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THE DESIGN OF LINEAR ELECTRON ACCELERATORS

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I. INTRODUCTION

The theory of operation of linear electron accelerators with beam loading and the design of such machines has been considered previously by Johnsen[1], Saxon[2], Leiss[3], and Neal.[4] The treatment given in these papers is somewhat preliminary, and it is therefore proposed to reconsider the problem.

The completely general solution of the mathematical problem of a relativistic beam interacting with a nonsynchronous ensemble of "space harmonic waves" with arbitrary boundary conditions has a form that precludes the possibility of easily grasping the physics of the situation. We therefore propose to consider the problem in several stages of increasing complexity.

The design of linear accelerators, on the other hand, can be accomplished on a much simpler basis, since in a machine design we do not have to accept the physical conditions that occur in a general statement of the problem.

II. BEAM-LOADING THEORY IN THE LINEAR ACCELERATOR

We propose to use the accepted definitions of waveguide properties, that is, the shunt impedance per unit length r, the field attenuation coefficient I and the energy velocity (equal to the group velocity v_g). Further, we will discuss only the case where the waveguide parameters are not varied. This therefore excludes two important cases, viz. the constant gradient accelerator and "bunching" sections. The latter constitutes a separate, quite involved problem. The former is omitted in the interest of briefness; moreover, the manner of handling this case is quite different.

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[1]K. Johnsen, Proc. Phys. Soc. 64, 1062 (1951).

[2]G. Saxon, Proc. Phys. Soc. 67, 705 (1954).

[3]J. Leiss, National Bureau of Standards, Internal Report.

[4]R. Neal, "Design of Linear Electron Accelerators With Beam Loading," Microwave Laboratory Report No. 379, W. W. Hansen Laboratories of Physics, Stanford University, Stanford, California, March 1957.

We take as our basic equation Poynting's theorem in the form

$$\frac{dP}{dz} = - \quad 2IP - \vec{i} \cdot \vec{E} \qquad (1)$$

where the left side, representing the net power flow out of the bounded volume consisting of a differential length of the cross section, is equated (by the conservation of energy) to the loss of stored electromagnetic energy to the conducting walls and to the momentum transferred to the beam current. The electric field at the location of the beam is related to the power flux and the properties (geometry and material) of the waveguide system. That is

$$E^2 = 2IrP$$

Let us suppose that the travelling wave is synchronous with the beam, and let us ignore the effects of space harmonics momentarily. We are still prevented from integrating the above expression directly, as we have not specified the phase relation between the beam and the wave. The initial value of phase angle is a boundary condition only.

We must consider the interaction from the electric field aspect in order to proceed further. Assume a constant current i is injected at a phase angle θ_o with respect to the rf wave crest. It is evident that the field component $E \cos \theta$ is both beam loaded and attenuated but that the component $E \sin \theta$ is only attenuated. The solution of the differential equations describing this condition

$$\frac{d(E \cos \theta)}{dz} = - I(E \cos \theta + ir)$$

$$\frac{d(E \sin \theta)}{dz} = - IE \sin \theta$$

subject to the boundary condition $\theta = \theta_o$ and $E = E_o$ at $z = 0$ is

$$E \cos\theta = E_o \cos\theta_o \, e^{-Iz} - ir\left(1 - e^{-Iz}\right) \qquad (2)$$

$$E \sin\theta = E_o \sin\theta_o \, e^{-Iz}$$

The electric wave will hence appear to change phase with respect to the beam during the passage down the waveguide, the relative phase being given by

$$\tan\theta = \frac{\tan\theta_o}{1 + m\left(1 - e^{Iz}\right)} \qquad (3)$$

where

$$m = ir/E_o \cos\theta_o$$

From Eq. (2) the power flow down the waveguide is

$$P(z) = P_o \, e^{-2Iz} + \sqrt{\frac{2P_o r}{I}}\, i \cos\theta_o \left(e^{-Iz} - 1\right)e^{-Iz} + \frac{i^2 r}{2I}\left(e^{-Iz} - 1\right)^2 \qquad (4)$$

In this form it is evident that the first term describes the attenuation due to the structure, the second term describes beam-loading, and the third term describes the reradiation of the beam into the structure. When $\theta_o = 0$, the beam remains in step with the wave and Eq. (1) could have been integrated directly, being Bernoulli's equation. When $\theta_o = \pi/2$, or even if there is no input power, the last term describes the reradiated power by relativistic beams into periodic structures.

The energy gain of particles passing through the waveguide is obtained from the integration of Eq. (2)

$$V = \int_0^{\ell} \left[\left(E_o \cos\theta_o + ir\right) e^{-Iz} - ir\right] dz$$

$$= \left(E_o \cos\theta_o + ir\right)\left(\frac{1 - e^{-I\ell}}{I}\right) - ir\ell \qquad (5)$$

In order to consider here more general aspects of the problem we pass now to the space harmonic representation. It is well-known that the electric field in periodic structures may be analyzed in terms of components of various amplitudes and phase velocities in the form

$$E_z = \sum_{-\infty}^{+\infty} E_n \, e^{-j\left(\omega t - \beta_n z\right)}$$

$$\beta_n = \beta_o + 2\pi n/p \tag{6}$$

(where p is the periodicity of the structure).

It is obvious by inspection that when a particle is synchronized with one of the waves of the set ($t = z/v$, $\beta_n = \omega/v$) the line integral over a period of the structure is $E_n p$ for that component and zero for all other components. This is of course a general property of expansions in orthogonal functions. Unfortunately, due to attenuation this is not precisely true in our case; that is, the effects of the space harmonics do not exactly cancel. Thus, the energy gain in the case of a particle synchronized with the crest of the zeroth space haromonic would be, neglecting beam loading,

$$V = \int_o^{\ell} \sum E_n(0) e^{-Iz-j\left(\omega t - \beta_n z\right)} \, dz = \sum E_n(0) \, \frac{I\left(1 - e^{-I\ell}\right)}{I^2 + \left(\frac{2\pi n}{p}\right)^2} \tag{7}$$

It is evident that for practical values of attenuation the higher space harmonics will not seriously affect the action of the synchronous wave. This remark, however, is somewhat superficial, since we have not discussed the manner in which the space harmonics transfer power to the beam-loaded component of the wave.

We know from the boundary-matching condition that all components must have the same attenuation. Therefore, if one component is beam-loaded we can only assume the remaining components transfer power to this component to maintain the relative harmonic amplitudes required by the boundary conditions. This solution is obtained by substituting the space harmonic expansion, Eq. (6), into Poynting's theorem, Eq. (1), where each harmonic amplitude is given by $E_m^2 = 2IPr_m$. In order to solve the resulting equation, we must be more specific about the beam current. We will, for example, assume the beam is highly relativistic and synchronous with the fastest component of the space harmonics $(n = 0)$. Then $t = z/c$ and $c = w/\beta_o$, or $(wt - \beta_n z) = - 2\pi nz/p$ and the differential equation

$$\frac{dP}{dz} = - 2IP - i \sum \sqrt{2IPr_n}\ e^{-j2\pi nz/p}$$

has the solution, with boundary conditions $t = 0,\ z = 0,\ P = P_o$,

$$\sqrt{P} = \sqrt{P_o}\ e^{-Iz} - i \sum \sqrt{\frac{Ir_n}{2}} \left[\frac{e^{j2\pi nz/p} - e^{-Iz}}{I + j2\pi n/p} \right] \tag{8}$$

With only the first term of the series $(n = 0)$ Eq. (8) is seen to give the solution found earlier, Eq. (4). In addition, the interaction between the beam-loaded component and the other components is demonstrated. The extension to cases where there is a phase difference between the beam and the rf wave or where the wave is nonsynchronous is obvious.

The energy acquired by the beam is the line integral of the electric field represented by the power flux of Eq. (8), of which the fundamental component is

$$E_o(z) = E_o(0)e^{-Iz} - iI \sum \sqrt{r_o r_n}\ \frac{e^{j2\pi nz/p} - e^{-Iz}}{I + j2\pi n/p} \tag{9}$$

Thus the energy gain is given by

$$V = \int_0^z E_0(z)dz$$

$$= \left(E_0(0) + ir_0\right)\frac{1 - e^{-Iz}}{I} - ir_0 z - \sum \frac{Ii\sqrt{r_0 r_n}}{I + j2\pi n/p}\left[\frac{e^{j2\pi nz/p} - 1}{j2\pi n/p} - \frac{1 - e^{-Iz}}{I}\right] \tag{10}$$

The first term in the series contributes nothing to the energy over any integral number of periods. However, it is evident that there is a small contribution to the energy from the higher space harmonics, which depends on the lossiness of the structure.

The only serious remaining question is whether the space harmonic amplitude ratios of the form given by Eq. (9) are independent of the distance down the guide with beam-loading. That they are is almost obvious by inspection but may be proved by taking the ratio of two arbitrary harmonics. This ratio will be seen to be independent of z.

The physical location in which the exchange of power between the space harmonics takes place is presumably in the obstructed regions of the waveguide, where the space harmonics do not individually satisfy the homogeneous wave equation.

III. DESIGN OF LINEAR ACCELERATORS

The concept of design implies the maximization of some property with respect to another, usually subject to certain conditions. We are interested in the choice of a waveguide of length ℓ, having the parameters of shunt impedance per unit length r, power attenuation per unit length $2I$ and group velocity v_g. The object to be accomplished is usually specified as the attainment of a specified energy V for a beam current i with an available power P_0. The design must therefore be based on Eq. (5). This analysis will now be undertaken for the principal types of machines.

A. MAXIMUM ENERGY ACCELERATOR

If we maximize the energy gain in the beam-loaded case, Eq. (5), with respect to the attenuation constant, we find the condition illustrated in Fig. 1

$$\frac{i^2 r}{P_o} = \frac{I}{2}\left[\frac{\left(e^{I\ell} - 1\right) - 2I\ell}{I\ell - \left(e^{I\ell} - 1\right)}\right]^2 \tag{11}$$

If we insert this condition in Eq. (5), we find the expression illustrated in Fig. 2

$$\frac{iV_m}{P_o} = \left[\frac{\left(e^{I\ell} - 1\right) - 2I\ell}{I\ell - \left(e^{I\ell} - 1\right)}\right]\left[1 - e^{-I\ell} + \frac{1 - e^{-I\ell} - I\ell}{2}\left(\frac{e^{I\ell} - 1 - 2I\ell}{I\ell - e^{I\ell} + 1}\right)\right] \tag{12}$$

Thus, given V, P_o and i, a kind of efficiency of the machine, we find the value of $I\ell$. With this value of $I\ell$ and a specified choice of ℓ we determine r from Eq. (11). There is no other machine of this length and "efficiency" that will produce a higher beam energy. The case of vanishing beam current ($I\ell = 1.255$) was pointed out by Harvie.[5]

We have illustrated in Fig. 3 the ratio of steady-state to no-load energy. The diagram gives some impression of the energy spread to be expected during the transient regime. This ratio is obtained by inserting the beam current of Eq. (11) in the energy equation, Eq. (5), and factoring out the no-load energy.

$$\frac{V_m}{V_o} = 1 + \frac{1 - e^{-I\ell} - I\ell}{2\left(1 - e^{-I\ell}\right)}\left[\frac{\left(e^{I\ell} - 1\right) - 2I\ell}{I\ell - \left(e^{I\ell} - 1\right)}\right] \tag{13}$$

How seriously this affects the energy spectrum depends upon the fill-time of the structure.

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[5]R.B.R.S. Harvie, Proc. Phys. Soc. 61, 255 (1948).

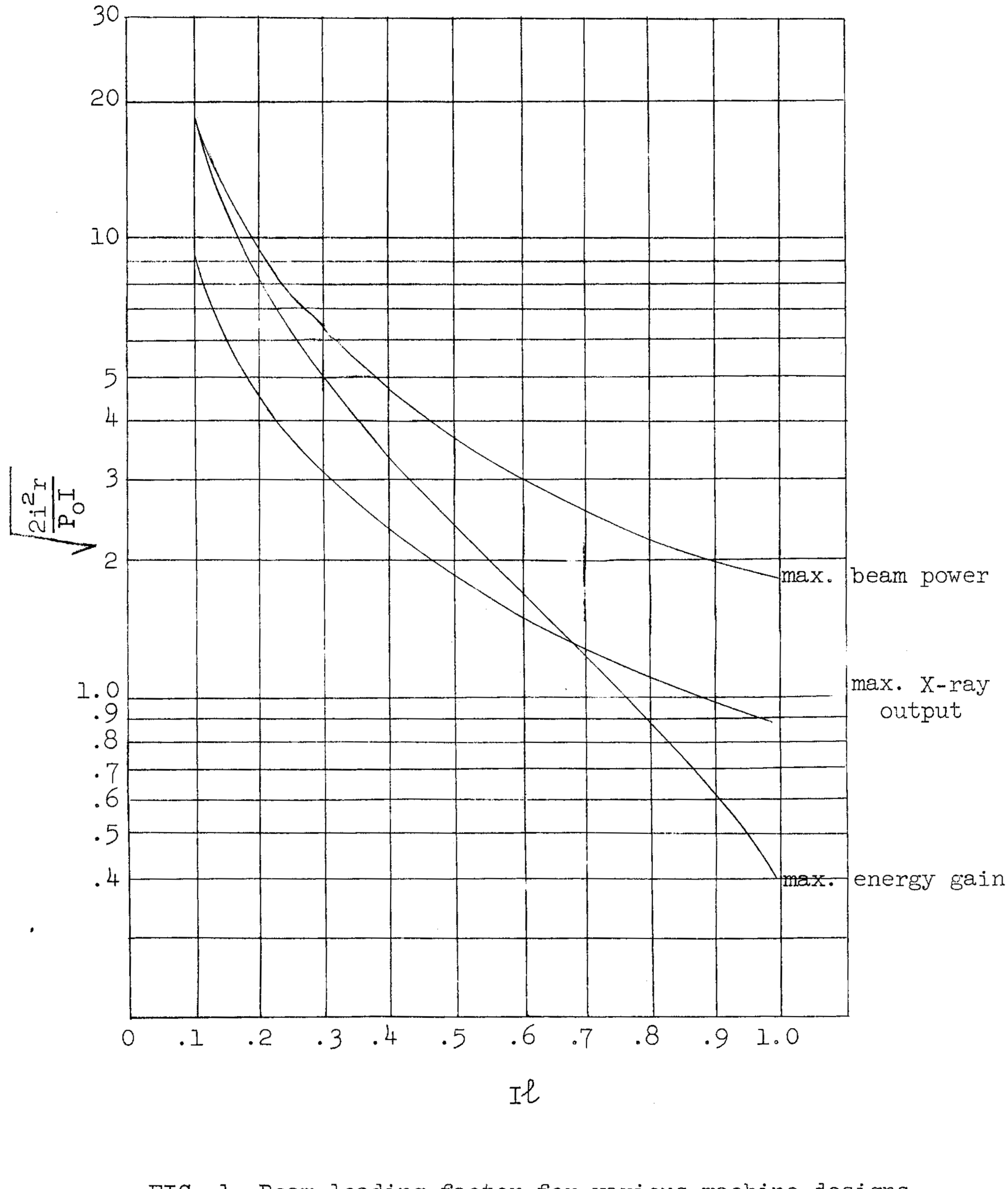

FIG. 1--Beam loading factor for various machine designs.

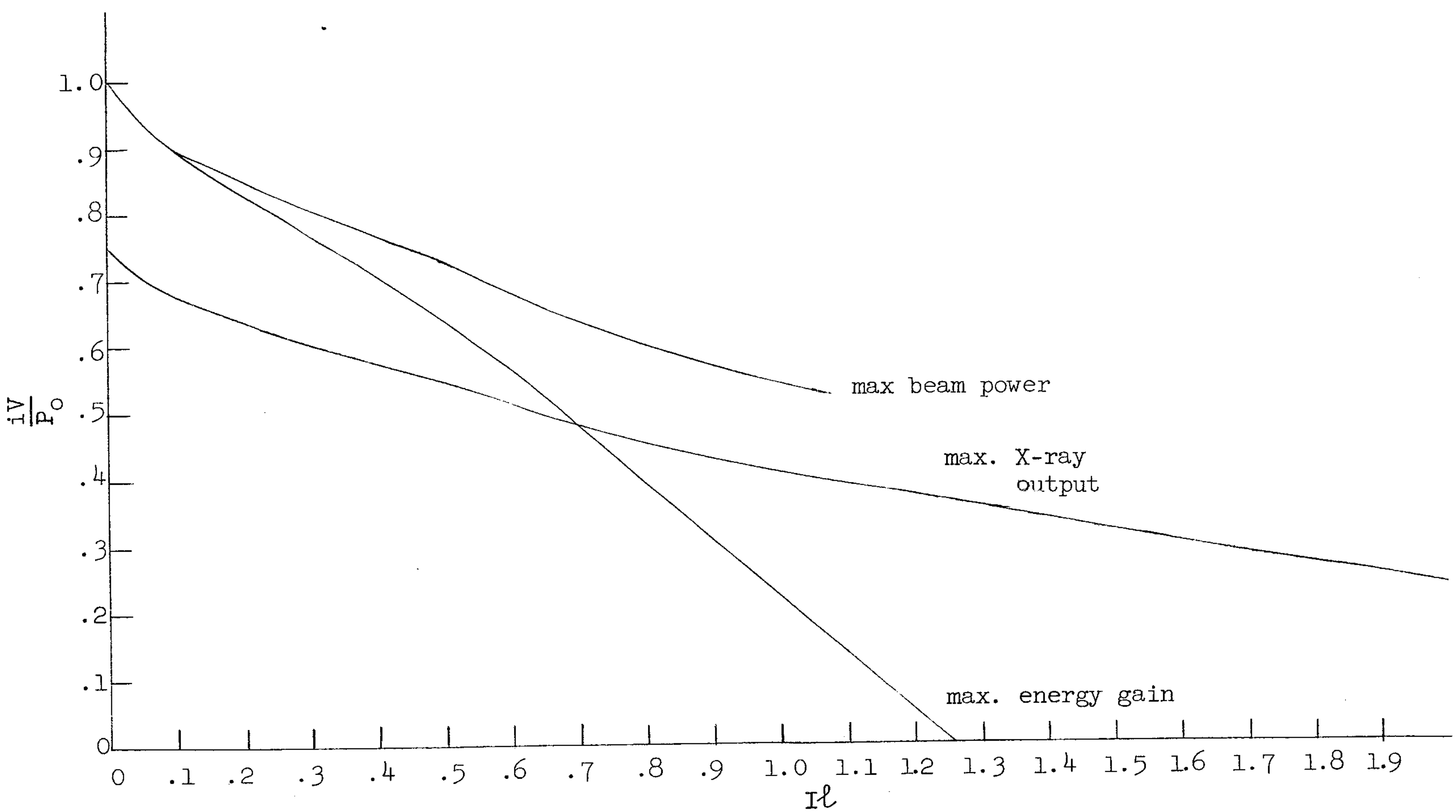

FIG. 2--"Efficiency" of various machines as a function of attenuation.

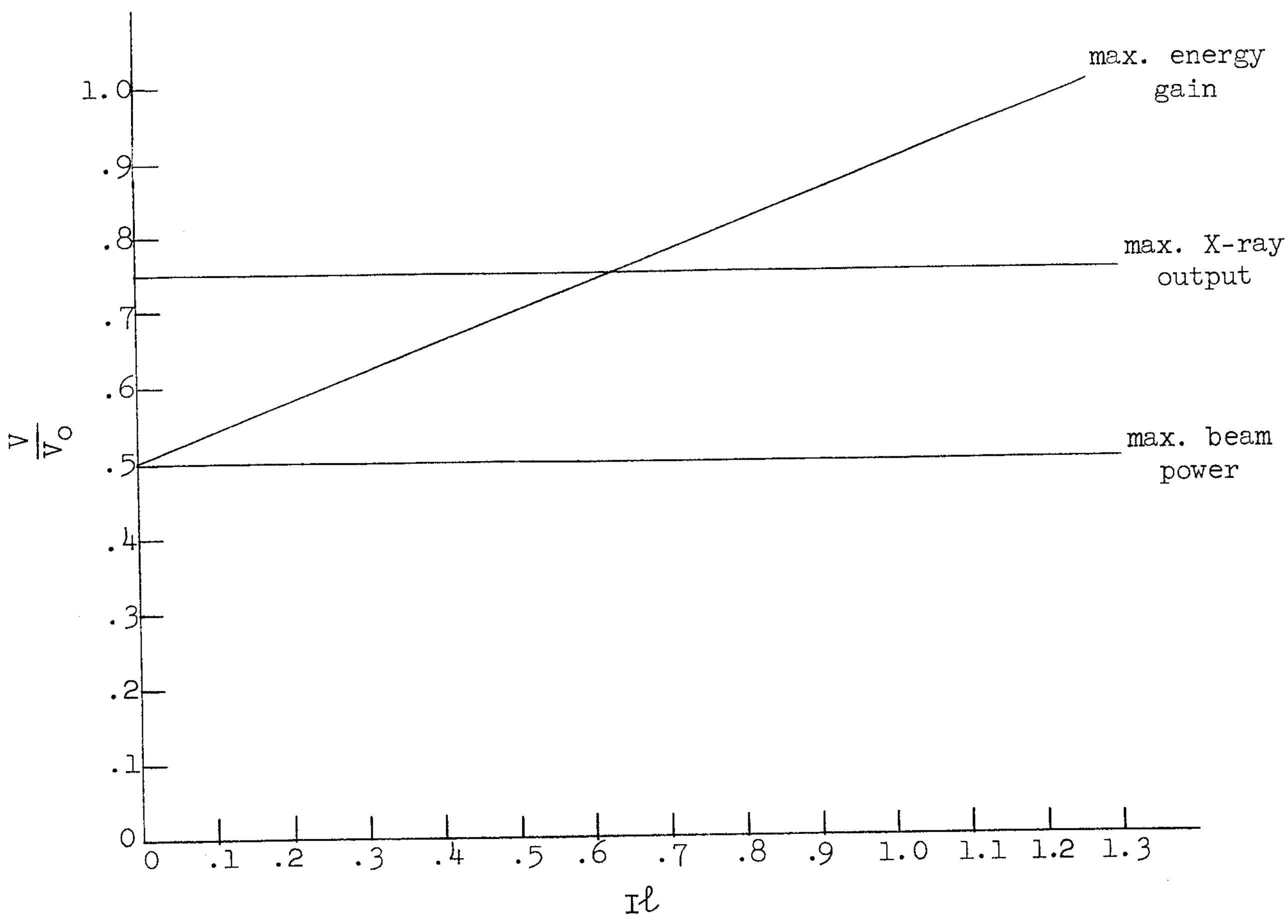

FIG. 3--Output energy of various machines as a function of attenuation.

B. MAXIMUM BEAM POWER ACCELERATOR (MAXIMUM EFFICIENCY ACCELERATOR)

Suppose now that we maximize the energy gain in the beam-loaded case with respect to the shunt impedance of the structure. We would find then the necessary condition, shown in Fig. 1

$$\frac{i^2 r}{P_o} = \frac{I}{2}\left[\frac{1 - e^{-I\ell}}{I\ell - \left(1 - e^{-I\ell}\right)}\right]^2 \qquad (14)$$

Inserting this condition in Eq. (5) the beam energy is seen to be

$$V_p = \frac{V_o}{2} \qquad (15)$$

and the power conversion efficiency

$$\frac{iV_p}{P_o} = \frac{1}{2}\frac{\left(1 - e^{-I\ell}\right)^2}{I\ell - \left(1 - e^{-I\ell}\right)} \qquad (16)$$

as illustrated in Figs. 3 and 2, respectively. As in the previous case when the desired beam power and available input rf power are specified, the value of $I\ell$ is determined by Eq. (16). With the choice of a specified length the required shunt impedance is given by Eq. (14). There is no machine of this length and efficiency giving a larger beam power.

This machine has been described by Saxon and Leiss, but their method is essentially the maximization of the beam power iV (where V is the beam loaded energy gain) with respect to the beam current. The condition is precisely the same as that given by Eq. (14).

C. MAXIMUM X-RAY OUTPUT ACCELERATOR

It has been shown by W. W. Buechner et al.[6] and C. W. Miller[7] that the X-ray intensity in the forward direction per unit beam current varies nearly as the cube of the beam energy, in the range 1-10 Mev. We therefore maximize the product iV^3 with respect to the beam current to

[6] W. W. Buechner et. al., Phys. Rev. 74, 1348 (1948).
[7] C. W. Miller, Jour. B.I.R.E. 14, 361 (1954).

obtain the greatest X-ray output. This condition, shown in Fig. 1, is

$$\frac{i^2 r}{P_o} = \frac{I}{8}\left[\frac{1 - e^{-I\ell}}{I\ell - \left(1 - e^{-I\ell}\right)}\right]^2 \tag{17}$$

Inserting this condition in Eq. (5) we find the beam energy

$$\frac{V_x}{V_o} = \frac{3}{4} \tag{18}$$

and a machine efficiency

$$\frac{iV_x}{P_o} = \frac{3}{8}\frac{\left(1 - e^{-I\ell}\right)^2}{I\ell - \left(1 - e^{-I\ell}\right)} \tag{19}$$

both shown in Figs. 2 and 3.

In each of these three types of machines it appears that having chosen the machine efficiency (i, V, P_o) we thereby choose the total attenuation of the guide $I\ell$. Similarly, we have chosen the ratio r/I, or indirectly the total shunt impedance $r\ell$ of the guide.

However, there is another restriction on the product Ir that arises from the allowable field strength in the machine, since $E^2_{max} = 2IrP_o$. We are now in a position to individually specify I, r and ℓ, since both r/I and Ir are determined.

It is perhaps noticeable that the group velocity of the structure did not enter into the description of the steady-state design. The role of the group velocity has been sufficiently discussed by Harvie. Briefly, the length of time required by transient processes and the required frequency stability of the rf drive are governed by the group velocity. The dephasing of the beam and the wave $\Delta\varphi$ due to a frequency error (or frequency modulation) Δf is well-known to be given very closely by

$$\frac{\Delta\varphi}{\varphi} = \frac{v_p}{v_g}\frac{\Delta f}{f}$$

where $\varphi = 2\pi \ell/\lambda$. Thus, the known instability of the source sets the lower limit on the group velocity. This same equation is used to set the dimensional tolerances on the structure, as has been described by Chodorow, et al.[8]

The transient regime of the machine is complicated by the numerous possibilities. In the past the beam has been fired when the machine was filled with rf power. However, it has been shown by Leiss that other firing times are superior from the point of view of output energy spectrum, e.g., firing both gun and rf power together or firing the gun a specified time after the rf power but before the fill time.

IV. THE DESIGN OF HIGH-ENERGY ACCELERATORS

While the above remarks appear reasonable, the fact is that linear accelerators are not designed on that basis. Furthermore, in order to construct high-energy accelerators, it is obvious that we will use many separate sources of rf power. In order to determine the total length of the machine and the number of sources required, we make the following observations:

a. The efficiency of the whole machine is the same as that of any section.

b. The number of sources depends only on the efficiency and is independent of the length of the machine.

c. The length of the machine depends upon the efficiency and allowable gradient. The higher the gradient, the shorter the machine, but the lower the efficiency for a fixed shunt impedance.

d. Given any length, the highest efficiency requires the highest shunt impedance and lowest field gradient possible. For the same efficiency, the higher the shunt impedance, the shorter the machine.

e. The length of the machine decreases nearly directly as the attenuation constant for constant efficiency; the lower the attenuation constant, the higher the efficiency for a fixed length.

It is evident therefore that we prefer the highest shunt impedance and the lowest attenuation constant available, but this is almost intuitively obvious. However, this changes the general aspect of the problem as given in Sec. III above.

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[8]M. Chodorow, E.L. Ginzton, W.W. Hansen, R.L. Kyhl, R.B. Neal, W.K.H. Panofsky, and the Staff of the W.W. Hansen Laboratories of Physics, Rev. Sci. Instru. 26, 134 (1955).

Note also that the energy gain equation given above, Eq. (5), may by substituting the definition of shunt impedance be put in the form

$$\frac{V}{E_0 \ell} = \frac{1 - e^{-I\ell}}{I\ell} + \frac{E_0 \ell}{V} \frac{\eta}{2I\ell} \left(\frac{1 - e^{-I\ell}}{I\ell} - 1 \right) \tag{20}$$

This quadratic equation is evidently true for the entire accelerator or any section of it. Since the attenuation of the guide $I\ell$ is a function of the machine efficiency for a given type of machine, the ratio given in Eq. (20) may be expressed as a function of efficiency (or attenuation) only. In Fig. 4, this ratio is given for general values of η and $I\ell$, as well as for specified types of machines.

In order to demonstrate the use of this analysis, consider the following example.[9] As mentioned above, an accelerator is customarily designed to produce a beam current i at some specified energy V_T using specified rf power sources P_0. When this is to be done using a structure whose parameters of attenuation coefficient I and shunt resistance r are given, Fig. 4 may be replotted as a function of V and ℓ only, as shown in Fig. 5. (A complete set of data for various currents is given to demonstrate the method.)

For high-energy accelerators the number of sources required is $N = V_T/V$, and the length of the machine is $L = N\ell$, so that a plot such as Fig. 6 may be given directly by Fig. 5. There are no other possibilities of constructing an accelerator with the specified parameters. The choice of which accelerator would be the optimum design is a matter for further consideration. The points A and B marked on the 0.100 ampere beam current curve are the maximum energy gain condition and the minimum of the product LN.

A possible criterion in the choice of a machine is the beam-loading characteristic, or energy variation with a change in beam current. For

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[9]The example taken is illustrative of the accelerator described in in the "Proposal for a Two-Mile Linear Electron Accelerator," Stanford University, Stanford, California, April 1954.

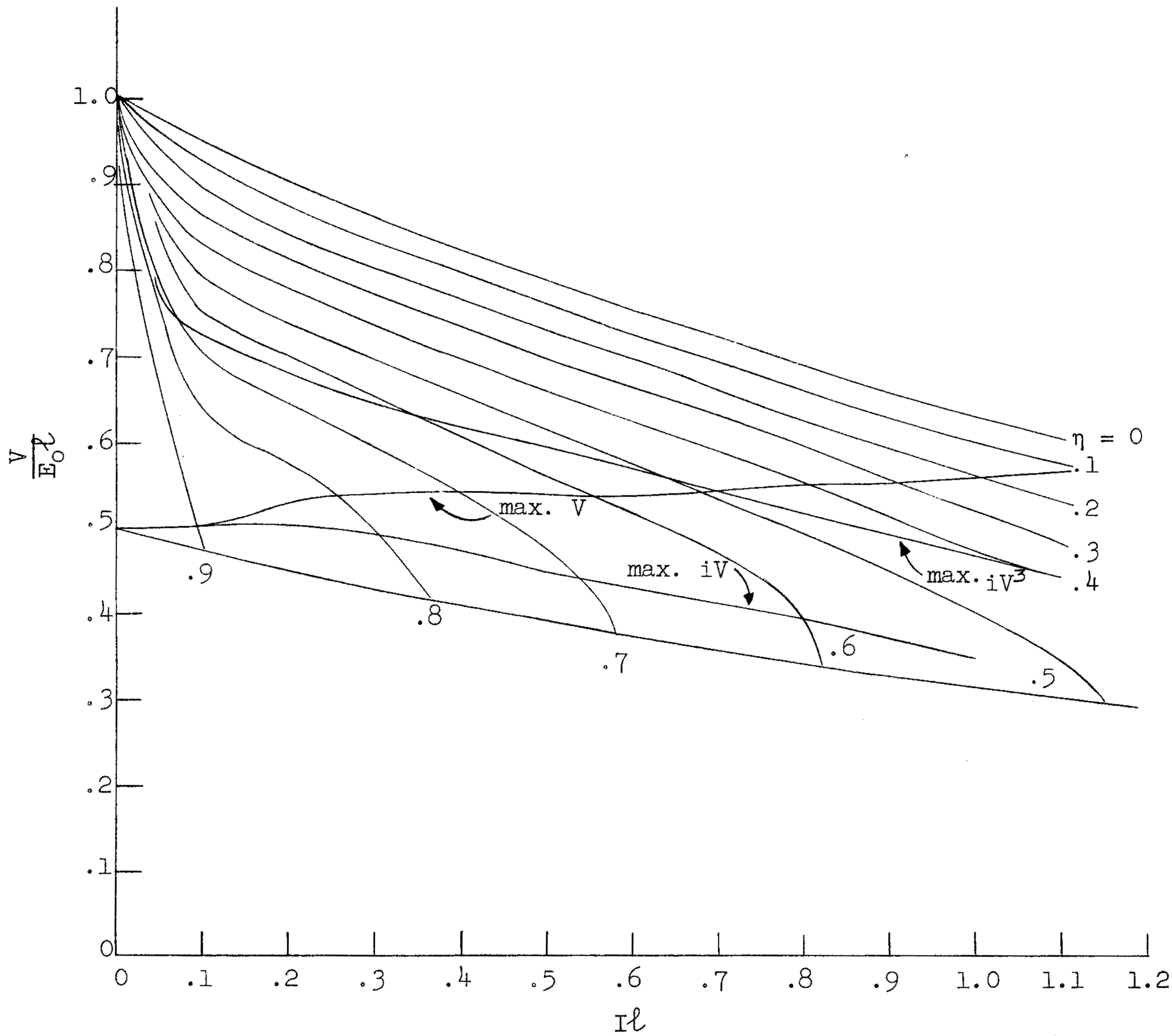


FIG. 4-- $$\frac{V}{E_o \ell} = \frac{1}{2}\left[\frac{1 - e^{-I\ell}}{I\ell} + \sqrt{\left(\frac{1 - e^{-I\ell}}{I\ell}\right)^2 + \frac{2\eta}{I\ell}\left(\frac{1 - e^{-I\ell}}{I\ell} - 1\right)}\right]$$

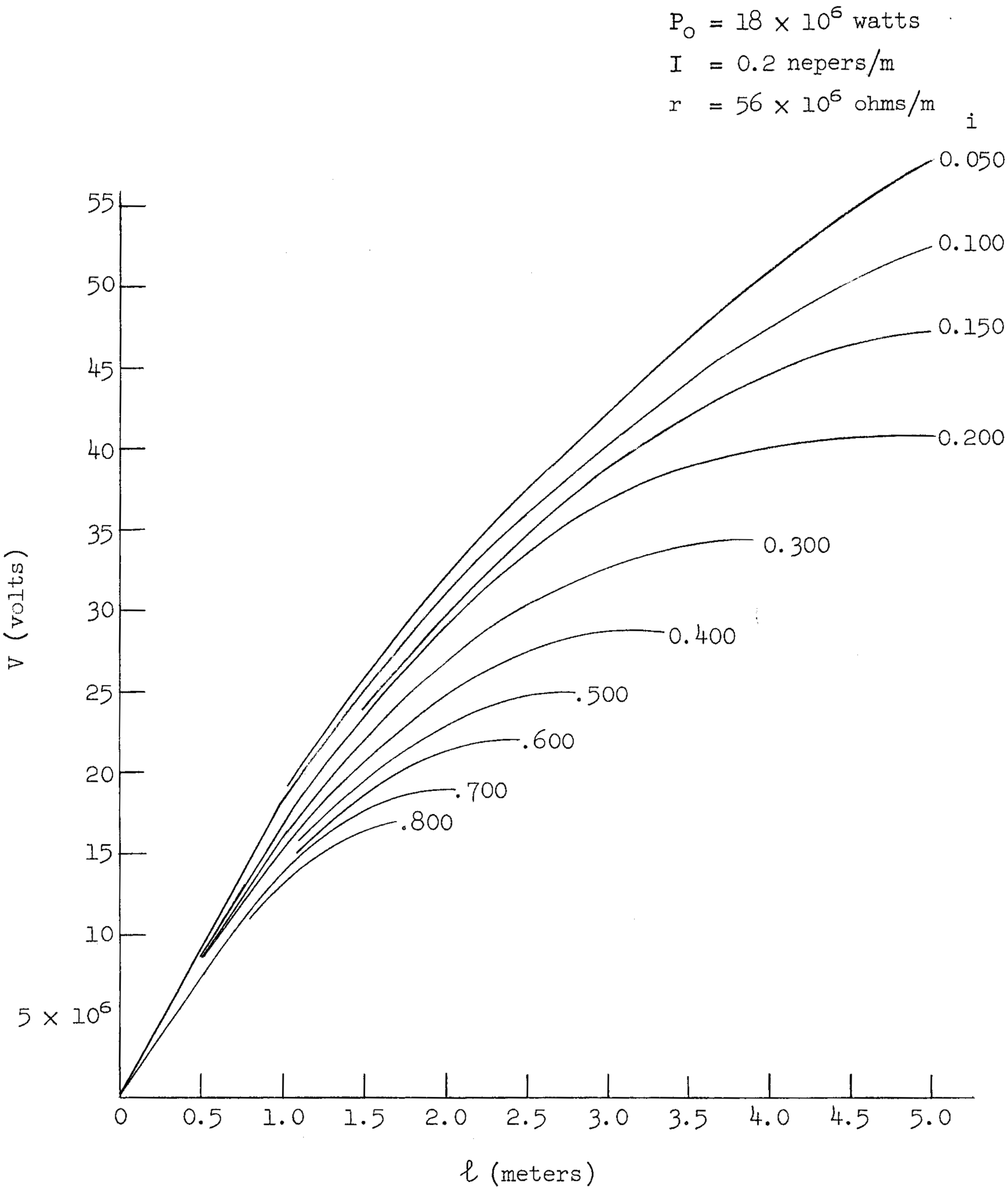
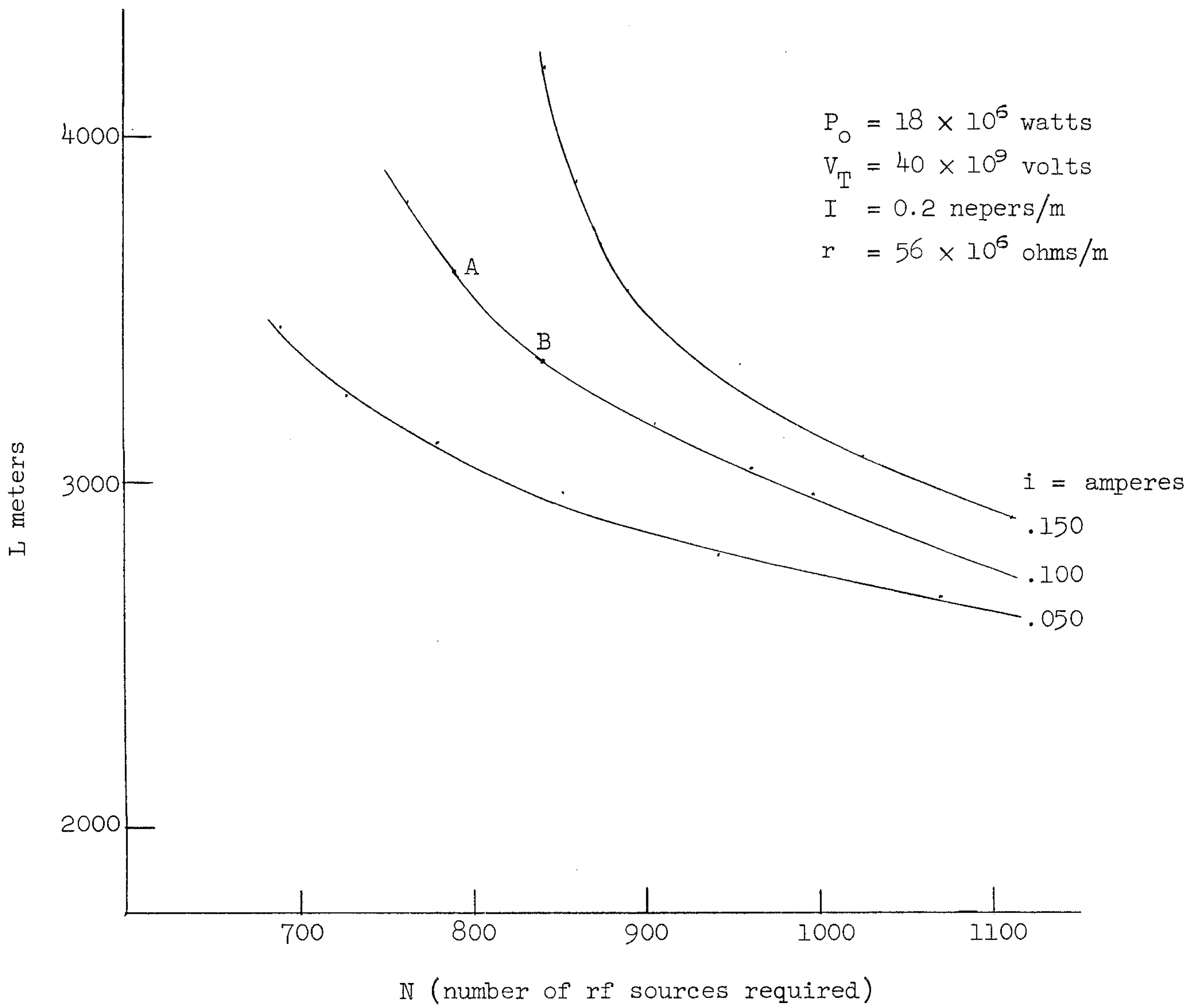

$P_o = 18 \times 10^6$ watts
$I = 0.2$ nepers/m
$r = 56 \times 10^6$ ohms/m
i
0.050
0.100
0.150
0.200
0.300
0.400
.500
.600
.700
.800
55
50
45
40
35
30
25
20
15
10
5×10^6
V (volts)
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
ℓ (meters)

FIG. 5



P_o = 18 x 10^6 watts
V_T = 40 x 10^9 volts
I = 0.2 nepers/m
r = 56 x 10^6 ohms/m
i = amperes
.150
.100
.050
A
B
4000
3000
2000
L meters
700
800
900
1000
1100
N (number of rf sources required)

FIG. 6

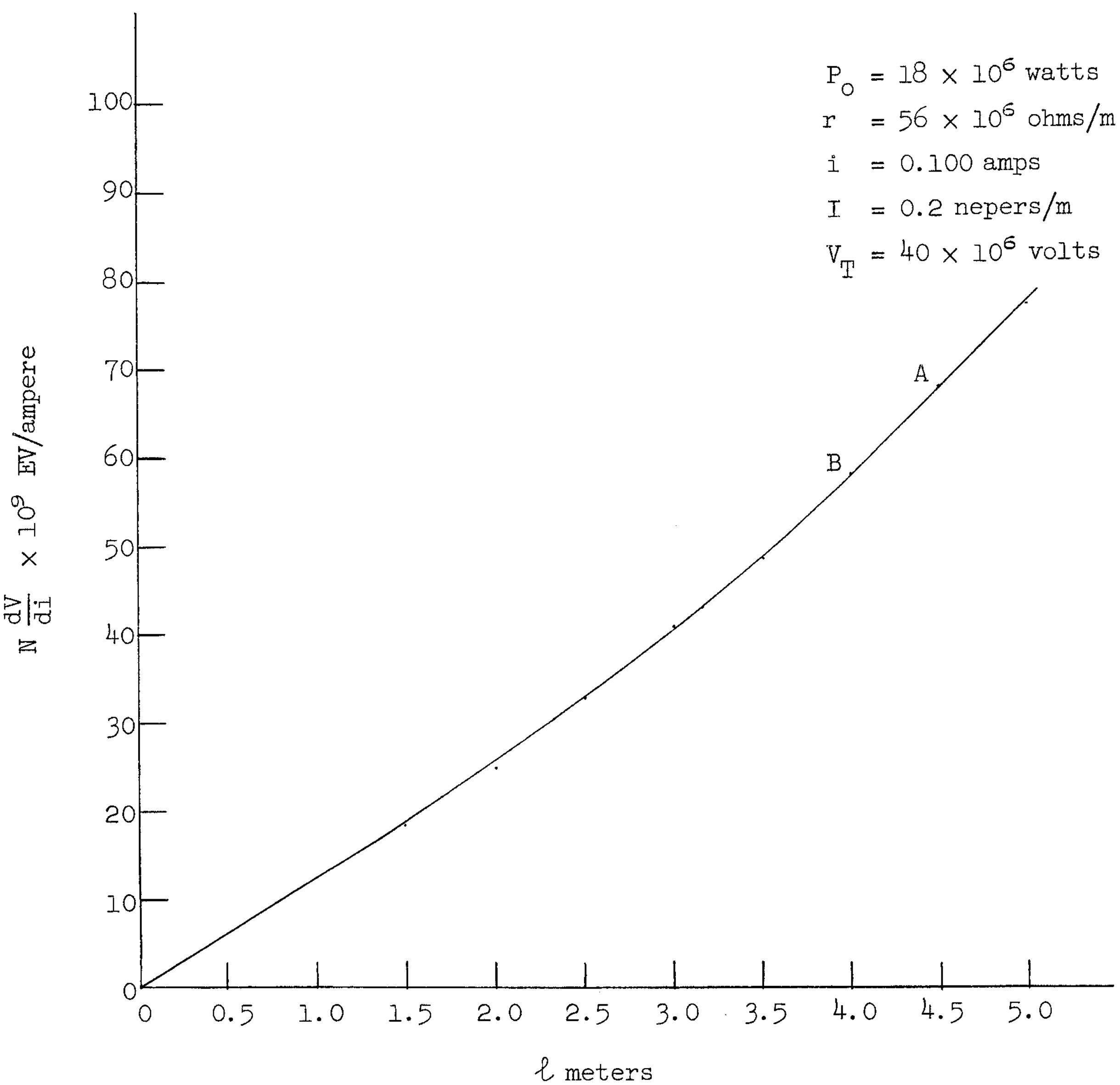
a multiple-section machine this is given by $N(dV/di)$ where

$$\frac{dV}{di} = - r\ell \left(1 - \frac{1 - e^{-I\ell}}{I\ell} \right) \tag{21}$$

for an individual section, and is obtained by differentiating the energy-gain equation, Eq. (5). Continuing the example considered above, Fig. 7 presents this characteristic. Examination of this graph reveals that, regardless of the length of the sections, a given energy excursion occurs for about the same value of beam-current variation. For instance, if we wish the output energy to remain within 1 percent of total energy, the allowable beam-current variation, given in Table I, shows that this is not a critical matter.

TABLE I

Section Length	Allowable Current Variation
2.5 m	0.012 amp
3.0	0.010
3.5	0.008
4.0	0.007
4.5	0.006
5.0	0.005



P_o = 18 × 10^6 watts
r = 56 × 10^6 ohms/m
i = 0.100 amps
I = 0.2 nepers/m
V_T = 40 × 10^6 volts
A
B
$N \frac{dV}{di} \times 10^9$ EV/ampere
0
10
20
30
40
50
60
70
80
90
100
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
ℓ meters

FIG. 7