T + \vec{p}_{av}/M$.

Let us identify the physical content of the terms in (5.9). The first term is obviously the electric dipole term $i\omega\vec{r}_p\cdot\hat{\epsilon} = (\vec{p}_p\cdot\hat{\epsilon})/M$. Before we discuss the second term, $-i\vec{r}_p\cdot\hat{\epsilon}\,\vec{p}_p\cdot\vec{k}/M$, we should add to it the contribution $-\omega\vec{r}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k} \simeq i(\vec{p}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k} + \vec{r}_p\cdot\hat{\epsilon}\,\vec{p}_p\cdot\vec{k})/M$, coming from the exponential factor. The sum is $i\vec{p}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k}$, the second term in the power-series expansion of $\vec{p}_p\cdot\hat{\epsilon}\ e^{i\vec{k}\cdot\vec{r}_p}$. Calculating non-relativistically, we identify this as a sum of electric quadrupole and orbital magnetic dipole terms:

$$\vec{p}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k} = \tfrac{1}{2}\ (\vec{p}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k} + \vec{r}_p\cdot\hat{\epsilon}\,\vec{p}_p\cdot\vec{k}) + \tfrac{1}{2}\ (\vec{p}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k} - \vec{r}_p\cdot\hat{\epsilon}\,\vec{p}_p\cdot\vec{k})$$

$$= \frac{M}{2}\,\frac{d}{dt}\,(\vec{r}_p\cdot\hat{\epsilon}\,\vec{r}_p\cdot\vec{k}) + \tfrac{1}{2}\ (\vec{k}\times\hat{\epsilon})\cdot(\vec{r}_p\times\vec{p}_p). \qquad (5.10)$$

The spin part of the magnetic dipole interaction is represented by the term $\frac{i}{2M}\ \vec{\sigma}\cdot\vec{k}\times\hat{\epsilon}$. Finally, the last "spin-orbit" terms contain the interaction with the motional magnetic field $-\vec{v}_p\times\vec{E}$ and the effects of the Thomas precession. These are the terms which would generate the spin-orbit interaction if the spin $\frac{1}{2}$ particle were bound about a center of force, instead of interacting with a radiation field. In fact, precisely the same spin-orbit terms

appear if we consider the matrix element of an electric potential $ZeA^0(\vec{r}_p)$, as they must by gauge invariance.

Turning our attention to the matrix element of $\vec{v}_\pi \cdot \vec{A}(\vec{r}_\pi)$, we calculate

$$\begin{aligned} &< f | \vec{v}_\pi \cdot \hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_\pi} | i > = < f | i[H_0, \vec{r}_\pi \cdot \hat{\epsilon}]\, e^{i\vec{k}\cdot\vec{r}_\pi} | i > \\ &= i\omega < f | \vec{r}_\pi \cdot \hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_\pi} (1 - \vec{v}_\pi \cdot \hat{k}) | i > \\ &= i\omega \Big(f \Big| e^{i\vec{k}\cdot\vec{r}_\pi} \Big\{ \vec{r}_\pi \cdot \hat{\epsilon} \left(1 - \vec{v}_\pi \cdot \hat{k} + \frac{\vec{p}_p^{\,2}}{2M^2} - \frac{\vec{p}_{av}^{\,2}}{2M^2} \right) \\ &\quad - \frac{1}{4MM_T} \vec{\sigma} \cdot \hat{\epsilon} \times \vec{p}_{av} - \frac{1}{4M_T^2} \vec{\sigma} \cdot \hat{\epsilon} \times \vec{P}_{av} \Big\} \Big| i \Big) . \end{aligned} \tag{5.11}$$

This matrix element also makes a contribution to the "spin-orbit" terms.

The total spin-dependent interaction coming from the matrix element $< f | Ze\vec{\alpha} \cdot \vec{A}(\vec{r}_p) + ze\vec{v}_\pi \cdot \vec{A}(\vec{r}_\pi) | i >$ is thus, to first order in the momenta,

$$-H_{spin}^{em} = \mu \vec{\sigma} \cdot \vec{B} + \left(2\mu - \frac{Ze}{2M} - \frac{Z_T e}{2M_T} \right) \vec{\sigma} \cdot \vec{E} \times \frac{\vec{p}_{av}}{2M} + \left(2\mu - \frac{Z_T e}{2M_T} \right) \vec{\sigma} \cdot \vec{E} \times \frac{\vec{P}_{av}}{2M_T} \tag{5.12}$$

where $\mu = Ze/2M$ is the magnetic moment of the Dirac particle, which is also equal to that of the system as a whole[20]. For completeness, we will also write down the leading spin-independent terms:

$$- H^{em}_{\text{spin-independent}} = \frac{Ze}{M}\left(\vec{p}_p \cdot \hat{\epsilon} + i\vec{p}_p \cdot \hat{\epsilon}\, \vec{r}_p \cdot \vec{k}\right)$$

$$+ \frac{ze}{m}\left(\vec{p}_\pi \cdot \hat{\epsilon} + i\vec{p}_\pi \cdot \hat{\epsilon}\, \vec{r}_\pi \cdot \vec{k}\right) \tag{5.13}$$

$$= \left(\frac{Z}{M} - \frac{z}{m}\right) e\vec{p}_{av} \cdot \hat{\epsilon} + \frac{Z_T e}{M_T} \vec{P}_{av} \cdot \hat{\epsilon} + \cdots .$$

It is appropriate here to discuss the relative sizes of the various terms we have considered. Our remarks will be valid for any sufficiently loosely bound system. The only requirement is that in the bound state, the velocity v of the particles is small compared to c. (In hydrogenic systems such as the "pionic atom" considered here, $v \simeq \alpha$.) For matrix elements involving bound or low-lying continuum states f and i, $\omega \equiv E_f - E_i \simeq m_r \alpha^2$, $r \simeq 1/m_r\alpha$, and $p \simeq m_r\alpha$, where $m_r = Mm/(M+m)$ is the reduced mass. It follows that for these matrix elements successive multipoles are smaller by a factor of $\vec{k}\cdot\vec{r} \simeq \alpha$. Relative to the electric dipole (El) term $i\omega\vec{r}_p \cdot \hat{\epsilon}$, the magnetic dipole (Ml) and electric quadrupole (E2) terms are smaller by a factor of α, and the "spin-orbit" (S0) terms are smaller by a factor of α^2, as are such spin-independent (SI2) terms as $i\omega\vec{r}\cdot\hat{\epsilon}\,(\vec{p}/M)^2$. For these remarks to be valid estimates of the contribution of these terms to the sum I in Eq. (5.3), we must argue that, for i = ground state, the higher continuum states f are unimportant in the sum. This is generally true for loosely bound systems because of the small overlap integral between the rapidly oscillating exponential of a high energy wavefunction and the smoothly spread wavefunction of a

low energy bound state. For example, for hydrogen the matrix element $|(f|\vec{r}|i)|$ is maximum for $E_f \simeq 0$ and decreases for large E_f faster than $(E_f - E_i)^{3/2}$ if non-relativistic wavefunctions are used[21].

Now let us consider which of the various terms of H^{em} can contribute to the integral in the DHG sum rules (5.1) to produce a result of first order in α. Using the estimates discussed above and recalling that $A \propto 1/\sqrt{2\omega}$, we have listed in Table III the contribution to I of each possible product of matrix elements through order α. Of the two terms which are not eliminated by general considerations, the $(M1)^2$ term vanishes because it can connect the ground state only with itself and not with any excited states. Thus for the model considered in this section, our non-relativistic calculation gives

$$I = 4\pi^2 e^2 \sum_{f>i} \frac{1}{\omega^2} \left[(i|\left(\frac{Z}{M} - \frac{z}{m}\right)\vec{p}\cdot\hat{\epsilon}^*|f)(f|\left(\frac{Z}{2M} - \frac{Z_T}{2M_T}\right)i\omega\vec{\sigma}\cdot\hat{\epsilon}\times\frac{\vec{p}}{2M}|i) + h.c.\right] \tag{5.14}$$

$$= 4\pi^2 e^2 \left(\frac{Z}{M} - \frac{z}{m}\right)\frac{mM}{M_T}\left(\frac{Z}{2M} - \frac{Z_T}{2M_T}\right)\frac{1}{2M}(i|\vec{r}\cdot\hat{\epsilon}^*\vec{\sigma}\cdot\hat{\epsilon}\times\vec{p} + \vec{\sigma}\cdot\hat{\epsilon}^*\times\vec{p}\,\vec{r}\cdot\hat{\epsilon}|i).$$

In obtaining this result we have dropped the $\vec{P}$ terms (which cannot contribute to matrix elements between i and an excited state f), used the non-relativistic identity $\vec{p} = i\frac{mM}{M_T}[H_0, \vec{r}]$, and used closure (we do not need to subtract the unexcited ground state contribution from the closure sum since the matrix element $(i|\vec{r}|i)$ vanishes by parity). In order to evaluate the remaining matrix element, we introduce the circular polarization basis $\hat{\epsilon}_\rho$, $\rho = \pm 1$, with the properties (recall that $\hat{k} = \hat{z}$)

$$\hat{\epsilon}_\rho \equiv - \rho \frac{\hat{x} + i\rho\hat{y}}{\sqrt{2}}, \quad \hat{\epsilon}^*_\rho \cdot \hat{\epsilon}_{\rho'} = \delta_{\rho,\rho'}, \quad \hat{z} \times \hat{\epsilon}_\rho = - i\rho\, \hat{\epsilon}_\rho, \quad \hat{\epsilon}^*_\rho \times \hat{\epsilon}_\rho = i\rho \hat{z}, \tag{5.15}$$

and designate the spin orientation of the ground state i by $\mu = \pm 1$:

$\sigma_z \mid i,\mu) = \mu \mid i,\mu)$. Then

$$(i,\mu \mid \vec{r} \cdot \hat{\epsilon}^*_\rho \vec{\sigma} \cdot \hat{\epsilon}_\rho \times \vec{p} + \vec{\sigma} \cdot \hat{\epsilon}_\rho \times \vec{p}\, \vec{r} \cdot \hat{\epsilon}_\rho \mid i,\mu)$$

$$= - i\mu\rho \; (i,\mu \mid \tfrac{1}{2} [x, p_x] + \tfrac{1}{2} [y, p_y] + i\rho L_z \mid i,\mu) = \mu\rho .$$

In the last step $L_z \mid i) = 0$ since $\mid i)$ is an S state. Finally,

$$\int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega) - \sigma_A(\omega)}{\omega} \, d\omega = I_P - I_A = 8\pi^2 e^2 \left(\frac{Z}{2M} - \frac{Z_T}{2M_T} \right)^2 \tag{5.16}$$

which agrees[22] with (1) for the model considered thus far, consisting of a Dirac particle ($\mu = Ze/2M$) and a zero spin particle.

If we wish to allow the spin $\frac{1}{2}$ constituent of our system to possess an anomalous magnetic moment λ so that $\mu = Ze/2M + \lambda$, we must introduce into Eq. (5.4) the Pauli term

$$\tfrac{1}{2} \lambda \beta \sigma_{\mu\nu} F^{\mu\nu}(\vec{r}_p) = \lambda\beta [\vec{\sigma} \cdot \vec{B}(\vec{r}_p) - i\vec{\alpha} \cdot \vec{E}(\vec{r}_p)] .$$

It is straightforward to verify that Eq. (5.12) remains correct, if we just

reinterpret μ in it as being the entire magnetic moment; the Thomas terms are unchanged. One might think, therefore, that the entire calculation presented above is unaffected. This is not correct, however; our non-relativistic treatment gives correctly only the low energy part of the DHG integral I. Suppose, for instance, that the anomalous magnetic moment of our spin $\frac{1}{2}$ particle arises from its coupling to a field of mass m' (where m' is much larger than the binding energy of the spin $\frac{1}{2}$ plus spin 0 system), just as the anomalous moment of the physical proton is associated with the existence of its meson cloud. Then one must separate I into two parts (4),

$$I = I^{low} + I^{high} = \int_{\omega_{th}}^{m'} \frac{\sigma(\omega)}{\omega} d\omega + \int_{m'}^{\infty} \frac{\sigma(\omega)}{\omega} d\omega. \tag{5.17}$$

One calculates I^{low} as before and finds

$$I_P^{low} - I_A^{low} = 8\pi^2 \left(\frac{Ze}{2M} - \frac{Z_T e}{2M_T} \right) \left(2\mu - \frac{Ze}{2M} - \frac{Z_T e}{2M_T} \right) \tag{5.18}$$

In order to evaluate I^{high}, assume that at the high proton energies $\omega > m'$, the forward scattering amplitude is just the sum of the forward scattering amplitudes for each particle, and apply the DHG sum rule to the spin $\frac{1}{2}$ particle:

$$I_P^{high} - I_A^{high} = \left(I^{high}_{spin\ \frac{1}{2}} \right)_P - \left(I^{high}_{spin\ \frac{1}{2}} \right)_A = \int_{m'}^{\infty} \frac{\sigma_P^{\frac{1}{2}}(\omega) - \sigma_A^{\frac{1}{2}}(\omega)}{\omega} d\omega \tag{5.19}$$

$$= 8\pi^2 \left(\mu - \frac{Ze}{} \right)^2$$

Combining this with (5.18) we find

$$\int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega) - \sigma_A(\omega)}{\omega} d\omega = I_P - I_A = 8\pi^2 \left(\mu - \frac{Z_T e}{2M_T}\right)^2 , \tag{5.20}$$

again in accord with the Drell-Hearn-Gerasimov sum rule.

B. Alternative Derivation of the DHG Sum Rule for Spin $\frac{1}{2}$ Composite System

It is advantageous in calculations such as that of the DHG integral and the Compton scattering amplitude to postpone the reduction to non-relativistic forms until after the sum over final states has been performed. The evaluation of the DHG integral by this method proceeds as follows:

$$\begin{aligned} I = 4\pi^2 \sum_{f>i} \frac{1}{\omega^2} | < f | Ze\vec{\alpha}\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_p} + i\lambda\beta(\vec{\sigma}\cdot\vec{k}\times\hat{\epsilon} - i\omega\vec{\alpha}\cdot\hat{\epsilon})\, e^{i\vec{k}\cdot\vec{r}_p} \\ + ze\vec{v}_\pi\cdot\hat{\epsilon}\, e^{i\vec{k}\cdot\vec{r}_\pi} | i > |^2 \\ = 4\pi^2 \sum_{f>i} | < f | Ze\vec{r}_p\cdot\hat{\epsilon}(1-\vec{\alpha}\cdot\hat{k})\, e^{i\vec{k}\cdot\vec{r}_p} + \lambda\beta(\vec{\sigma}\cdot\hat{k}\times\hat{\epsilon} - i\vec{\alpha}\cdot\hat{\epsilon})\, e^{i\vec{k}\cdot\vec{r}_p} \\ + ze\vec{r}_\pi\cdot\hat{\epsilon}(1-\vec{v}_\pi\cdot\hat{k})\, e^{i\vec{k}\cdot\vec{r}_\pi} | i > |^2 . \end{aligned} \tag{5.21}$$

Again we will neglect the retardation factors. The error made in doing this corresponds to binding corrections to the magnetic moment on the rhs of Eq. (5.1)[23]. Since the matrix elements are now independent of $\omega = E_f - E_i$, we are able to use closure on the states f. We must insert the (hermitian) positive energy projection operator

$$P_+ \equiv \sum_{E_j > 0} | j > < j |$$

since physical photoabsorption transitions require $E_f > E_i$. Defining the (hermitian) operator

$$\vec{h} = Ze\vec{r}_p(1 - \vec{\alpha}\cdot\hat{k}) + \lambda\beta(\vec{\sigma}\times\hat{k} - i\vec{\alpha}) + ze\vec{r}_\pi \quad (1 - \vec{v}_\pi\cdot\hat{k}), \tag{5.22}$$

we find

$$I = 4\pi^2 < i | \vec{h}\cdot\hat{\epsilon}^* P_+ \vec{h}\cdot\hat{\epsilon} | i > - 4\pi^2 \sum_{i'} | < i' | \vec{h}\cdot\hat{\epsilon} | i > |^2$$

$$= I^{closure} + I^{ground} \tag{5.23}$$

where the i' sum in the I^{ground} term runs over both polarizations of the ground state.

The $I^{closure}$ term makes no contribution when we take the difference $I_P - I_A$. Using the notation $I_{\mu\rho}$ corresponding to (5.15), we note that $I_P - I_A = \frac{1}{2}(I_{++} - I_{+-}) - \frac{1}{2}(I_{-+} - I_{--})$. We easily verify, using the reality of the ground

state, that the $I^{closure}$ contribution to each of these parentheses is zero:

$$I^{closure}_{\mu+} - I^{closure}_{\mu-} = \langle i,\mu \mid \vec{h}\cdot\hat{\epsilon}^*_+ P_+ \vec{h}\cdot\hat{\epsilon}_+ \mid i,\mu \rangle - \langle i,\mu \mid \vec{h}\cdot\hat{\epsilon}^*_- P_+ \vec{h}\cdot\hat{\epsilon}_- \mid i,\mu \rangle$$

$$= \langle i,\mu \mid \vec{h}\cdot\hat{\epsilon}^*_+ P_+ \vec{h}\cdot\hat{\epsilon}_+ \mid i,\mu \rangle - \langle i,\mu \mid \vec{h}\cdot\hat{\epsilon}_+ P_+ \vec{h}\cdot\hat{\epsilon}^*_+ \mid i,\mu \rangle \quad (5.24)$$

$$= 0.$$

Now we must evaluate I^{ground}. The relevant matrix element is

$$\langle i' \mid \vec{h}\cdot\hat{\epsilon} \mid i \rangle = (i' \mid Z_T e \vec{R}\cdot\hat{\epsilon} + \mu\vec{\sigma}\cdot\hat{k}\times\hat{\epsilon} + \left(2\mu - \frac{Z_T e}{2m}\right)\vec{\sigma}\cdot\hat{\epsilon}\times\frac{\vec{P}}{2m} \mid i). \quad (5.25)$$

Terms linear in $\vec{r}$ and $\vec{p}$ vanish because i and i' are both of the same parity in the relative coordinate $\vec{r}$; various higher order terms which do not contribute to $I_P - I_A$ have also been omitted in writing (5.25). Now note that none of the terms in the matrix element of Eq. (5.25) can connect $\mid i)$ to any excited state $\mid n)$ because of the orthogonality of the $\vec{r}$ wavefunctions. Thus we can do closure[24] in the $\mid\ )$ states (at this point, we can regard the states $\mid\ )$ as being the eigenstates of a Pauli Hamiltonian), and after a little calculation again obtain the DHG sum rule:

$$I_P - I_A = I_P^{\text{ground}} - I_A^{\text{ground}} = -4\pi^2 \sum_{i'} | < i' | \vec{h}\cdot\hat{\epsilon} | i > |^2 \Big|_{P-A}$$

$$= -4\pi^2 \sum_{(|} | (\quad | Z_T e \vec{R}\cdot\hat{\epsilon} + \mu\vec{\sigma}\cdot\hat{k}\times\hat{\epsilon} + \left(2\mu - \frac{Z_T e}{2m}\right)\frac{1}{2m}\vec{\sigma}\cdot\hat{\epsilon}\times\vec{P} | i) |^2 \Big|_{P-A} \tag{5.26}$$

$$= -4\pi^2 (i | \mu^2[1 - i\vec{\sigma}\cdot\hat{\epsilon}^*\times\hat{\epsilon}] + \frac{Z_T e}{2m}\left(2\mu - \frac{Z_T e}{2m}\right)(\vec{R}\cdot\hat{\epsilon}^*\vec{\sigma}\cdot\hat{\epsilon}\times\vec{P} + \vec{\sigma}\cdot\hat{\epsilon}\times\vec{P}\,\vec{R}\cdot\hat{\epsilon}) | i) \Big|_{P-A}$$

$$= 8\pi^2\left(\mu - \frac{Z_T e}{2m}\right)^2 .$$

This method of calculating the DHG integral has several advantages over the traditional methods used in atomic and nuclear physics[25]. In the usual treatments, which the derivation of Section 5A is supposed to parallel, one must use non-relativistic reductions - in which the Dirac operator $\vec{\alpha}$ gets replaced by $\vec{p}$'s and $\vec{\sigma}$'s - at an early point in the calculation. That is because all dynamical operators must in these treatments be split into CM and relative coordinate parts, and the CM parts dropped. This is trivial for $\vec{p}_p = \vec{P}(M/M_T) + \vec{p}$, but impossible for $\vec{\alpha}$. In the derivation presented in this section we were able to avoid such calculational gymnastics. In particular, we utilized the completeness of the relativistic states; and we used our approximate wavefunctions, derived in Section 4 under the assumption that $\vec{V} \simeq \vec{P}/M_T$ is small, only in evaluating the matrix element (5.25), where such an approximation is entirely justified. A further advantage of the calculation of this subsection, which we sometimes refer to as "relativistic closure" in the following, is that it emphasizes the fact that the DHG integral equals the "Born term" I_{ground}, which depends only on the static properties of the system: the total mass, charge, and magnetic moment.

C. Spin 1 Composite System

The analogue of the DHG sum rule for a spin-S system of charge $Z_T e$, mass $\mathcal{M}$, and magnetic moment μ reads (18)

$$\int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega) - \sigma_A(\omega)}{\omega} d\omega = I_P - I_A = \frac{4\pi^2}{S}\left(\mu - \frac{Z_T e S}{\mathcal{M}}\right)^2 . \tag{5.27}$$

Here the P and A configurations correspond to (photon helicity, target S_z) $\equiv$ (μ, ρ) equal respectively to $(\pm 1, \pm S)$ and $(\pm 1, \mp S)$. In this section we are considering the case S = 1, the model consisting of two spin $\frac{1}{2}$ particles in an S-state. The electromagnetic interaction is

$$-H^{em} = \sum_{s=a,b}\left[z_s e\vec{\alpha}_s \cdot \vec{A}(\vec{r}_s) + \tfrac{1}{2}\lambda_s \beta_s \sigma_{s\mu\nu} F^{\mu\nu}(\vec{r}_s)\right] . \tag{5.28}$$

The notation is defined in Table II.

Introducing a notation for states analogous to that of Eq. (5.8), and utilizing the results of Section 4, we write

$$\mathcal{\varphi}_i(\vec{r},\vec{R})_\rho = \langle \vec{r},\vec{R} \mid i,\rho \rangle$$

$$\equiv \int d^3p d^3P\, N^{(2)}_{p,P}\begin{pmatrix} 1 + \vec{\sigma}_a \cdot \frac{\vec{P}}{2E_i}\vec{\sigma}_a \cdot \frac{\vec{p}}{2m_a} \\ \vec{\sigma}_a \cdot \left(\frac{\vec{P}}{2E_i} + \frac{\vec{p}}{2m_a}\right) \end{pmatrix} \otimes \begin{pmatrix} 1 - \vec{\sigma}_b \cdot \frac{\vec{P}}{2E_i}\vec{\sigma}_b \cdot \frac{\vec{p}}{2m_b} \\ \vec{\sigma}_b \cdot \left(\frac{\vec{P}}{2E_i} - \frac{\vec{p}}{2m_b}\right) \end{pmatrix} \tag{5.29a}$$

$$\phi_i(\vec{p})\Phi_i(\vec{P})\; e^{i(\vec{p}\cdot\vec{r} + \vec{P}\cdot\vec{R})}\chi_\rho$$

$$< \vec{r}, \vec{R} \mid i, \rho > \equiv \int d^3p\, d^3P\, \phi_i(\vec{p})\Phi_i(\vec{P})\, e^{i(\vec{p}\cdot\vec{r} + \vec{P}\cdot\vec{R})}\chi_\rho$$

$$N^{(2)}_{p,P} = 1 - \frac{p^2}{8m_a^2} - \frac{p^2}{8m_b^2} + 0(\text{momenta}^4). \tag{5.29b}$$

These expressions for the spinors are correct through second order in momenta, and are adequate for our purposes. With

$$H_0 = \vec{\alpha}_a\cdot\vec{p}_a + \beta_a m_a + \vec{\alpha}_b\cdot\vec{p}_b + \beta_b m_b + U(\vec{r}_a - \vec{r}_b) \tag{5.30}$$

we again have the identity

$$\vec{\alpha}_{a,b} = i[H_0, \vec{r}_{a,b}] \tag{5.31}$$

and we evaluate the matrix elements of H^{em} as before:

$$< f \mid z_a e \vec{\alpha}_a\cdot\vec{A}(\vec{r}_a) + \tfrac{1}{2}\lambda_a\beta_a\sigma_{a\mu\nu}F^{\mu\nu}(\vec{r}_a) \mid i > =$$

$$i\omega < f \mid [z_a e \vec{r}_a\cdot\hat{\epsilon}(1 - \vec{\alpha}_a\cdot\hat{k}) + \lambda_a\beta_a(\vec{\sigma}_a\cdot\hat{k}\times\hat{\epsilon} - i\vec{\alpha}_a\cdot\hat{\epsilon})]\, e^{i\vec{k}\cdot\vec{r}_a} \mid i > . \tag{5.32}$$

The spin-independent part of the corresponding non-relativistic Hamiltonian is the same as for the previous model (see Eq. (5.13)), and the spin-dependent part is the obvious generalization of Eq. (5.12):

$$- H^{em}_{spin} = \mu_a \vec{\sigma}_a \cdot \vec{B} + \mu_b \vec{\sigma}_b \cdot \vec{B} + \left[\left(2\mu_a - \frac{z_a e}{2m_a} - \frac{Z_T e}{2M_T} \right) \vec{\sigma}_a \cdot \vec{E} \times \frac{\vec{p}_{av}}{2m_a} - (a \rightarrow b) \right]$$

$$+ \left[\left(2\mu_a - \frac{Z_T e}{2M_T} \right) \vec{\sigma}_a \cdot \vec{E} \times \frac{\vec{P}_{av}}{2M_T} + (a \rightarrow b) \right] . \tag{5.33}$$

We will now exhibit the generalization of the relativistic closure derivation of the sum rule. We begin with Eq. (5.32). Again we must drop the exponential factors in order to sum over the final states by closure. Thus we define

$$P_+ \equiv \sum_{E_j > 0} | j > < j |$$

and

$$\vec{h} \equiv \sum_{s=a,b} [z_s e \vec{r}_s (1 - \vec{\alpha}_s \cdot \vec{k}) + \lambda_s \beta_s (\vec{\sigma}_s \times \vec{k} - i\vec{\alpha}_s)] . \tag{5.34}$$

As before,

$$I = 8\pi^2 \sum_{f > i} \frac{1}{\omega} | < f | H^{em} | i > |^2 , \tag{5.35}$$

so that

$$I_{\mu\rho} = 4\pi^2 < i,\mu | \vec{h} \cdot \hat{\epsilon}^*_\rho P_+ \vec{h} \cdot \hat{\epsilon}_\rho | i,\mu > - 4\pi^2 \sum_{\mu'} | < i,\mu' | \vec{h} \cdot \hat{\epsilon}_\rho | i,\mu > |^2$$

$$\equiv I^{closure}_{\mu\rho} + I^{ground}_{\mu\rho} . \tag{5.36}$$

As before, $I^{closure}$ does not contribute to $I_P - I_A$. In calculating I^{ground}, only one slight subtlety changes the calculation from that of the spin $\frac{1}{2}$ model considered previously. The ground state here is the spin triplet; the spin singlet is regarded as having higher energy and hence is not contained in the μ' sum in I^{ground}. Consequently the total spin $\vec{S} \equiv \frac{1}{2}(\vec{\sigma}_a + \vec{\sigma}_b)$ can contribute to the matrix elements in I^{ground}, but the operator $\vec{T} \equiv \frac{1}{2}(\vec{\sigma}_a - \vec{\sigma}_b)$ cannot. Then, with $\mu = \mu_a + \mu_b$,

$$
\begin{aligned}
I_P - I_A &= I_P^{ground} - I_A^{ground} \\
&= 4\pi^2 \sum_{\mu} \Big| (i,\mu' \mid Z_T e \vec{R}\cdot\hat{\epsilon} + \mu \vec{S}\cdot\hat{k}\times\hat{\epsilon} + (\mu_a - \mu_b)\vec{T}\cdot\vec{k}\times\hat{\epsilon} \\
&\quad + \left(2\mu - \frac{Z_T e}{m}\right)\vec{S}\cdot\vec{\epsilon}\times\frac{\vec{P}}{2m} + (\mu_a - \mu_b)\vec{T}\cdot\hat{\epsilon}\times\frac{\vec{P}}{m} \mid i) \Big|^2 \Big|_{P-A} \qquad (5.37) \\
&= 4\pi^2 \sum_{n} \Big| (n \mid Z_T e \vec{R}\cdot\hat{\epsilon} + \mu \vec{S}\cdot\hat{k}\times\hat{\epsilon} + \left(2\mu - \frac{Z_T e}{m}\right)\vec{S}\cdot\hat{\epsilon}\times\frac{\vec{P}}{2m} \mid i) \Big|^2 \Big|_{P-A} \\
&= 4\pi^2 \left(\mu - \frac{Z_T e}{m}\right)^2 .
\end{aligned}
$$

It is clear from these calculations that proofs of the DHG sum rule could be constructed for any loosely bound composite system. Multiparticle systems could be treated by pairwise induction[26]. In particular, the sum rules are valid for H^3 and He^3 by treating them as composite states of spin $\frac{1}{2}$ + spin 0 constituents.

Section 6

Low Energy Forward Compton Scattering

A. Spin $\frac{1}{2}$ Composite System

It is possible to prove from field theory (3), or directly from S-matrix theory (20), that the amplitude for Compton scattering is completely determined to first order in the photon frequency by the static properties of the discrete system. In particular, for a spin $\frac{1}{2}$ system characterized by mass $\mathcal{M}$, charge $Z_T e$, and magnetic moment μ, the S-matrix for low-energy forward Compton scattering must take the form

$$S_{fi} = -\, 2\pi i\, \delta(E_f - E_i) M_{fi} \tag{6.1}$$

where

$$M_{fi} = \frac{1}{2\omega}(2\pi)^3 \delta^3(P_f - P_i) \left[\frac{Z_T^2 e^2}{\mathcal{M}} \hat{e}' \cdot \hat{e}\, \delta_{fi} + 2i\omega \left(\mu - \frac{Z_T e}{2\mathcal{M}} \right)^2 \vec{\sigma}_{fi} \cdot \hat{e}' \times \hat{e} + 0(\omega^2) \right]. \tag{6.2}$$

This result depends essentially only on relativistic invariance, and is valid for atoms and nuclei as well as elementary particles[27]. As a check on the consistency of our formalism, we rederive it in this section for the spin $\frac{1}{2}$ system of a "proton" (mass M, charge Z, magnetic moment $\mu = \frac{Ze}{2M} + \lambda$) loosely bound in an S-state to a "pion" (mass m, charge z) so that $Z_T = Z + z$, and $\mathcal{M} = M + m - W$.[28] (The notation is summarized in Table II.) We use techniques very similar to those employed in deriving the DHG sum rule for this system.

The first and second order perturbation theory[29] contributions to the S-matrix elements for forward Compton scattering give

$$M_{fi} = \langle f, \vec{k}\hat{e}' \mid H^{em} \mid i, \vec{k}\hat{e} \rangle + \sum_j \frac{\langle f, \vec{k}\hat{e}' \mid H^{em} \mid j \rangle \langle j \mid H^{em} \mid i, \vec{k}\hat{e} \rangle}{E_i + \omega - E_j + i\epsilon}$$

$$+ \sum_j \frac{\langle f, \vec{k}\hat{e}' \mid H^{em} \mid j, \vec{k}\hat{e}', \vec{k}\hat{e} \rangle \langle j, \vec{k}\hat{e}', \vec{k}\hat{e} \mid H^{em} \mid i, \vec{k}\hat{e} \rangle}{E_i - \omega - E_j} . \tag{6.3}$$

The electromagnetic interaction Hamiltonian

$$-H^{em} = Ze\vec{\alpha}\cdot\vec{A}(\vec{r}_p) + \lambda\beta[\vec{\sigma}\cdot\vec{B}(\vec{r}_p) - i\vec{\alpha}\cdot\vec{E}(\vec{r}_p)] + ze\vec{v}_\pi\cdot\vec{A}(\vec{r}_\pi) - \frac{z^2e^2}{2m}\vec{A}(\vec{r}_\pi)^2 \tag{6.4}$$

is the same as that employed in the last section — with the addition of the quadratic term, which was irrelevant there. It is convenient here to treat $\vec{A}$ as an operator and use linear, rather than circular, polarization vectors $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$; viz.,

$$\vec{A}(\vec{r}) = \sum_{\vec{k},\alpha} \frac{1}{\sqrt{2\omega}} \left(a_{\vec{k},\alpha} \hat{e}_\alpha e^{i\vec{k}\cdot\vec{r}} + a^\dagger_{\vec{k},\alpha} \hat{e}_\alpha e^{-i\vec{k}\cdot\vec{r}} \right). \tag{6.5}$$

We use i, j, f to designate states of the spin $\frac{1}{2}$ system and E_i, E_j, E_f to designate the energies of these states, including recoil energy. For forward scattering, $E_f = E_i$, so $f = i$ except possibly for spin orientation.

Our technique for calculating the sums in (6.3) will be to remove factors of energy from the matrix elements in order to perform closure. For example,

$$\sqrt{2\omega}\ <j|H^{em}|i,\vec{k}\hat{e}> = <j|[Ze\vec{\alpha}\cdot\hat{e} + i\lambda\beta(\vec{\sigma}\cdot\vec{k}\times\hat{e} - i\omega\vec{\alpha}\cdot\hat{e})]\, e^{i\vec{k}\cdot\vec{r}_p} + ze\vec{v}_\pi\cdot\hat{e}\, e^{i\vec{k}\cdot\vec{r}_\pi}|i>$$

$$= i<j|[H_0, Ze\vec{r}_p\cdot\hat{e}]\, e^{i\vec{k}\cdot\vec{r}_p} + [H_0, ze\vec{r}_\pi\cdot\hat{e}]\, e^{i\vec{k}\cdot\vec{r}_\pi}|i> + i\omega <j|\lambda\beta(\vec{\sigma}\cdot\hat{k}\times\hat{e} - i\vec{\alpha}\cdot\hat{e})\, e^{i\vec{k}\cdot\vec{r}_p}|i>$$

$$= i(E_j - E_i)<j|Ze\vec{r}_p\cdot\hat{e}\, e^{i\vec{k}\cdot\vec{r}_p} + ze\vec{r}_\pi\cdot\hat{e}\, e^{i\vec{k}\cdot\vec{r}_\pi}|i> \qquad (6.6)$$

$$- i\omega <j|Ze\vec{r}_p\cdot\hat{e}\,\vec{\alpha}\cdot\hat{k}\, e^{i\vec{k}\cdot\vec{r}_p} + ze\vec{r}_\pi\cdot\hat{e}\,\vec{v}_\pi\cdot\hat{k}\, e^{i\vec{k}\cdot\vec{r}_\pi}|i>$$

$$+ i\omega <j|\lambda\beta(\vec{\sigma}\cdot\hat{k}\times\hat{e} - i\vec{\alpha}\cdot\hat{e})\, e^{i\vec{k}\cdot\vec{r}_p}|i> .$$

The unperturbed Hamiltonian H_0 is defined in Eq. (5.5). It is convenient to introduce the following notation[30]

$$\vec{H} \equiv \left[Ze\vec{\alpha} + i\lambda\beta(\vec{\sigma}\times\vec{k} - i\omega\vec{\alpha})\right] e^{i\vec{k}\cdot\vec{r}_p} + ze\vec{v}_\pi\, e^{i\vec{k}\cdot\vec{r}_\pi}$$

$$\vec{g} \equiv Ze\vec{r}_p\, e^{i\vec{k}\cdot\vec{r}_p} + ze\vec{r}_\pi\, e^{i\vec{k}\cdot\vec{r}_\pi} \qquad (6.7)$$

$$\vec{h} \equiv [Ze\vec{r}_p(1 - \vec{\alpha}\cdot\hat{k}) + \lambda\beta(\vec{\sigma}\times\hat{k} - i\vec{\alpha})]\, e^{i\vec{k}\cdot\vec{r}_p} + ze\vec{r}_\pi(1 - \vec{v}_\pi\cdot\hat{k})\, e^{i\vec{k}\cdot\vec{r}_\pi} .$$

Then we can express the results of (6.5) and similar calculations for the other matrix elements as follows:

$$\sqrt{2\omega} < j \mid H^{em} \mid i, \vec{k}\hat{e} > = < j \mid \vec{H}\cdot\hat{e} \mid i > = i(E_j - E_i - \omega) < j \mid \vec{g}\cdot\hat{e} \mid i > + i\omega < j \mid \vec{h}\cdot\hat{e} \mid i >$$

$$\sqrt{2\omega} < f, \vec{k}\hat{e}' \mid H^{em} \mid j, \vec{k}\hat{e}', \vec{k}\hat{e} > = < f \mid \vec{H}\cdot\hat{e} \mid j > = i(E_f - E_j - \omega) < f \mid \vec{g}\cdot\hat{e} \mid j > + i\omega < f \mid \vec{h}\cdot\hat{e} \mid j >$$

$$\sqrt{2\omega} < f, \vec{k}\hat{e}' \mid H^{em} \mid j > = < f \mid \vec{H}^{\dagger}\cdot\hat{e}' \mid j > = i(E_f - E_j + \omega) < f \mid \vec{g}^{\dagger}\cdot\hat{e}' \mid j > - i\omega < f \mid \vec{h}^{\dagger}\cdot\hat{e}' \mid j > \qquad (6.8)$$

$$\sqrt{2\omega} < j, \vec{k}\hat{e}', \vec{k}\hat{e} \mid H^{em} \mid i, \vec{k}\hat{e} > = < j \mid \vec{H}^{\dagger}\cdot\hat{e}' \mid i > = i(E_j - E_i + \omega) < j \mid \vec{g}^{\dagger}\cdot\hat{e}' \mid i > - i\omega < j \mid \vec{h}^{\dagger}\cdot\hat{e}' \mid i > .$$

Substituting the first two lines of (6.8) into the sums over intermediate states j in (6.3) and using closure[31], we obtain

$$2\omega M_{fi} = \frac{z^2e^2}{m}\,\hat{e}'\cdot\hat{e} < f \mid i > + i < f \mid [\vec{g}\cdot\hat{e}, \vec{H}^{\dagger}\cdot\hat{e}'] \mid i > + i\omega \sum_j \frac{< f \mid \vec{H}^{\dagger}\cdot\hat{e}' \mid j >< j \mid \vec{h}\cdot\hat{e} \mid i >}{E_i + \omega - E_j} + \frac{< f \mid \vec{h}\cdot\hat{e} \mid j >< j \mid \vec{H}^{\dagger}\cdot\hat{e}' \mid i >}{E_i - \omega - E_j} . \qquad (6.9)$$

At this point we have assumed that there is a finite energy gap ΔE between the ground state and the continuum; and, since we are interested in the limit $\omega \to 0$, we consider $0 < \omega < E_2 - E_i$, where E_2 is the energy of the first excited state. Thus we have dropped the $i\epsilon$ in the denominator, since there are no poles in this region of ω.

Referring to the definitions (6.7), we observe that the "seagull" term in (6.9) is exactly cancelled by the commutator. Lest the wary reader notice that only the sum proportional to ω survives, and worry that we have lost the Thomson limit, we will point out that the recoiling ground state term $j = i'$ in the sum has a denominator proportional to ω and contains the Thomson term. We now substitute the third and fourth lines of (6.8) and (6.9) and again use the completeness of the states j to perform closure:

$$2\omega M_{fi} = \omega < f | [\vec{h}\cdot\hat{e}, \vec{g}^{\dagger}\cdot\hat{e}'] | i >$$

$$+ \omega^2 \sum_j \frac{< f | \vec{h}^{\dagger}\cdot\hat{e}' | j > < j | \vec{h}\cdot\hat{e} | i >}{E_i + \omega - E_j} + \frac{< f | \vec{h}\cdot\hat{e} | j > < j | \vec{h}^{\dagger}\cdot\hat{e} | i >}{E_i - \omega - E_j} \quad . \tag{6.10}$$

The first term in (6.10) is of order ω^2. Moreover, for $\omega << E_2 - E_i$, all the terms in the sum except $j = i'$ are also of order ω^2. Thus

$$2\omega M_{fi} = \omega^2 \sum_{i'} \left(\frac{< f | \vec{h}^\dagger \cdot \hat{e}' | i' > < i' | \vec{h} \cdot \hat{e} | i >}{\omega + (E_i - E_{i'})} - \frac{< f | \vec{h} \cdot \hat{e} | i' > < i' | \vec{h}^\dagger \cdot \hat{e}' | i >}{\omega - (E_i - E_{i'})} \right) + 0(\omega^2)$$

$$= \omega \sum_{i'} \left(< f | \vec{h}^\dagger \cdot \hat{e}' | i' > < i' | \vec{h} \cdot \hat{e} | i > - < f | \vec{h} \cdot \hat{e} | i' > < i' | \vec{h}^\dagger \cdot \hat{e}' | i > \right) \tag{6.11}$$

$$+ \sum_{i'} (E_{i'} - E_i) \left(< f | \vec{h}^\dagger \cdot \hat{e}' | i' > < i' | \vec{h} \cdot \hat{e} | i > + < f | \vec{h} \cdot \hat{e} | i' > < i' | \vec{h}^\dagger \cdot \hat{e}' | i > \right) + 0(\omega^2)$$

$$\equiv T_1 + T_0 + 0(\omega^2).$$

The second term, T_0, in (6.11) may appear to be of order ω^2, since $E_i' - E_i \simeq \omega^2/2$. However the recoil transition matrix element $< i' | R | i >$ is of order $1/\omega$, so in fact a portion of this term is of order 1. Bringing the spinor factors into the matrix elements $< i' | \vec{h} \cdot \hat{e} | i >$, etc., using the wave function of Eq. (5.8), we find (with sufficient accuracy)

$$< i' | \vec{h} \cdot \hat{e} | i > = (i' | \left\{ Ze \vec{r}_p \cdot \hat{e} \left[1 - \hat{k} \cdot \left(\frac{\vec{P}}{m} + \frac{\vec{p}}{M} \right) \right] + \mu \vec{\sigma} \cdot \hat{k} \times \hat{e} + \left(2\mu - \frac{Ze}{2M} - \frac{Ze}{2m} \right) \vec{\sigma} \cdot \hat{e} \times \frac{\vec{p}}{2M} + \left(2\mu - \frac{Ze}{2m} \right) \vec{\sigma} \cdot \hat{e} \times \frac{\vec{P}}{2m} \right\} e^{i\vec{k} \cdot \vec{r}_p}$$

$$+ \left\{ ze \vec{r}_\pi \cdot \hat{e} \, (1 - \hat{k} \cdot \vec{v}_\pi) - \frac{ze}{2m} \vec{\sigma} \cdot \hat{e} \times \frac{\vec{p}}{2M} - \frac{ze}{2m} \vec{\sigma} \cdot \hat{e} \times \frac{\vec{P}}{2m} \right\} e^{i\vec{k} \cdot \vec{r}_\pi} | i) \equiv (i' | \tilde{\vec{h}} \cdot \hat{e} | i). \tag{6.12}$$

The only difference between $\vec{h}$ (or $\tilde{\vec{h}}$) and $\vec{h}^\dagger$ (or $\tilde{\vec{h}}^\dagger$) is that, in the exponentials, $\vec{k}$ is replaced by $-\vec{k}$.

Using the leading terms in the center of mass energy, we find that

$$(E'_i - E_i) < i | \vec{h}\cdot\hat{e} | i > = (i' | \left[\frac{\vec{P}^2}{2m}, \vec{\tilde{h}}\cdot\hat{e}\right] | i) \quad . \tag{6.13}$$

Thus

$$T_0 = \sum_{i'} \left[(f | \vec{\tilde{h}}^{\dagger}\cdot\hat{e}' | i') (i' | \left[\frac{\vec{P}^2}{2m}, \vec{\tilde{h}}\cdot\hat{e}\right] | i) - (f | \left[\frac{\vec{P}^2}{2m}, \vec{\tilde{h}}\cdot\hat{e}\right] | i') (i' | \vec{\tilde{h}}^{\dagger}\cdot\hat{e}' | i) \right]$$

$$= - i \frac{Z_T^2 e^2}{m} \sum_{i'} \left[(f | \vec{R}\cdot\hat{e}' | i') (i' | \vec{P}\cdot\hat{e} | i) - (f | \vec{P}\cdot\hat{e} | i') (i' | \vec{R}\cdot\hat{e}' | i) \right] + 0(\omega^2) \tag{6.14}$$

$$= (2\pi)^3 \delta^3 (\vec{P}_f - \vec{P}_i) \frac{Z_T^2 e^2}{m} \hat{e}'\cdot\hat{e}\ \delta_{fi} + 0(\omega^2).$$

In obtaining from T_1 the term of order ω we can drop the exponentials and proceed as in Eq. (5.26). The result is

$$T_1 = \omega (f | [\mu^2(\hat{e}'\cdot\hat{e} + i\vec{\sigma}\cdot\hat{e}'\times\hat{e}) + \frac{Z_T e}{2m}(2\mu - \frac{Z_T e}{2m})(\vec{R}\cdot\hat{e}'\vec{\sigma}\cdot\hat{e}\times\vec{P} + \vec{\sigma}\cdot\hat{e}'\times\vec{P}\,\vec{R}\cdot\hat{e})] - [\hat{e}' \longleftrightarrow \hat{e}] | i) + 0(\omega^2)$$

$$= (2\pi)^3 \delta^3 (\vec{P}_f - \vec{P}_i) 2i\omega \left(\mu - \frac{Z_T e}{2m}\right)^2 \vec{\sigma}_{fi}\cdot\hat{e}'\times\hat{e} + 0(\omega^2). \tag{6.15}$$

Thus we have verified the low energy theorem (6.2).

B. Spin $\frac{1}{2}$ Model, Alternative Treatment

It is of some interest to check how (6.2) is derived if the two component reduced matrix elements are employed from the start in (6.3). In analogy with expression (6.8),

$$\sqrt{2\omega} < j \mid H_{em} \mid i, \vec{k}\hat{e} > = i(E_j - E_i - \omega)(j \mid \vec{g}_{NR} \cdot \hat{e} \mid i) + i\omega\,(j \mid \vec{h}_{NR} \cdot \hat{e} \mid i), \text{ etc.} \tag{6.16}$$

where, to the required accuracy for this calculation ,

$$\begin{aligned} \vec{g}_{NR} \cdot \hat{e} &= Ze\vec{r}_p \cdot \hat{e} + ze\vec{r}_\pi \cdot \hat{e} + \left(\frac{Ze}{2M} - \frac{Z_T e}{2M_T}\right) \vec{\sigma} \cdot \hat{e} \times \frac{\vec{p}}{2M} + \left(\frac{Ze}{M} - \frac{Z_T e}{2M_T}\right) \vec{\sigma} \cdot \hat{e} \times \frac{\vec{P}}{2M_T} \\ \vec{h}_{NR} \cdot \hat{e} &= \vec{g}_{NR} \cdot \hat{e} + \mu\vec{\sigma} \cdot \hat{k} \times \hat{e} + 2\lambda\vec{\sigma} \cdot \hat{e} \times \frac{\vec{p}_p}{2M} . \end{aligned} \tag{6.17}$$

Proceeding as in (6.9) and (6.10) we obtain

$$\begin{aligned} 2\omega M_{fi} = \; & T_{NE} + \frac{z^2 e^2}{m} \hat{e}' \cdot \hat{e}\,(f \mid i) + i\,(f \mid \left[\vec{g}_{NR} \cdot \hat{e},\, Ze\frac{\vec{p}_p}{M} \cdot \hat{e}' + ze\frac{\vec{p}_\pi}{m} \cdot \hat{e}' - i\omega\,(\mu\vec{\sigma} \cdot \hat{k} \times \hat{e}' + 2\lambda\vec{\sigma} \cdot \hat{e}' \times \frac{\vec{p}_p}{2M})\right] \mid i) \\ & + \omega\,(f \mid \left[\vec{h}_{NR} \cdot \hat{e},\, \vec{g}_{NR} \cdot \hat{e}'\right] \mid i) \quad + T_0 + T_1 \\ = \; & \left\{ T_{NE} + \left[-\frac{Z^2 e^2}{M} \hat{e}' \cdot \hat{e}\, \delta_{fi} + i\omega \frac{Ze}{M} (2\lambda + \frac{Ze}{2M}) \vec{\sigma}_{fi} \cdot \hat{e}' \times \hat{e} \right] (2\pi)^3 \delta^3 (\vec{P}_f - \vec{P}_i) \right\} \quad + T_0 + T_1, \end{aligned} \tag{6.18}$$

where T_{NE} is the contribution of the intermediate states in which the spin $\frac{1}{2}$ particle has negative energy.[32] It will be recalled that the sum of T_o and T_1, defined in Eq. (6.12), equals the full low energy amplitude (6.2). We must thus show that the sum of the terms in the brace in (6.19) is zero. We can calculate the leading terms in T_{NE} by inserting the free Dirac negative energy projection operator and using closure to sum over all intermediate states:

$$T_{NE} = \tfrac{1}{2}\sum_j \frac{< f | \vec{H}^\dagger \cdot \hat{e}'(1-\beta-\vec{\alpha}\cdot\vec{p}_p/M) | j > < j | \vec{H}\cdot\hat{e} | i >}{(m+M) - (m-M) + \omega} + \frac{< f | \vec{H}\cdot\hat{e}(1-\beta-\vec{\alpha}\cdot\vec{p}_p/M) | j > < j | \vec{H}^\dagger\cdot\hat{e}' | i >}{(m+M) - (m-M) - \omega}$$

$$= \left[\frac{Z^2e^2}{M}\,\hat{e}'\cdot\hat{e}\,\delta_{fi} - i\omega\frac{Ze}{M}\,(2\lambda + \frac{Ze}{2M})\vec{\sigma}_{fi}\cdot\hat{e}'\times\hat{e}\right](2\pi)^3\delta^3(\vec{P}_f - \vec{P}_i)\,. \tag{6.19}$$

Thus the quantity inside the curly brackets in Eq. (6.18) is indeed zero.

We are now in a position to write down an effective interaction Hamiltonian for the two particle system, which is correct to order v/c, and which takes into account the negative energy state contribution through second order perturbation theory. The result[33] is

$$-H^{em}_{NR} = \left[-ZeA^o_p + \frac{Ze}{M}\vec{A}_p\cdot\vec{p}_p - \frac{Z^2e^2}{2M}\vec{A}^2_p + \mu\vec{\sigma}\cdot\vec{B}_p + \frac{1}{2M}(2\mu - \frac{Ze}{2M})\vec{\sigma}\cdot\vec{E}_p\times(\vec{p}_p - Ze\vec{A}_p)\right]$$
$$+\left[-zeA^o_\pi + \frac{ze}{m}\vec{A}_\pi\cdot\vec{p}_\pi - \frac{z^2e^2}{2m}\vec{A}^2_\pi\right] + \left[\frac{Ze}{4MM_T}\vec{\sigma}\cdot\vec{E}_p\times(\vec{p}_\pi - ze\vec{A}_\pi) - \frac{ze}{4MM_T}\vec{\sigma}\cdot\vec{E}_\pi\times(\vec{p}_p - Ze\vec{A}_p)\right]. \tag{6.20}$$

If $A^o = 0$ and if we neglect retardation — thus setting $\vec{A}_p = \vec{A}_\pi = \vec{A}$, which is sufficiently accurate for the calculation of the DHG integral and the low energy limit of Compton scattering — we can combine terms and obtain

$$-H_{NR}^{em} = -\left[\frac{Z^2e^2}{2M}\vec{A}^2 + \frac{Ze}{2M}(2\mu - \frac{Ze}{2M})\vec{\sigma}\cdot\vec{E}\times\vec{A}\right] - \frac{z^2e^2}{2m}\vec{A}^2 + \mu\vec{\sigma}\cdot\vec{B}$$

$$+ \frac{Z_Te}{M_T}\vec{A}\cdot\vec{P} + \frac{1}{2M_T}(2\mu - \frac{Z_Te}{2M_T})\vec{\sigma}\cdot\vec{E}\times\vec{P} + (\frac{Z}{M} - \frac{z}{m})e\vec{A}\cdot\vec{p} \qquad (6.21)$$

$$+ \frac{1}{2M}(2\mu - \frac{Ze}{2M} - \frac{Z_Te}{2M_T})\vec{\sigma}\cdot\vec{E}\times\vec{p}.$$

Let us discuss the significance of the various terms of this effective Hamiltonian. The terms in the first bracket of (6.20) are identical with the usual terms arising from the Foldy-Wouthuysen reduction of the Dirac equation, including an anomalous moment (21). The last bracket of (6.20) contains terms that can only be obtained from a proper relativistic treatment of the two-particle system, and are traceable from the effects of the boost in the two-particle wavefunctions[34]. We discuss the physical origin of these "spin-orbit" terms in Section VII. In (6.21) we have separated the terms into those proportional to the total momentum $\vec{P} = \vec{p}_p + \vec{p}_\pi$ and those proportional to the relative momentum $\vec{p} = (m\vec{p}_p - M\vec{p}_\pi)/M_T$. The terms quadratic in $\vec{A}_p$ or $\vec{E}_p$ in (6.20), which are collected in the bracket in (6.21), reproduce in non-relativistic perturbation theory the effects of negative energy states in second order relativistic perturbation theory. For example, these quadratic terms give a contribution to Compton scattering exactly equal to T_{NE}, which was calculated in (6.19).[32]

C. Spin 1 Composite System

For a spin-S target of charge $Z_T e$, mass $\mathcal{M}$, and magnetic moment μ, the low energy limit of the forward Compton scattering amplitude is (3, 22)

$$M_{fi} = \frac{1}{2\omega}(2\pi)^3 \delta^3(P_f - P_i)\left[\frac{Z_T^2 e^2}{\mathcal{M}}\hat{e}'\cdot\hat{e}\,\delta_{fi} + i\omega\left(\frac{\mu}{S} - \frac{Z_T e}{\mathcal{M}}\right)^2 \vec{S}_{fi}\cdot\hat{e}'\times\hat{e} + 0(\omega^2)\right]. \quad (6.22)$$

We can derive this result explicitly for the spin 1 composite system considered in Section III, consisting of two spin $\frac{1}{2}$ particles in a spatial S state and spin triplet state, by the same method used for the spin $\frac{1}{2}$ model above. The analogue of (6.7) for this case is[30]

$$\vec{H} \equiv \sum_{s=a,b}\left[z_s e\vec{\alpha}_s + i\lambda_s\beta_s(\vec{\sigma}_s\times\vec{k} - i\omega\vec{\alpha}_s)\right]e^{i\vec{k}\cdot\vec{r}_s}$$

$$\vec{g} \equiv z_a e\vec{r}_a e^{i\vec{k}\cdot\vec{r}_a} + z_b e\vec{r}_b e^{i\vec{k}\cdot\vec{r}_b} \quad (6.23)$$

$$\vec{h} \equiv \sum_{s=a,b}\left[z_s e\vec{r}_s(1 - \vec{\alpha}_s\cdot\hat{k}) + \lambda_s\beta_s(\vec{\sigma}_s\times\hat{k} - i\vec{\alpha}_s)\right]e^{i\vec{k}\cdot\vec{r}_s}.$$

The remaining calculations are exactly analogous to those following (6.7), so they will not be reproduced here.

On the same basis[33] as Eq. (6.20), we can obtain the effective interaction for the spin $\frac{1}{2}$ - spin $\frac{1}{2}$ system; it appears as Eq. (1.1).

Section 7 - Comments

A. Revision of the Foldy-Wouthuysen Method

It is worth noting that there is nothing wrong in principle with using a F-W transformation to eliminate "odd" operators in the relativistic Hamiltonian (to a given order in m^{-1}). What is incorrect is to assume that this F-W unitary operator reduces the bound state wavefunction to a simple Pauli form. Unless the wavefunction is written in the CM frame, the transformed wavefunction contains extra kinematical terms, as is apparent from Eq. (4.13). From a physical point of view, what appears as a triplet wavefunction of two spin $\frac{1}{2}$ particles in the CM, contains singlet contributions in the boosted wavefunctions required for the matrix element[13]. The usual treatment employing the sum of F-W Hamiltonians with the incorrect Pauli wavefunctions misses this singlet contribution.

On the other hand, we can recover the correct interaction (1.1) by extending the F-W technique, as we have shown elsewhere.[1] The usual F-W result is obtained, plus a contribution due to the presence of the boost operator in the bound state wavefunction.

B. Semiclassical Derivation of H_{NR}^{em}

A somewhat curious feature of the interaction Hamiltonians H_{NR}^{em} for composite systems developed in the proceeding sections — see Eqs. (1.1), (5.12), (5.33), (6.20) and (6.21) — is the appearance of additional "spin-orbit" terms beyond those which would appear if H_{NR}^{em} were simply the sum of Foldy-Wouthuysen or Pauli Hamiltonians for the constituent particles. For example,

for the spin 0 - spin $\frac{1}{2}$ system, one might have thought that the "spin-orbit" term would be just that of the free spin $\frac{1}{2}$ particle

$$- \left(2\mu - \frac{Ze}{2M}\right) \frac{\vec{\sigma}}{2} \cdot \vec{E} \times \left(\frac{\vec{p}}{M} + \frac{\vec{P}}{\mathcal{m}}\right) .$$

In fact, the correct term is

$$- \left(2\mu - \frac{Ze}{2M} - \frac{Z_T e}{2\mathcal{m}}\right) \frac{\vec{\sigma}}{2} \cdot \vec{E} \times \frac{\vec{p}}{M} - \left(2\mu - \frac{Z_T e}{2\mathcal{m}}\right) \frac{\vec{\sigma}}{2} \cdot \vec{E} \times \frac{\vec{P}}{\mathcal{m}} .$$

In this sub-section we will give, for a system of arbitrary composition and arbitrary spin, a heuristic derivation of the part of H_{NR}^{em} which does not depend on the internal structure of the system. (We will refer to this part of the interaction as H_{ext}^{em}.) Thus we will explain the appearance of the coefficient $(2\mu - \frac{Z_T e}{2\mathcal{m}})$ in the second term of the above equation, instead of the coefficient $(2\mu - \frac{Ze}{2M})$ which one might naively expect. We will also show, in the next sub-section, that knowledge of H_{ext}^{em} is adequate to derive the DHG sum rule for any spin.

Our analysis rests on the following simple observation: A system's interaction with radiation is known in any reference frame once it is known in the system's rest frame, the CM frame of its constituents. In this frame the interaction is simply

$$(H_{ext}^{em})_{CM} = Z_T e A^0_{CM} - \frac{\mu}{S} \vec{S} \cdot \vec{B}_{CM} + (\text{higher-moment interactions}), \qquad (7.1)$$

where Z_T is the total charge, μ is the total magnetic moment, and S is the spin of the system; and $A^{\mu}_{CM} = (A^0_{CM}, \vec{A}_{CM})$ and $\vec{B}_{CM}$ are the electromagnetic vector potential and magnetic field in the CM frame. We will neglect the electric quadrupole interaction and all other higher-moment interactions; it will be obvious that they can be discussed in the same manner. We will also neglect terms quadratic in $\vec{A}$. The interaction with a field A^{μ}_L specified in any other reference frame L ("laboratory") is determined by transforming A^{μ}_L to the CM frame and using $(H^{em}_{ext})_{CM}$. $(H^{em}_{ext})_{CM}$ is not by itself a suitable interaction Hamiltonian, however, since the CM frame is in general accelerated. We recall the well-known result of relativity that an accelerated frame rotates with respect to any inertial frame, with instananeous angular velocity ω_T, the "Thomas precession" frequency (23). Since we are writing the Hamiltonian in a rotating frame we must add the usual $\vec{\omega}_T \cdot \vec{J}$ term[35], where $\vec{J}$ is the total angular momentum; of course, $\vec{J} = \vec{S}$ in the CM frame. Thus

$$H^{em}_{ext} = Z_T e A^0_{CM} - \frac{\mu}{S} \vec{S} \cdot \vec{B}_{CM} + \vec{\omega}_T \cdot \vec{S}. \tag{7.2}$$

Let us suppose that the system moves with velocity $\vec{V} = \frac{\vec{\pi}}{E} = \frac{\vec{P} - Z_T e \vec{A}}{\gamma m}$ in the L frame. We must do a Lorentz transformation with velocity $-\vec{V}$ in order to go to the CM frame. Hence if $A^{\mu}_L = (0, \vec{A})$, then $A^{\mu}_{CM} = (-\gamma \vec{V} \cdot \vec{A}, \vec{A} + (\gamma - 1)\hat{V}\hat{V} \cdot \vec{A})$, and if the electromagnetic fields in the L frame are denoted by $\vec{E}$ and $\vec{B}$, then $\vec{B}_{CM} = \gamma \vec{B} - (\gamma - 1)\hat{V}\hat{V} \cdot \vec{B} - \gamma \vec{V} \times \vec{E}$. As usual, $\gamma = (1 - \vec{V}^2)^{-\frac{1}{2}}$. The Thomas precession frequency is

$$\vec{\omega}_T = - \frac{\gamma}{1+\gamma} \vec{V} \times \vec{a}$$

where the acceleration of the CM frame with respect to the L frame is

$$\vec{a} = \frac{Z_T e}{M} (\vec{E} + \vec{V} \times \vec{B}).$$

Thus,

$$-H^{em}_{ext} = Z_T e \gamma \vec{V} \cdot \vec{A} + \frac{\mu}{S} \vec{S} \cdot [\gamma \vec{B} - (\gamma - 1) \hat{V}\hat{V} \cdot \vec{B} - \gamma \vec{V} \times \vec{E}]$$

$$+ \frac{Z_T e}{m} \frac{\gamma}{1+\gamma} \vec{V} \times (\vec{E} + \vec{V} \times \vec{B}) \cdot \vec{S} \qquad (7.3)$$

$$= \frac{Z_T e}{m} \vec{P} \cdot \vec{A} + \frac{\mu}{S} \vec{S} \cdot \vec{B} + \left(\frac{\mu}{S} - \frac{Z_T e}{2m} \right) \vec{S} \cdot \vec{E} \times \frac{\vec{P}}{m} + 0(\vec{V}^2).$$

The reason that the coefficient $(\frac{\mu}{S} - \frac{Z_T e}{2m})$ appears instead of $(\frac{\mu}{S} - \frac{Ze}{2M})$ is that Eq. (7.1) was necessarily written in the frame in which the total spin of the system is well-defined — namely the rest, or CM, frame — not in the spin $\frac{1}{2}$ particle's rest frame. Thus the Thomas precession must be that of the CM frame.

C. The DHG Sum Rule

The derivations of the DHG sum rules presented in Sections 3B and 3C show very clearly that these sum rules have the superconvergent form[36]:

$$0 = 8\pi \int \frac{\mathrm{Im}\, f_2(\omega)}{\omega}\, d\omega \equiv B + \int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega) - \sigma_A(\omega)}{\omega}\, d\omega, \tag{7.4}$$

where the Born term B is determined entirely by H_{ext}^{em}:

$$B = \lim_{\omega \to 0} 8\pi^2 \sum_{i'} \frac{|\,(i'|\, H_{ext}^{em}\,|\, i)\,|^2}{\omega}\Bigg|_{P\text{-}A}. \tag{7.5}$$

The sum in (7.5) is over states i' degenerate with the ground state, i.e., over the 2S +1 spin orientations of the ground state. Such expressions have been worked out above. There are two terms which contribute to B, one coming from the magnetic dipole term $\frac{\mu}{S}\vec{S}\cdot\vec{B}$ of H_{ext}^{em}, specifically from the transitions from $S_z = S$ (or $-S$) to $S_z = S - 1$ (or $-S+1$) upon absorption of a zero-energy photon, and one coming from the cross term between $\vec{A}\cdot\vec{P}$ and $\vec{S}\cdot\vec{E}\times\vec{P}$. The former contributes only for the A ("antiparallel") case, not for the P case, so that its contribution to B is negative. The latter contributes for both A and P. The final result is

$$B = 4\pi^2 \left[-\frac{\mu^2}{S} + 2\,\frac{Z_T e}{m}\left(\frac{\mu}{S} - \frac{Z_T e}{2m}\right)\right] = -4\pi^2 S\left(\frac{\mu}{S} - \frac{Z_T e}{m}\right)^2 . \tag{7.5'}$$

Thus

$$\int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega) - \sigma_A(\omega)}{\omega} d\omega = \frac{4\pi^2}{S}\left(\mu - \frac{Z_T e S}{m}\right)^2 . \tag{7.6}$$

Let us assume the usual unsubtracted dispersion relation for $f_2(\omega^2)$;

$$f_2(\omega^2) = - \frac{1}{4\pi^2} \int_{\omega_{th}}^{\infty} \frac{\sigma_P(\omega') - \sigma_A(\omega')}{\omega'^2 - \omega^2} \omega' d\omega' . \tag{7.7}$$

Evaluating (7.7) at $\omega = 0$, using (5.27), we prove that

$$f_2(0) = - S\left(\frac{\mu}{S} - \frac{Z_T e}{m}\right)^2 . \tag{7.8}$$

This low energy limit is also obtained directly in Section 6 for the models considered (which have spin $S = \frac{1}{2}$ and 1).[37]

We emphasize that we have proved the DHG sum rule for the models we have considered, without requiring any assumptions on the high energy behavior of $\mathrm{Im}\, f_2(\omega^2) = -[\sigma_P(\omega) - \sigma_A(\omega)]/8\pi$. Thus it will apply to systems, like positronium and muonium, for which our model Hamiltonian (5.28) adequately describes the electromagnetic interactions. There is no reason to expect that the sum rule will fail for systems like the hydrogen atom in which one or more of the constituents have strong interactions, but the questions of convergence and subtractions in connection with the DHG integral for such systems require more information on the high energy behavior of the hadronic Compton scattering amplitudes.

D. Subtractions

In connection with their calculation of the DHG sum rule for loosely bound composite systems, which was based on the assumption — here shown to be incorrect — that H_{NR}^{em} equals the sum of the F-W Hamiltonians of the constituent particles, Barton and Dombey (4) raised the question of the necessity of subtraction constants in such sum rules. There are really two separate questions to be considered: (1) whether there is an additional constant, or even a polynomial, on the rhs of the dispersion relation (7.7) for f_2 (such terms could come from the integral over the circle at ∞ which closes the contour in the Cauchy integral), and (2) whether the DHG integral on the lhs of (5.27) actually converges.[40]

We will discuss the convergence question first. For high energy photons, the scattering amplitude f_2 should be well approximated by the sum of the amplitudes for the constituent particles ("additivity" or "impulse approximation"). Thus the asymptotic behavior of f_2 for a composite system can be no worse than that of the constitutents. For example, if the DHG integral converges for the proton — which is not inconsistent with experiment (1) — then it will converge for the hydrogen atom.

Now it could conceivably happen that the DHG integral converges, and yet the dispersion relation (7.7) nevertheless requires the addition of a constant, sometimes called a " subtraction at ∞ ". Abarbanel and Goldberger (20) have shown that such a "subtraction at ∞" in the DHG sum rule would correspond to a fixed pole at J=1 in the complex angular momentum plane. It is a very interesting experimental question whether such a singularity exists in the complex momentum plane for Compton scattering on elementary particles such as the proton.

The calculations of Barton and Dombey (4), if correct, would have proved that a "subtraction at ∞" is necessary in the DHG sum rule for a composite system, even if it is not for the constituents. Such a state of affairs would be physically most unreasonable, since a "subtraction at ∞" is associated with the asymptotic behavior of $\mathrm{Re}\, f_2(\omega)$ for $|\omega| \to \infty$; as we have argued above, the asymptotic behavior of the Compton amplitude for the composite system should be no worse than that of the sum of the amplitudes for the constituents. In fact, by deriving explicitly both the DHG integral and the low energy limit of the Compton amplitude, and demonstrating that they are equal[41], we have shown that there is nothing in the treatment of such composite systems as we have considered here which introduces into the dispersion relation a real constant.

Section 8

Conclusions

The central conclusions of this paper are the following:

(1) The external electromagnetic interactions of a loosely bound composite system are well described by the sum of the relativistic interactions of the constituents (for spin $\frac{1}{2}$ particles, for instance, these are the Dirac plus anomalous moment interactions). The basis for this result is a systematic reduction of the Bethe-Salpeter and Salpeter equations extended to include the effect of an external field.

(2) In particular, the Zeeman spectrum of atomic hydrogen is accurately described to 1 ppm if this interaction Hamiltonian is used.

(3) The non-relativistic reduction of this Hamiltonian is not given by the sum of the F-W Hamiltonians of the constitutents if the constitutents have spin. Additional terms arise from the spin transformation of the composite state wavefunction. Obviously, calculations based on the assumption of simple F-W additivity should be re-examined.

(4) In particular, previous incorrect non-relativistic calculations of the DHG sum rule (4) and the low energy theorem for Compton scattering[3] are corrected when the proper interaction Hamiltonian is used.

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FOOTNOTES

1. A derivation of Eq. (1.1) and a brief pedagogical presentation of some of our other results is given in S. J. Brodsky and J. R. Primack, Phys. Rev. 16, Section 5.

2. Here e_a, m_a, μ_a, and $\frac{1}{2}\vec{\sigma}_a$ are the charge, mass, total magnetic moment, and spin of fermion a. Note that we take $\vec{S} = \frac{1}{2}(\vec{\sigma}_a + \vec{\sigma}_b)$ to be the total spin in the c.m. frame. The relative and total four-momentum are given by $M_T p = m_b p_a - m_a p_b$, $P = p_a + p_b$, with $M_T = m_a + m_b$. As indicated below, the wavefunction $\varphi(\vec{x}'_a, \vec{x}'_b)$ to be used for evaluating matrix elements of (1.1) must include the Lorentz contraction $\vec{x}' = \Lambda\vec{x}$. This is important for evaluating the DHG sum rule and low energy theorem for bound states with $\ell \geq 1$. Equation (1) includes only terms involving the external field $A_s^\mu \equiv (A_s^0, A_s) \equiv A^\mu(x_s)$; there are consequently no Darwin terms. The Hamiltonian for the atom in zero external field is assumed to be known. Binding correction factors of order (1 + W/m) are neglected here as well as cross terms in the binding potential U and the external field such as $\vec{\sigma}\cdot\vec{\nabla}U \times \vec{A}$.

3. We wish to thank Dr. H. R. Pagels for suggesting that the low energy theorem might be violated if the sum of Foldy-Wouthuysen Hamiltonians is used. This was also independently discovered by G. Barton, "Apparent Clash Between the Foldy-Wouthuysen Transformation and the Threshold Theorem for Compton Scattering", September 1967 (unpublished).

4. See Section 7B.

5. For simplicity, we assume the two fermions are distinct and have non-derivative coupling via neutral spin 0 or spin 1 fields. Normal ordering is understood in the definition of the current.

6. Throughout this discussion we suppress renormalization constants, for simplicity. All masses and charges have their physical values. In this equation, and throughout this paper, we use the notation and metric of J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1965).

7. Effective potentials can often be designed to simulate even higher-order irreducible BS kernels, as in the Breit equation: G. Breit, Phys. Rev. 51, 248 (1937). See also H. Grotch and D. R. Yennie, Z. Phys. 202, 425 (1967).

8. The interaction kernel is actually assumed to be instantaneous in the CM frame, and it is not instantaneous in any other frame. Eq. (2.16) consequently involves a further approximation: neglect of the induced non-instantaneous Coulomb interaction ($\delta_{\mu o}\delta_{\nu o}/\vec{k}^2$ in the CM frame becomes $V_\mu V_\nu / [(k\cdot V)^2 - k^2]$ in a frame where V is the total four-velocity). The usually negligible effect of the induced non-instantaneous term can be treated with the other neglected kernels in perturbation theory.

9. The contour prescription comes from the negative imaginary parts of m_a and m_b, δ being infinitesimal, real, and positive.

10. We use Eqs. (2.19) and (2.21), which give the connection between ψ and φ. In the first line of (2.23) we have used the orthogonality of Λ_+^b and Λ_-^b. The contour prescription in the second line of (2.23) corresponds to the fact that $\bar{\psi}$ is obtained from ψ by complex conjugation and antichronological ordering.

11. Conversely, the BS equation also describes the interaction of antifermions; then $\varphi \approx \varphi_{--}$, and charge conjugation is reflected in the change of sign in Eq. (2.24).

12. In particular, in the experiments of Robiscoe et al. (14), a magnetic field was applied to a beam of metastable atoms in a definite hyperfine component of the $2S_{\frac{1}{2}}$ state and adjusted until the energy of this level became degenerate with one of the $2P_{\frac{1}{2}}$ components. By knowing the magnetic field at which the crossing occurs, one can extrapolate back to zero magnetic field and determine the $2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}$ separation at $H = 0$.

13. Ian J. McGee, ref (18), have an illuminating discussion of the relativistic transformation of a zero-binding wavefunction. We wish to thank Professor L. Durand for calling this work to our attention.

14. In fact, we have shown in Section 2 that to first order in the binding the Breit equation correctly yields the electromagnetic interactions of the composite system. The wavefunction (4.8) which we obtain solves the Salpeter equation through first order in the binding. We also note that the Breit equation reduces to the Dirac equation for $m_a/m_b \to 0$ whereas the Salpeter equation must be augmented by crossed graph contributions to obtain the correct limit. This has been shown to all orders in perturbation theory by D. R. Yennie (private communication). Also see ref. (7).

15. For example, g in Eq. (2.16) can be the zeroth component of a four-vector interaction, such as the instantaneous Coulomb interaction $g_c(\vec{k}) = \gamma_a^o \gamma_b^o / \vec{k}^2$.

16. For example, $\chi_{11} = \chi_a^+ \otimes \chi_b^+$, $\chi_{10} = \frac{1}{2}(\chi_a^+ \otimes \chi_b^- + \chi_a^- \otimes \chi_b^+)$, etc., where $\sigma_{za}\chi_a^\pm = \pm \chi_a^\pm$, $(\chi_a^\pm)^\dagger(\chi_a^\pm) = 1$.

17. The normalization condition (4.9) for ladder approximation was first stated by Salpeter (8). For comparison with (4.8), note that a single-particle wave-packet is written

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^o}}\, u(p)\, \phi(p)\, e^{-ip\cdot x}$$

where

$$u(p) = \sqrt{\frac{p^o + m}{2m}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^o + m} \end{pmatrix} \chi \ .$$

18. This is similar to the approximation made in deriving Eq. (2.16). See Footnote (8).

19. If, in the calculation (5.9), we do not use the identity $\vec{\alpha} = i[H_0, \vec{r}_p]$, then the proper inclusion of the kinetic energy terms k_i (see Section 4) and $K_i \equiv \vec{P}^2/2E_i$ is essential in obtaining the correct coefficients for the $\vec{\sigma}\cdot\vec{E}\times\vec{p}$ and $\vec{\sigma}\cdot\vec{E}\times\vec{P}$ terms. In the calculations presented in the text, however, k_i and K_i can be dropped.

20. As stated in Section 2, we consider only binding arising from exchange of <u>neutral</u> particles with non-derivative coupling, so that "exchange currents" cannot contribute to μ. Furthermore, we ignore "Breit" binding corrections to the magnetic moments of the constituent particles. These are of order $\mu\, W/\mathcal{M}$, and we have assumed $W/\mathcal{M} << 1$ ("loose binding").

21. H. A. Bethe and E. E. Salpeter, ref. (<u>5</u>), p. 300. Cf. also H. A. Bethe, <u>Intermediate Quantum Mechanics</u> (Benjamin, N. Y., 1964), p. 152.

22. We do not distinguish $\mathcal{M}$ and $M_T = \mathcal{M} + W$ in this section. First order binding corrections are correctly taken into account in the alternative treatment given in Section 5B.

23. Cf. discussion following Eq. (5.13).

24. An alternative way of presenting the argument is to factor the wavefunctions of states i and i' into relative and CM coordinate dependence. Since the states i' and i have identical dependence on $\vec{r}$, the relative coordinate, and since there is no relevant $\vec{r}$ dependence in the operators whose matrix element (eq.(5.25)) is being calculated, the integral on $\vec{r}$ gives 1. The remaining integral on $\vec{R}$ effectively just sets $\vec{P}_{i'} = \vec{P}_i$, and Eq. (5.26) obtains.

25. See, for example, J. S. Levinger, Nuclear Photodisintegration (Oxford University Press, London, 1960), for a general exposition of non-relativistic sum rule techniques. Barton and Dombey, ref. 4, give in their Section 6 a partial catalog of the delicacies and perversities of the usual treatments; these include, for instance, the question whether closure (over positive energy states) and the Foldy-Wouthuysen transformation commute.

26. Three particle composite systems are discussed in a recent preprint by V. P. Shelest, ITP, Kiev (1967).

27. It is necessary, however, that the system to which we apply Eq. (6.2) satisfy the following requirements: (1) There is no state "accidentally" degenerate in energy with the ground state. (2) There is a finite gap in energy between the ground state and the continuum.

28. We remind the reader that W is the binding energy and that we assume $W << M, m$.

29. See Eq. (2.11).

30. Note that the definition of $\vec{h}$ in Eq. (6.7) [or (6.23)] differs from that in Eq. (5.22) [or (5.35)] by retardation factors.

31. Low (3) has shown that the photoproduction channels in the sum over intermediate states do not contribute through order ω, so we ignore them here.

32. The "negative energy" expression (6.19) corresponds to the "Z" diagram contribution to (2.11) in which the intermediate states contain three free spin $\frac{1}{2}$ particles (two fermions and one antifermion).

33. The $\vec{E} \times \vec{A}$ terms in the last bracket of (6.20) are given here for completeness although they give no contribution to low energy Compton scattering. They are derived by replacing the canonical momenta by the mechanical momenta in the CM equation of motion (4.1) and in the boost operator (4.12). (See also S. J. Brodsky and J. R. Primack, footnote 1.) Darwin terms vanish for external fields. Terms proportional to the binding potential are omitted.

34. Cf. Section 4, and Appendix III of I. J. McGee, ref. (18).

35. One can show that this term must be added to the Hamiltonian by doing a canonical transformation. Alternatively, we recall that $(d\vec{G}/dt)_{\text{inertial}} = (d\vec{G}/dt)_{\text{rotating}} + \vec{\omega} \times \vec{G}$ for any vector $\vec{G}$. (H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, Mass., 1950), p. 133). Thus $H_{\text{inertial}} = H_{\text{rotating}} + \vec{\omega} \cdot \vec{J}$, so that $(d\vec{G}/dt)_{\text{inertial}} = i[H_{\text{inertial}}, \vec{G}] = i[H_{\text{rotating}}, \vec{G}] + i[\vec{\omega} \cdot \vec{J}, \vec{G}] = (dG/dt)_{\text{rotating}} + \vec{\omega} \times \vec{G}$.

36. We write the forward Compton amplitude in the traditional form

$$f(\omega) = f_1(\omega^2)\, \hat{e}' \cdot \hat{e} + i\omega f_2(\omega^2)\, \vec{S} \cdot \hat{e}' \times \hat{e},$$

where $f_1(0) = -Z_T^2 e^2/\mathcal{M}$.

37. The quantity $\mu - Z_T eS/\mathcal{M} \equiv \lambda$, the square of which is proportional to the low energy limit of the $\Delta S_z = 1$ Compton amplitude, coincides with the usual definition of the anomalous magnetic moment for $S = \frac{1}{2}$ and is a reasonable definition for all S.[38] A simple semiclassical dynamical interpretation can be given for this choice of $\mu_N \equiv Z_T eS/\mathcal{M}$ as the "normal" part of the magnetic moment.[39] The covariant equation for the motion of a particle

in an electromagnetic field is

$$\frac{du^{\mu}}{d\tau} = \frac{Z_T e}{m} F^{\mu}{}_{\nu} u^{\nu}$$

and the BMT (23) equation for the spin four-vector is

$$\frac{ds^{\mu}}{d\tau} = \frac{\mu}{S} F^{\mu}{}_{\nu} s^{\nu} + \frac{\lambda}{S} F_{\sigma\nu} s^{\sigma} u^{\nu} u^{\mu} \ .$$

For the case $\lambda = 0$, i.e. $\mu = \mu_N$, both equations assume the same form. In this special case, u and s can be regarded as simultaneously undergoing an infinitesimal Lorentz transformation with $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \frac{Z_T e}{m} F^{\mu}{}_{\nu} d\tau$. If we attach a "natural" instantaneous coordinate frame to each point of the world line — this frame being defined so that the time axis is along u and the space axes rotate in an interval $d\tau$ according to $\Lambda^{\mu}{}_{\nu}$ —then in this natural coordinate system the spin is a constant vector for $\mu = \mu_N$.

38. One usually introduces an electromagnetic interaction into the wave equation of a spin S particle by making the minimal substitution $p_{\mu} \rightarrow p_{\mu} - eA_{\mu}$ and adding terms proportional to $F_{\mu\nu}$ or its derivatives. Those parts of the interaction coming from the minimal substitution we call "normal", the others, "anomalous". In general, however, there are numerous wave equations for given spins, and consequently this approach will not lead to a unique definition of the "anomalous magnetic moment".

39. A. S. Wightman, summer lectures at Stanford, 1967; cf. Bargmann, Michel, Telegdi (23). Note that the gyromagnetic ratio $g = (\mu/S)(e/2m)^{-1}$. Thus $g = 2$ is the "normal" value for particles with spin.

40. The convergence of the DHG sum rule is discussed by A. H. Mueller and T. L. Trueman, Phys. Rev. 160, 1306 (1967).

41. Even for the incorrect Hamiltonian used by Barton and Dombey (4) — (a sum of two F-W Hamiltonians)— the (wrong) DHG integral equals the (wrong) low energy limit[3] of f_2.

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Figure Captions

Figure 1 - Exact calculation of the hydrogen spectrum. The typical kernels required for calculation of the energy levels of the H-atom to the present precision are shown. The one photon exchange contribution can be separated into Coulomb and transverse parts in the CM frame. The effects of strong interactions are summarized by form factors in $G_{1\gamma}$ plus nuclear polarization contributions as indicated in $G_{NUC-POL}$. The main effects of adding the higher order kernels are listed below the diagrams. The available small expansion parameters are also given.

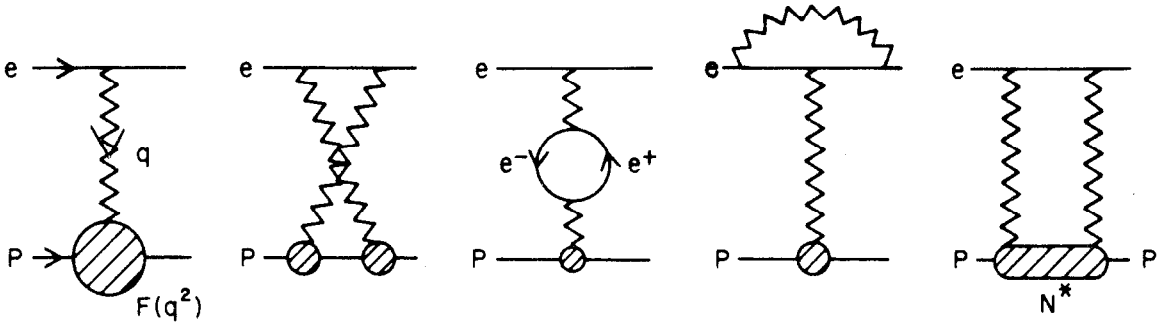
Figure 2 - (a) Diagrammatic representation of the BS equation in ladder approximation. (b) Lowest order electromagnetic interaction in ladder approximation. After isolating the contribution of the bound state to the two-particle BS propagator and absorbing the Feynman propagators S_F^a and S_F^b, we are left with the matrix element $\gamma_\mu^{(a)} (S_F^b)^{-1}$, Eq. (2.12). This is represented in (c). For simplicity, we consider only particle a to be charged.

Figure 3 - (a) The full BS equation, showing examples of graphs omitted in ladder approximation. (b) The lowest order electromagnetic interaction for the full BS equation.

Bethe-Salpeter Equation

$$(\not{p}_e - m_e)\,(\not{p}_p - m_p)\,X = GX$$

$$G = G_{1\gamma} + G_{\text{CROSSED}} + G_{\text{VAC.POL.}} + G_{\text{SELF ENERGY}} + G_{\text{NUC-POL}} + \cdots$$



$$G_{1\gamma} = G_{\text{COULOMB}} + G_{\text{TRANSVERSE}}$$

$$-\epsilon_\mu \frac{1}{q^2}\,\epsilon^\mu = \epsilon_0 \frac{1}{q^2}\,\epsilon_0 + \sum_{\substack{\text{TRAN} \\ i=1,2}} \epsilon_i \frac{1}{q^2}\,\epsilon_i$$

G_{COULOMB} ⟶ Schrödinger equation, proton finite size correction

$+ G_{\text{TRANS}}$ ⟶ reduced mass corrections, HFS splittings

$+ G^{(\text{all})}_{\text{CROSSED}}$ ⟶ Dirac equation, relativistic reduced mass correction

$+ G_{\text{VAC-POL}} + G_{\text{SELF ENERGY}}$ ⟶ Lamb shift, radiative corrections to HFS

$+ G_{\text{NUC-POL}}$ ⟶ correction to HFS

Expansion Parameters: α, $Z\alpha$, m_e/M_p, R_p/a_0

Fig. 1

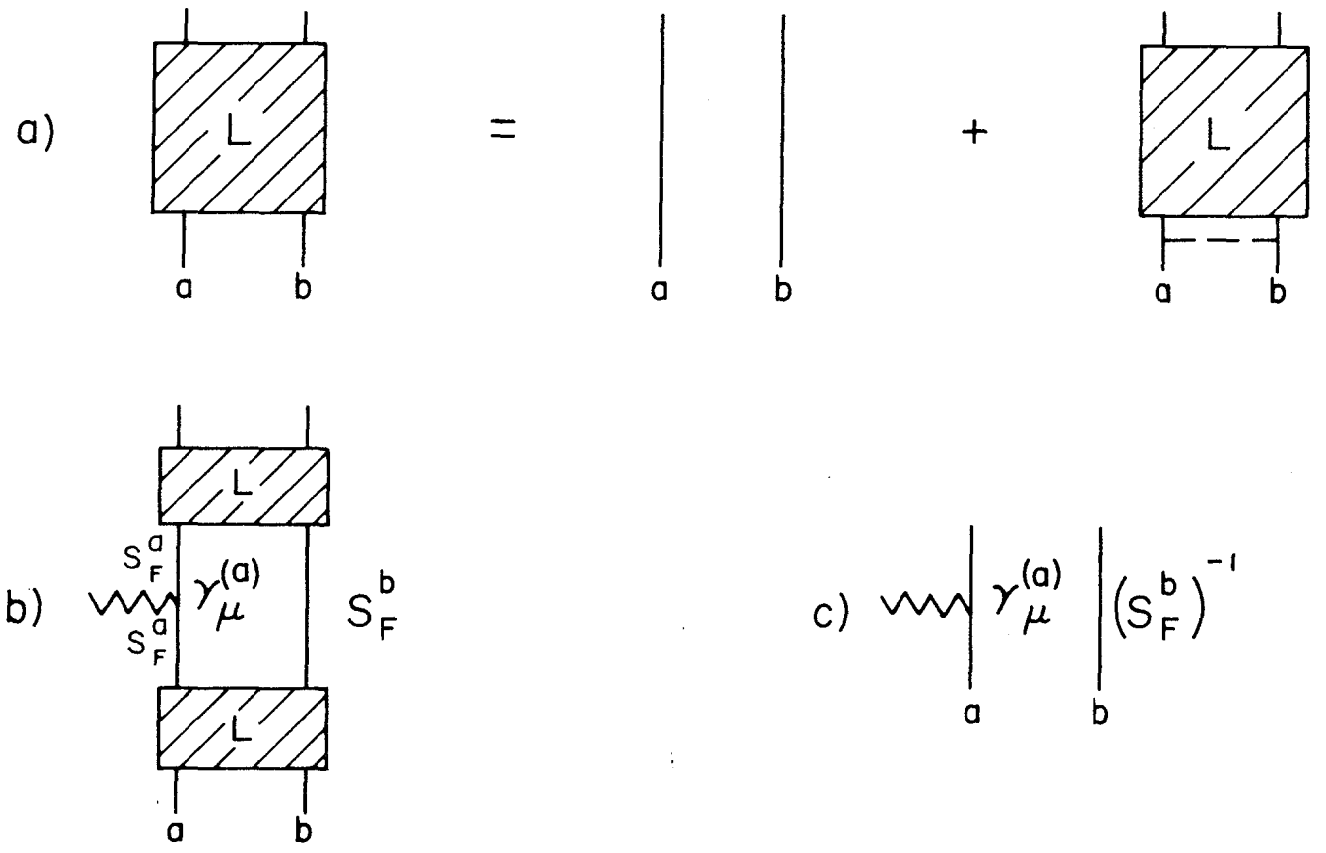


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Fig. 2

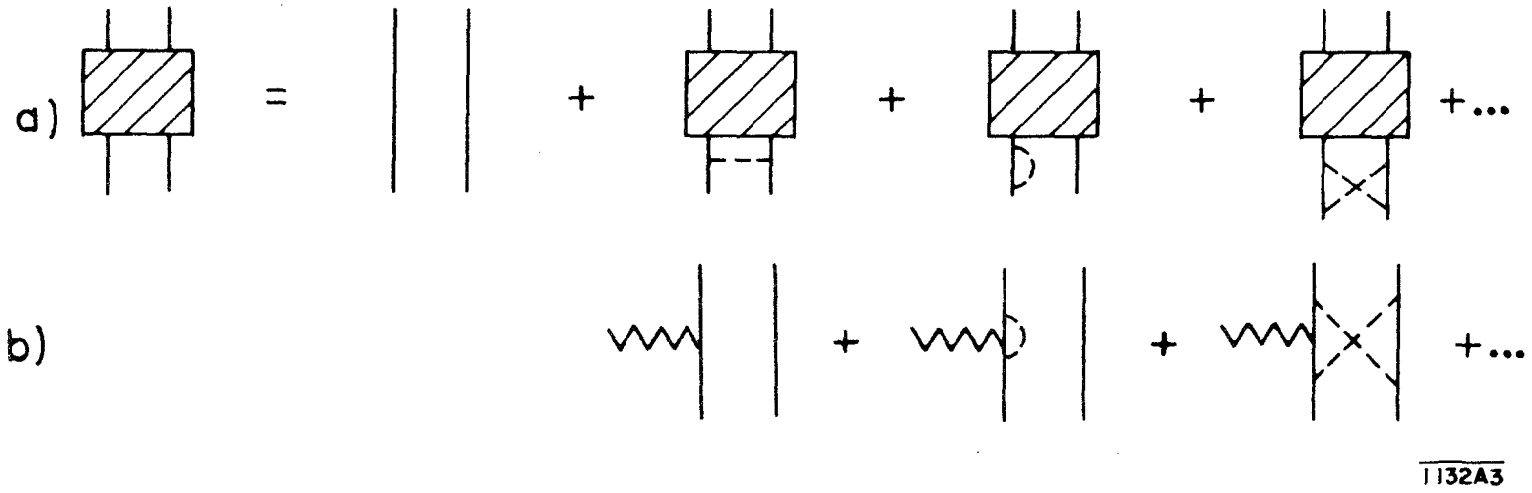

Fig. 3

TABLE I

Corrections to the Zeeman Interaction of Hydrogen-Like Atoms from a Bethe-Salpeter Analysis

Reduction Step	Contribution to Zeeman Interaction
1. Neglect self-energy and vacuum polarization kernels	anomalous moment terms + error of $0[\alpha(Z\alpha)^2\mu_e H]$
2. Neglect crossed graph kernel	Corrections $0[(Z\alpha)^4\mu_e H]$ + error of $0[(Z\alpha)^4(m_e/M_p)\mu_e H]$
3. Retain instantaneous kernel only	error of $0[(Z\alpha)^4(m_e/M_p)\mu_e H]$ + $0[(Z\alpha)^2\vec{V}^2\mu_e H]$
4. Adopt Breit formalism	error of $0[(Z\alpha)^4(m_e/M_p)\ \mu_e H]$

The expansion parameters are the fine structure constant α, $(Z\alpha)$ = the coupling strength of the binding interaction, and (m_e/M_p) = the electron to nucleus mass ratio. After these four reduction steps are made, the resulting interaction is given by Eq. (2.2) augmented by anomalous moment terms.

TABLE II

Notation Used for Bound Systems

	Spin 0 "pion"	- Spin $\frac{1}{2}$ "proton"	Spin $\frac{1}{2}$	- Spin $\frac{1}{2}$
Individual Masses	m	M	m_a	m_b
Total Mass				
Binding Energy	$W = M + m - \mathcal{m}$		$W = m_a + m_b - \mathcal{m}$	
Individual Charges (in units of e)	z	Z	z_a	z_b
Total Charge	$Z_T = z + Z$		$Z_T = z_a + z_b$	
Individual Magnetic Moments	0	$\mu = \frac{Ze}{2M} + \lambda$	$\mu_{a,b} = \frac{z_{a,b}e}{2m_{a,b}} + \lambda_{a,b}$	
Total Magnetic Moment		μ	$\mu = \mu_a + \mu_b$	
Coordinates	$\vec{r}_\pi$	$\vec{r}_p$	$\vec{r}_a$	$\vec{r}_b$

TABLE III

Contribution of Various Products of Terms to I

Term	$(E1)^2$	(E1)(M1)	(E1)(E2)	(E1)(SO)	(E1)(SI2)	$(E2)^2$	$(M1)^2$	(M1)(E2)	$(E2)^2$
Contribution	$\frac{1}{\alpha}$	1	1	α	α	α	α	α	α
Remarks	SI	P	P	$\surd$	SI	SI	$\surd$	S	SI

Explanation of Remarks:

SI - spin independent, hence vanishes when we calculate $I_P - I_A$,

P - vanishes by parity considerations,

S - vanishes because of its spin structure,

$\surd$ - could contribute.