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SCATTERING OF BEAM ELECTRONS BY THE RESIDUAL GAS IN THE ACCELERATOR

By

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I. INTRODUCTION AND SUMMARY

In an electron linear accelerator the scattering of beam electrons by the residual gas in the machine places a limit on the vacuum required. In practice, the actual vacuum seems to be determined by experience with rf breakdown (pressures $\lesssim 10^{-5}$ mm Hg), so this note is mainly of academic interest. However, it seems that the limits previously derived from multiple scattering calculations, although approximately correct numerically, are probably wrong in principle, for two reasons: first, because there is so little gas in a machine that multiple scattering theory does not apply; and second, because the multiple scattering formula frequently used in previous calculations seems to be wrong.

In this memo we calculate the beam loss caused by single Coulomb scattering for an infinitely narrow, parallel initial beam in a constant-gradient machine under two different focusing conditions: first, that there is no radial focusing; and second, that there is radial focusing which is strong enough to contain a transverse momentum p_T. The results are summarized in the table below, where Q is the pressure in units of 10^{-5} mm Hg; R is the radius of the exit aperture in centimeters; E_f is the final energy in units of 40 Bev; p_T is the transverse momentum in units of 0.1 Mev/c.

Condition	Percent Beam Loss	For Different Conditions Multiply Loss by This Factor
No focusing	.04%	$\left(\dfrac{Q}{10^{-5}\text{ mm Hg}}\right)\left(\dfrac{1\text{ cm}}{R}\ \dfrac{40\text{ Bev}}{E_f}\right)^2$
With focusing	.03%	$\left(\dfrac{Q}{10^{-5}\text{ mm Hg}}\right)\left(\dfrac{0.1\text{ Mev/c}}{p_T}\right)^2$

These numbers are similar to those given by Helm and Panofsky in M-201. Finite beam size would increase the loss, perhaps substantially.

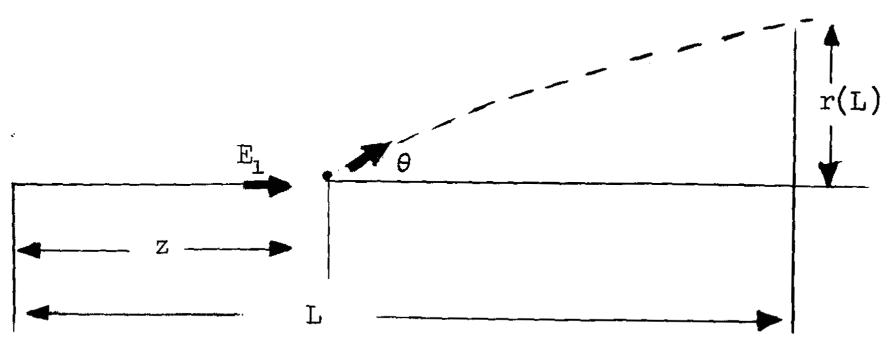
For no focusing, we calculate the distribution and strength of the radiation source caused by the scattered particles.

We discuss the multiple-scattering situation briefly.

II. SINGLE SCATTERING WITH NO FOCUSING

A. Total Beam Loss

The geometry of the problem is shown in the figure.



At a distance z down the machine an electron of energy $E_1(z)$ scatters from a gas atom through an angle θ. The question is whether the scattered electron hits the machine. We assume that the electron's trajectory before scattering is exactly along the axis of the machine. If

$r(L)$ is the radial displacement of the electron at the end of the machine, the electron hits if $r(L) \geqq R$ where R is the radius of the hole in the disks, or where R is the radius of a collimator at the end of the machine.

In the case of relativistic electrons and a simplified machine which has a uniform accelerating field and no radial forces (e.g., no quadrupoles or coupler asymmetries), the radial displacement at the end of the machine has the well-known logarithmic form

$$r(L) = \left(\frac{\theta E_1}{\epsilon}\right) \ln \left(\frac{E_f}{E_1}\right) \cong L\theta \left(\frac{E_1}{E_f}\right) \ln \left(\frac{E_f}{E_1}\right) = \theta z \ln \frac{L}{z}$$

Here E_f is the energy at the end of the machine, E_1 is the energy at the scattering point, and ϵ is the energy gain per unit length. If the injection energy is much less than the final energy, $\epsilon \cong E_f/L$, and the second equality holds.

For the atomic-scattering cross section we use the calculation of Goudsmit and Saunderson for an exponentially screened Coulomb field $[(Z/r) \exp(-r/r_a)]$ as given in Rossi, High Energy Particles, p. 65. The probability that a beam electron scatters into the solid angle $d\Omega$ at the angle θ as it passes through dz centimeters of a material with density ρ (atomic number Z and atomic weight A) is

$$f(z, \theta)\, dz d\Omega = 4\, Z^2\, r_e^2 \left(\frac{m_e c}{p_1 \beta}\right)^2 \frac{1}{\left(\theta^2 + \theta_1^2\right)^2}\ \frac{N}{A}\ \rho\, dz d\Omega$$

Here N is Avogadro's number, r_e is the classical radius of the electron (2.8×10^{-13} cm), m_e is the electron mass, c is the velocity of light, p_1 is the momentum of the beam electron (a function of z), β is its velocity relative to c, and

$$\theta_1 = \frac{m_e c}{p_1(z)} \left(Z^{1/3} \alpha\right)$$

where $\alpha = 1/137$. Here θ_1 is a kind of atomic form factor, and it is equal to the de Broglie wavelength (divided by 2π) of the particle divided by the screening radius of the Fermi-Thomas atom, $r_a = r_e/\alpha^2 Z^{1/3}$. Note that θ_1 is a function of z. We choose units such that $c = 1$. For relativistic electrons $p_1 \cong E_1$, and we rewrite the probability

$$f(z,\theta)\ dz d\Omega = 4\ Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \frac{\theta_1^2}{\left(\theta^2 + \theta_1^2\right)^2} \frac{N}{A} \rho\ dz d\Omega$$

At a fixed value of z we integrate over all values of θ for which the scattered electron hits the machine:

$$F(z)\ dz = dz \int_{\theta_m}^{\infty} f(z,\theta)\ d\Omega$$

All angles greater than θ_m give orbits which hit:

$$\theta_m = \left(\frac{R}{L}\right) \frac{E_f/E_1}{\ln\left(E_f/E_1\right)}$$

In the small-angle approximation, $d\Omega = 2\pi\theta\ d\theta$ (at large angles where this approximation is poor, the integrand is very small),

$$F(z)\ dz = 4 \frac{N}{A} \rho\ dz\ Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \theta_1^2 \int_{\theta_m}^{\infty} \frac{2\pi\theta\ d\theta}{\left(\theta^2 + \theta_1^2\right)^2}$$

$$= 4\pi \frac{N}{A} \rho\ dz\ Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \frac{\theta_1^2}{\left(\theta_m^2 + \theta_1^2\right)}$$

To get the total probability that a beam electron is lost, we integrate z over the full length of the machine:

$$P = \int_0^L F(z)\, dz = 4\pi \frac{N}{A} \rho L\, Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \int_0^L \frac{dz}{L} \frac{\theta_1^2}{\left(\theta_m^2 + \theta_1^2\right)}$$

To do the integral we first simplify it by neglecting θ_1^2 compared with θ_m^2 in the denominator. Usually this is a fairly good approximation, and at any rate it makes the probability of scattering larger than if both terms were kept. We have, with ϵ the energy gain per unit length,

$$\left(\frac{\theta_1}{\theta_m}\right)^2 = \left[\alpha Z^{1/3} \frac{L}{R} \frac{m_e}{E_f} \ln(E_f/E_1)\right]^2 = \left(\frac{m_e \alpha Z^{1/3}}{R\epsilon}\right)^2 \left(\ln \frac{L}{z}\right)^2$$

$$P = 4\pi \frac{N}{A} \rho L\, Z^2\, r_e^2 \left(\frac{m_e}{R\epsilon}\right)^2 \int_0^1 d\eta \left(\ln \frac{1}{\eta}\right)^2$$

where $\eta = z/L$, and

$$\int_0^1 d\eta \left(\ln \frac{1}{\eta}\right)^2 = 2$$

So finally

$$P = 8\pi \frac{N}{A} \rho L\, Z^2\, r_e^2 \left(\frac{m_e}{R\epsilon}\right)^2$$

To complete the numerical evaluation we take the scattering material as oxygen $(Z = 8)$ at a pressure Q measured in units of 10^{-5} mm Hg, and we use a density of 1.3×10^{-3} g cm^{-3} for air at STP.

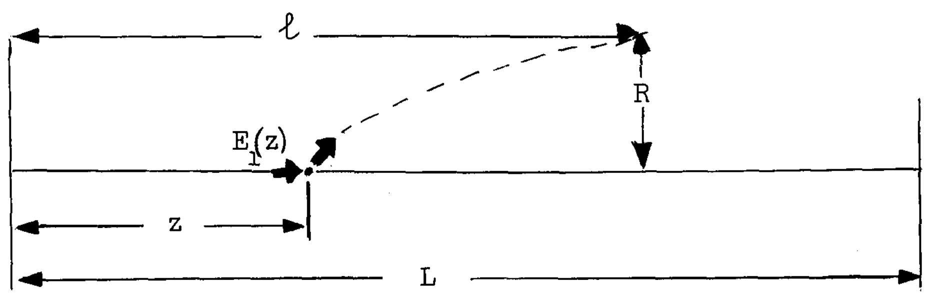
$$P = 8\pi \frac{6 \times 10^{23}}{16} 1.3 \times 10^{-3} \frac{Q}{760 \times 10^{5}} 30 \times 10^{4}(8^{2}) \left(2.8 \times 10^{-13}\right)^{2} \left(\frac{0.511}{1 \times \frac{4}{30}}\right)^{2}$$

$$P = 3.6 \times 10^{-4}$$

This means that for the conditions used ($Q = 10^{-5}$ mm Hg, $R = 1$ cm, $E_f = 40$ Bev) the fraction of the beam which is lost is 3.6×10^{-4} or about 0.04%. Although the scattering in the early part of the machine contributes most of the beam loss (where the energy is low and the distance to the end of the machine is large), it does not completely dominate.

B. Radiation Problem Arising from Gas Scattering

The strength of the line source of radiation is proportional to the number of beam electrons which hit the machine per unit length times the energy they have when they hit. We do a calculation which is similar to that just finished except we keep track of where the scattered particles hit the machine.



Suppose that a scattering occurs at z and that the electron hits the machine at R.

$$r(\ell) = R = \frac{\theta\, E_1(z)}{\epsilon}\, \ell n \frac{E(\ell)}{E(z)}$$

$$R = \theta\, z\, \ell n\left(\frac{\ell}{z}\right)$$

The probability that an electron scatters from dz into $d\Omega$ at θ is

$$f(z,\theta)\, dz d\Omega = 4\, \frac{N}{A}\, Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \rho\, \frac{\theta_1^2}{\left(\theta_1^2 + \theta^2\right)^2}\, dz d\Omega$$

As before, in the denominator we neglect θ_1^2 compared with θ^2. We want to integrate over all (θ, z) such that the scattered particle hits in $d\ell$ at ℓ. To do this, we change variables from θ to ℓ:

$$\theta = \frac{R}{z\, \ell n(\ell/z)}$$

$$d\theta = \left(\frac{\partial\theta}{\partial\ell}\right)_z d\ell = - \frac{R}{z\ell\, (\ell n\, \ell/z)^2}$$

We have

$$f(\ell) d\ell = \int f(z,\theta)\, d\theta dz$$

$$= 4\, \frac{N}{A}\, Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \rho \int \frac{\theta_1^2}{\theta^4}\, 2\pi\theta\, d\theta\, dz$$

$$= 8\pi\, \frac{N}{A}\, Z^2\, r_e^2\, \rho \left(\frac{m_e}{\epsilon}\right)^2 \int_{z=0}^{\ell} \frac{1}{z^2} \left(\frac{z\, \ell n\, \ell/z}{R}\right)^3 \left[\frac{R}{Z\ell\, (\ell n\, \ell/z)^2}\right] dz d\ell$$

$$f(\ell) d\ell = 8\pi\, \frac{N}{A}\, Z^2\, r_e^2\, \rho \left(\frac{m_e}{R\epsilon}\right)^2 d\ell \int_0^{\ell} \frac{dz}{\ell}\, \ell n\, (\ell/z)$$

The integral is 1, and we have the result that the probability per unit length that an electron hits the accelerator at a given point ℓ is constant. As a check we see that $\int_0^L f(\ell)d\ell$ gives the same expression for the total beam loss, P, that we had before.

When an electron hits the accelerator at ℓ, its energy is $E(\ell) = \epsilon\ell$. If we have a beam current I_0 (which is attenuated negligibly by scattering), the power absorbed per unit length is

$$W(\ell) = I_0 \, E(\ell) \, f(\ell)$$

$$= (I_0 \epsilon L) \, 8\pi \frac{N}{A} Z^2 \, r_e^2 \, \rho L \left(\frac{m_e}{R\epsilon}\right)^2 \left(\frac{\ell}{L}\right) \frac{1}{L}$$

The final energy is $E_f = \epsilon L$. The fractional power loss per unit length is

$$\frac{W(\ell)}{(E_f I_0)} = \frac{P}{L}\left(\frac{\ell}{L}\right)$$

where P, which depends on pressure, is the total fractional current loss calculated in the previous section. The radiation source strength increases linearly with distance down the machine. For the conditions used in the previous section (a pressure of 10^{-5} mm Hg, R = 1 cm, E_f = 40 Bev) the fraction of the power absorbed per unit length at the high-energy end of the machine is .04%/L. For orientation, the 35-foot shielding thickness was arrived at by assuming that the rate of power absorption per unit length was constant and equal to 3% $E_f I_0/L$.

We do not carry through a calculation of the radiation source strength in the presence of radial focusing because it would be too complicated and because the design of the radial focusing system is not finished.

III. SINGLE SCATTERING WITH FOCUSING

We suppose that the machine is equipped with a hypothetical radial focusing system of such properties that if a particle gets a kick which gives it a transverse momentum p_T, it remains in the beam. If the transverse momentum is greater than p_T, the particle is lost from the beam.

To get the beam loss we proceed as in the previous section except that now the minimum scattering angle required to lose a particle is

$$\theta_m (z) = \frac{p_T}{p_1(z)}$$

Note that the radius of the exit aperture does not enter explicitly here.

We want to do the following integral:

$$P = 4\pi \frac{N}{A} \rho L Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \int_0^L \frac{dz}{L} \frac{\theta_1^2}{\left(\theta_m^2 + \theta_1^2\right)}$$

where

$$\theta_1 = \frac{m_e c}{p_1(z)} \left(\alpha Z^{1/3}\right)$$

Here θ_1 and θ_m both depend on $p_1(z)$ in the same way, so the integrand is independent of z, and the integral is trivial:

$$P = 4\pi \frac{N}{A} \rho L Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \frac{\theta_1^2}{\theta_m^2 + \theta_1^2}$$

It is a good approximation to neglect θ_1^2 compared with θ_m^2 for p_T of the order of one-tenth $m_e c$. We have

$$\frac{\theta_1}{\theta_m} = \frac{m_e c \ \alpha Z^{1/3}}{p_T}$$

and

$$P = 4\pi \frac{N}{A} \rho L Z^2 \ r_e^2 \left(\frac{m_e c}{p_T}\right)^2$$

We evaluate P under the same conditions as before: $Z = 8$, pressure = $Q \times 10^{-5}$ mm Hg, $p_T = 0.1$ Mev/c. Note that the exit radius and the final energy do not enter.

$$P = 4\pi \frac{6 \times 10^{23}}{16} 1.3 \times 10^{-3} \frac{Q}{760 \times 10^5} 30 \times 10^4 \ (8^2) \ \left(2.8 \times 10^{-13}\right)^2 \left(\frac{5}{p_T}\right)^2$$

$$P = 3.0 \times 10^{-4}$$

IV. MULTIPLE SCATTERING

If we integrate the cross section of Goudsmit and Saunderson over all angles, we have the total screened cross section for scattering. This is

$$\sigma = 4\pi \ Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \theta_1^2 \int_0^\infty \frac{2\theta \ d\theta}{\left(\theta^2 + \theta_1^2\right)^2}$$

$$= 4\pi \ Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 = 3.0 \times 10^{-19} \ \text{cm}^2/\text{oxygen atom}$$

Note that this is independent of energy. If we multiply this by the total amount of gas (air) in the machine at a pressure Q, we have the total probability of any scattering, P_t:

$$P_t = 4\pi Z^2 \left(\frac{r_e}{\alpha Z^{1/3}}\right)^2 \frac{N}{A} \rho L Q$$

Taking the same numerical values as before, ($Z = 8$, $Q = 10^{-5}$ mm Hg, $\rho = 1.3 \times 10^{-3}$), we have

$$P_t = 5.8 \times 10^{-2}$$

This means that the probability of an electron making a collision is about 6%. Alternatively, 94% of the electrons make no collision at all, 6% make one collision, 0.2% make two collisions, etc. As the name implies, the theory of multiple scattering applies when a particle makes many collisions. In the case of thin scatterers, the exact meaning of "many" is important, but in our case it seems clear that 0.1 collision is not "many" and that the theory of multiple scattering does not apply. At a pressure of 10^{-4} mm Hg the average number of collisions is about one, but I think that it is still fairly evident that the results of multiple scattering theory are not applicable. The general scattering distribution consists of a Gaussian part at small angles which is connected by a plural scattering region to a single scattering distribution at large angles. For reasonable pressures in the Monster, the scattering distribution consists entirely of the single scattering tail.

For the record, we give the answer for the lateral distribution caused by multiple scattering of particles which are continuously being accelerated. Rossi (High Energy Particles, pp. 69-71) has a derivation of the transport equation relating scattering and lateral displacement. The solution of this equation for a general dependence of energy on depth of penetration is given by Eyges [Phys. Rev. 74, 1534 (1948)].

Suppose a particle is normally incident on a slab t radiation lengths thick. The distribution of y (measured in radiation lengths), the projected lateral displacement, is Gaussian when integrated over all angles. The mean-square displacement for the case of uniform energy gain from E_1 to E_2 is

$$< y^2 > \cong \frac{1}{2} \left(\frac{E_s^2\ t^3}{E_1\ E_2} \right)$$

The equality is approximate because we have assumed $E_2 \gg E_1$ (and that the particle is relativistic). E_s is a constant equal to 21 Mev.

The corresponding formula usually used (Neal, ML Report No. 185, p.77) is, in the present notation,

$$< y^2 > = \frac{1}{2} \frac{E_s^2\ t^3}{(E_2 - E_1)^2} \cong \frac{1}{2} \frac{E_s^2\ t^3}{E_2^2}$$

This is different from Eyges's result and gives $< y^2 >$ smaller by a factor (E_1/E_2). The trouble probably lies in the simple derivation used for the second expression. It does not contain the initial energy, which is somewhat surprising because at low energies the scattering is greater and the particle is farther from the observation point, so there is a larger lever arm in which a lateral displacement can develop; however, the single scattering result does not contain the initial energy either.