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STUDY OF THE BEAM PHASE COHERENCE

By

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The phase position of the electron bunches is determined by the first section of the machine. Any perturbation in the rf pulse in this section will introduce a phase incoherence between bunches. As shown in the Beam Specifications Table in Sec. II.B of the Project M Source Book,* the phase incoherence should be held to $\pm 5^{o}$.

In this memorandum we shall study the effect on the beam phase coherence of phase and amplitude variation of the high-voltage modulation on the sub-booster and main power-amplifier klystrons.

I. GENERAL EQUATIONS

The motion of an axial electron in a field which does not vary with time is described by the equations

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$$\left.\begin{aligned}\frac{d\gamma}{d\zeta} &= -\alpha \sin\Delta \\ \frac{d\Delta}{d\zeta} &= 2\pi\left(\frac{1}{\beta_w} - \frac{1}{\beta_e}\right) \\ \beta_e &= \frac{\sqrt{\gamma^2 - 1}}{\gamma}\end{aligned}\right\} \tag{1}$$

where

$\alpha = \dfrac{eE_o\lambda_o}{m_oc^2}$, with E_o the amplitude of the wave,

$\beta_w = \dfrac{v_o}{c}$, with v_o the phase velocity of the wave,

$\beta_e = \dfrac{v_e}{c}$, with v_e the velocity of the electron,

$\zeta = z/\lambda_o$,

and Δ is the phase angle between the electron and the wave.

When α and β_w do not depend on ζ, Δ and γ are related by the following expression:*

$$\cos\Delta - \frac{2\pi}{\alpha}\left(\frac{\gamma}{\beta_w} - \sqrt{\gamma^2 - 1}\right) = \cos\delta_o - \frac{2\pi}{\alpha}\left(\frac{\gamma_o}{\beta_w} - \sqrt{\gamma_o^2 - 1}\right)$$

where δ_o is the injection phase of the electron. If $\beta_w = 1$, the quantity $\gamma/\beta_w - \sqrt{\gamma^2 - 1}$ goes to zero when ζ increases, and there exists an asymptotic phase Δ_∞ so that

*G. Dome, "Electron Bunching by Uniform Sections of Disk-Loaded Waveguide," M Report No. 242-A, Project M, Stanford University, Stanford, California, December, 1960.

$$\cos \Delta_\infty = \cos \delta_0 - \frac{2\pi}{\alpha}\left(\gamma_0 - \sqrt{\gamma^2 - 1}\right) = \cos \delta_0 - \frac{2\pi}{\alpha}\sqrt{\frac{1 - \beta}{1 + \beta}} \qquad (2)$$

II. ACTION OF A SLOW AMPLITUDE MODULATION, OR A PULSE-TO-PULSE AMPLITUDE MODULATION

Calculation of the electron orbits shows that the quantity $\Delta - \Delta_\infty$ is very small after a few wavelengths $(\zeta \leq 4)$. This means that any perturbation of the electric field on this length during a time greater than the corresponding filling time $(t = 4\lambda/v_g = 0.1\ \mu\text{sec})$, will produce a perturbation of the asymptotic phase.

From Eq. (2) we can obtain

$$- \sin \Delta_\infty \, d\,\Delta_\infty = \frac{2\pi}{\alpha}\sqrt{\frac{1 - \beta_0}{1 + \beta_0}}\,\frac{d\alpha}{\alpha} = - (\cos \Delta_\infty - \cos \delta_0)\,\frac{d\alpha}{\alpha}$$

$$\left| d\,\Delta_\infty \right| \leq \left| \frac{\cos \Delta_\infty - \cos \delta_0}{\sin \Delta_\infty} \right| \left| \frac{d\alpha}{\alpha} \right|$$

Generally, Δ_∞ is very close to $- 90^\circ$, and

$$\left| d\,\Delta_\infty \right| \leq \left| \frac{d\alpha}{\alpha} \right|$$

But at the output of a klystron an amplitude modulation cannot appear without a phase modulation and the effect is more serious.

III. MOTION OF ELECTRONS IN A PHASE-MODULATED FIELD

The equation of motion is in the general case

$$\frac{dp}{dt} = e\,E(t,z)$$

The most general form of the field is

$$E(t,0) = E_0\left[e_1(t) \cos \omega_0 t + e_2(t) \sin \omega_0 t\right]$$

and

$$E(t,z) = E_o\left[e_1\left(t - \frac{z}{v_g}\right)\cos\omega_o\left(t - \frac{z}{v_o}\right) + e_2\left(t - \frac{z}{v_g}\right)\sin\omega_o\left(t - \frac{z}{v_o}\right)\right]$$

where ω_o is the nominal angular frequency, v_o is the phase velocity at the nominal frequency, and v_g is the group velocity.

The perfect case corresponds to $e_1 = 0$ and $e_2 = 1$. An amplitude modulation corresponds to $e_1 = 0$ and $de_2/dt \neq 0$. A frequency shift without modulation corresponds to $e_1 = \sin\Omega t$ and $e_2 = \cos\Omega t$. In the general case, then, the equation of the motion is

$$\frac{dp}{dt} = e\,E_o\left[e_1\left(t - \frac{z}{v_g}\right)\cos\omega_o\left(t - \frac{z}{v_o}\right) + e_2\left(t - \frac{z}{v_g}\right)\sin\omega_o\left(t - \frac{z}{v_o}\right)\right]$$

where p is now a function of t and z, and it is therefore impossible to derive equations such as equation (1).

Let us make the following changes:

$$\omega_o\left(t - \frac{z}{v_o}\right) = \delta$$

$$t = t_o + \int_0^z \frac{dz}{v_e}$$

$$t - \frac{z}{v_g} = t_o - \frac{z}{v_g} + \int_0^z \frac{dz}{v_e}$$

If $v_g \ll v_e$ (and this is generally true since $v_g/c \sim .01$ and $v_e/c > 0.5$), then

$$t - \frac{z}{v_g} = t_o - \frac{z}{v_g}$$

$$\frac{dp}{dt} = e\, E_o \left[e_1 \left(t_o - \frac{z}{v_g} \right) \cos \delta + e_2 \left(t_o - \frac{z}{v_g} \right) \sin \delta \right]$$

Now p is a function only of z, with t_o (the injection time) as a parameter, and we can derive the following equations:

$$\frac{d\gamma}{d\zeta} = - \alpha_o \left[e_2(t_o - k\zeta) \cos \delta + e_2(t_o - k\zeta) \sin \delta \right]$$

$$\frac{d\delta}{d\zeta} = 2\pi \left[\frac{1}{\beta_w} - \frac{\gamma}{\sqrt{\gamma^2 - 1}} \right] \qquad (3)$$

where $k = \lambda_o/v_g$. By combining these equations we obtain, for $\beta_w = 1$,

$$\frac{2\pi}{\alpha_o} \left(\gamma - \sqrt{\gamma^2 - 1} \right) - \frac{2\pi}{\alpha_o} \left(\gamma_o - \sqrt{\gamma_o^2 - 1} \right) =$$

$$- \int_{\delta_o}^{\delta} e_1(t_o - k\zeta) \cos \delta \, d\delta - \int_{\delta_o}^{\delta} e_2(t_o - k\zeta) \sin \delta \, d\delta$$

$$\delta(\zeta) = \Delta(\zeta) + \mu(\zeta)$$

$$\gamma(\zeta) = \Gamma(\zeta) + g(\zeta)$$

where Δ and Γ are the functions given by Eq. (1), $\mu(\zeta)$ and $g(\zeta)$ are small compared to Δ and Γ, and

$$\frac{2\pi}{\alpha_o} g \left(1 - \frac{\Gamma}{\Gamma^2 - 1} \right) =$$

$$- \int_{\delta_o}^{\Delta} e_1(t_o - k\zeta) \cos \Delta \, d\Delta - \int_{\delta_o}^{\Delta} \epsilon_2(t_o - k\zeta) \sin \Delta \, d\Delta - \mu \sin \Delta$$

where $e_2 = 1 + \epsilon_2$. For the asymptotic phase, $\left[1 - \Gamma \Big/ (\Gamma^2 - 1)\right]$ goes to zero and

$$\mu \sin \Delta_\infty = -\int_{\delta_0}^{\Delta_\infty} e_1(t_0 - k\zeta) \cos \Delta \, d\Delta - \int_{\delta_0}^{\Delta_\infty} \epsilon_2(t_0 - k\zeta) \sin \Delta \, d\Delta \qquad (4)$$

$$= - e_1(t_0 - k\zeta_1) \int_{\delta_0}^{\Delta_\infty} \cos \Delta \, d\Delta - \epsilon_2(t_0 - k\zeta_2) \int_{\delta_0}^{\Delta_\infty} \sin \Delta \, d\Delta$$

$$= - e_1(t_0 - k\zeta_1)(\sin \Delta_\infty - \sin \delta_0) + \epsilon_2(t_0 - k\zeta_2)(\cos \Delta_\infty - \cos \delta_0)$$

As the asymptotic phases are reached within four wavelengths, ζ_1 and ζ_2 will have values between 0 and 4. For $\delta_0 = 0$,

$$\mu = - e_1(t_0 - k\zeta_1) + \epsilon_2(t_0 - k\zeta_2) \left(\frac{\cos \Delta_\infty - 1}{\sin \Delta_\infty} \right) \qquad (5)$$

and $|\mu| \leq a_1 + a_2$, if a_1 and a_2 are the maximum amplitudes of the functions e_1 and ϵ_2.

IV. STUDY OF SOME SPECIAL CASES

1. Action of Constant Functions

We set

$$e_1(t_0 - k\zeta) = e_1(t_0) = e_1$$

and

$$\epsilon_2(t_0 - k\zeta) = \epsilon_2(t_0) = \epsilon_2$$

This case corresponds to a pulse-to-pulse jitter amplitude or to a pulse-amplitude drift of the klystron voltage

$$\mu = - e_1 \frac{\sin \Delta_\infty - \sin \delta_o}{\sin \Delta_\infty} + \epsilon_2 \frac{\cos \Delta_\infty - \cos \delta_o}{\sin \Delta_0}$$

The phase coherence from pulse to pulse is

$$\Delta\mu = \text{max variation of } \left| - e_1 + \epsilon_2 \right|$$

2. Action of a Constant Slope

In this case e_1 and ϵ_2 can be written

$$e_1(t_o - k\zeta) = e_1(t_o) - k\zeta s_1$$

$$\epsilon_2(t_o - k\zeta) = \epsilon_2(t_o) - k\zeta s_2$$

We can now calculate the quantity involved in Eq. (4):

$$\int_{\delta_o}^{\Delta_\infty} e_1(t_o - k\zeta) \cos \Delta \, d\Delta = e_1(t_o)(\sin \Delta_\infty - \sin \delta_o) - ks_1 \int_{\delta_o}^{\Delta_\infty} \zeta \cos \Delta \, d\Delta$$

$$\int_{\delta_o}^{\Delta_\infty} \epsilon_2(t_o - k\zeta) \sin \Delta \, d\Delta = - \epsilon_2(t_o)(\cos \Delta_\infty - \cos \delta_o) + ks_2 \int_{\delta_o}^{\Delta_\infty} \zeta \sin \Delta \, d\Delta$$

And thus

$$\mu = - e_1(t_o) \frac{\sin \Delta_\infty - \sin \delta_o}{\sin \Delta_\infty} + \epsilon_2(t_o) \frac{\cos \Delta_\infty - \cos \delta_o}{\sin \Delta_\infty} + ks_1 b_1 - ks_2 b_2 \tag{6}$$

If the slopes s_1 and s_2 remain constant during the pulse, the variation of μ as t_o varies gives the phase coherence $\Delta\mu$ in the pulse:

$$\Delta\mu \simeq \Delta \left[\epsilon_2(t_o) - e_1(t_o)\right]$$

We have calculated b_1 and b_2 in the case $\alpha = 4$, $\beta_{eo} = .5$, and $\delta_o = 0$:

$$b_1 = \int_0^{-84^\circ.66} \zeta \cos \Delta \, d\Delta = -\tfrac{1}{4}$$

$$b_2 = \int_0^{-84^\circ.66} \zeta \sin \Delta \, d\Delta = \tfrac{1}{2} .$$

3. Action of Monotonic Functions

By using Eq. (4) it is possible to see that the phase coherence within the pulse is given by the variation $\epsilon_2 - e_1$, because $k\zeta_1 < 0.1$ μsec and $0 < t_o < 2.1$ μsec.

4. Action of a Ripple

We can write the two functions in this form:

$$e_1(\tau) = \int_{-\Omega_o}^{+\Omega_o} F_1(\Omega) \, e^{j\Omega\tau} \, d\Omega$$

$$\epsilon_2(\tau) = \int_{-\Omega_o}^{+\Omega_o} F_2(\Omega) \, e^{j\Omega\tau} \, d\Omega$$

Thus the quantities in Eq. (4) are

$$\int_{\delta_o}^{\Delta_\infty} e_1(t_o - k\zeta) \cos \Delta \, d\Delta = \int_{-\Omega_o}^{\Omega_o} e^{j\Omega t_o} F_1(\Omega) \, d\Omega \int_{\delta_o}^{\Delta_\infty} e^{-jk\Omega\zeta} \cos \Delta \, d\Delta$$

$$\int_{\delta_o}^{\Delta_\infty} \epsilon_2(t_o - k\zeta) \sin \Delta \, d\Delta = \int_{-\Omega_o}^{\Omega_o} e^{j\Omega t_o} F_2(\Omega) \, d\Omega \int_{\delta_o}^{\Delta_\infty} e^{-jk\Omega\zeta} \sin \Delta \, d\Delta$$

We have to calculate the four quantities

$$\int_0^{\Delta_\infty} \cos k\Omega\zeta \cos \Delta \, d\Delta \qquad \int_0^{\Delta_\infty} \cos k\Omega\zeta \sin \Delta \, d\Delta$$

$$\int_0^{\Delta_\infty} \sin k\Omega\zeta \cos \Delta \, d\Delta \qquad \int_0^{\Delta_\infty} \sin k\Omega\zeta \sin \Delta \, d\Delta$$

In general the frequency of the ripple is low (< 5 MHz), however, and we can make the same assumptions as in Sec. 3.

V. INFLUENCE OF HIGH-VOLTAGE MODULATION ON THE RF MODULATION*

The high voltage applied to the klystron is

$$V(t) = V_o \left[1 + \frac{\Delta V}{V_o} v(t) \right]$$

where $\Delta V/V_o$ is the maximum rate of variation of $V(t)$ during the pulse, and $|v(t)| \leq 1$.

1. Amplitude Modulation

If we assume that the efficiency of the klystron does not change between $V_o + \Delta V$ and $V_o - \Delta V$, the rf power varies as $V^{5/2}$, and the amplitude of the electric field varies as $V^{5/4}$. Thus

$$E = E_o \left[1 + \frac{5}{4} \frac{\Delta V}{V} v(t) \right]$$

2. Phase Modulation

If φ_o is the phase shift across the klystron,

$$\frac{\Delta\varphi}{\varphi_o} = - \frac{\Delta\beta_e}{\beta_e}$$

*Several other causes can affect the phase at the output of the klystron (modulation of the beam by the focusing coil, by the magnetic field of the heater...) but all these effects are at low frequency (60 cycles) and will not affect the beam phase coherence during the pulse, but will affect the coherence from pulse to pulse which is not analyzed here.

and

$$\beta_e = \sqrt{x}\sqrt{\frac{2 + x}{1 + x}}$$

where $x = eV_o/m_o c^2$. Thus

$$\frac{\Delta\beta_e}{\beta_e} = \frac{\Delta x}{x}\,\frac{1}{(1 + x)(2 + x)} = \frac{\Delta V}{V_o}\,\frac{1}{(1 + x)(2 + x)} = k\,\frac{\Delta V}{V}$$

and

$$\Delta\varphi = -\,k\varphi_o\,\frac{\Delta V}{V_o}\,v(t)$$

The phase shift across the klystron is

$$\varphi_o - k\varphi_o\,\frac{\Delta V}{V_o}\,v(t)$$

The electric field at the input of the section is given by

$$E(t,0) = E_o\left[1 + \frac{5}{4}\frac{\Delta V}{V_o}\,v(t)\right]\sin\left\{\omega_o t - \varphi_o\left[1 - k\,\frac{\Delta V}{V_o}\,v(t)\right]\right\}$$

or, by changing the origin of time,

$$E(t,0) = E_o\left[1 + \frac{5}{4}\frac{\Delta V}{V_o}\,v(t)\right]\sin\left[\omega_o t + k\varphi_o\,\frac{\Delta V}{V_o}\,v(t)\right]$$

$$= E_o\left\{k\varphi_o\,\frac{\Delta V}{V_o}\,v(t)\,\cos\omega_o t + \left[1 + \frac{5}{4}\frac{\Delta V}{V_o}\,v(t)\right]\sin\omega_o t\right\}$$

With this notation, the functions e_1 and ϵ_2 (defined before) are

$$e_1(t) = k\varphi_o \frac{\Delta V}{V_o} v(t) \simeq 10 \frac{\Delta V}{V_o} v(t)^{*}$$

$$\epsilon_2(t) = \frac{5}{4} \frac{\Delta V}{V_o} v(t) = 1.25 \frac{\Delta V}{V_o} v(t)$$

It is possible to calculate the phase coherence corresponding to the tolerances on the modulators as shown in the Modulator Specifications Table in Sec. II.B of the Source Book:

Specifications		Phase Incoherence
Pulse height deviation from flatness (max) ± 0.5%		$\pm 2.5^\circ$ **
Pulse-to-pulse amplitude jitter ± 0.25%		$\pm 0.6^\circ$ **
Pulse amplitude drift		
Long term	± 1.5% per hour	$\pm 7.5^\circ$
Short term	± 0.25% per 5 min	$\pm 1.2^\circ$

The total phase coherence to be considered is $\pm (2.5^\circ + 0.6^\circ) = \pm 3.1^\circ$. The long and short drift have to be compensated by some device.

In the drive system there are other causes of modulation:

(a) The phase modulation in the driver and the first main booster does not affect the beam phase coherence because the rf signal in the drive line is taken as the phase reference for the machine.

(b) We have to take into account only the phase modulation of the sub-booster because its amplitude modulation is suppressed by the high-power klystron which works at saturation level. The characteristics of this sub-booster are not yet known, but it is possible to assume some

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*This is for a klystron voltage of 250 kv. At 200 kv the function $e_1 = 10.2\ \Delta V/V_o\ v(t)$.

**At 200 kv these numbers are $\pm 2.8^\circ$ and $\pm 0.7^\circ$.

reasonable values for the voltage and the length:*

$$V_o = 30 \text{ kv}$$

$$L = 2.5\ \lambda_o$$

Thus

$$e_1 \simeq 22 \frac{\Delta V}{V_o} v(t)$$

which gives 12.5^o per 1% of voltage variation. As shown in the Drive System Specifications in Sec. II.D, Table 3.1, the phase fluctuation due to the sub-booster has to be less than 3^o; this means that the power supply for this tube has to have a stability better than 0.25% (total variation within a pulse and from pulse to pulse).

VI. INFLUENCE OF BEAM LOADING ON THE PHASE COHERENCE

If the fundamental component of the current is constant through a section, the electric field in that section is in the steady case

$$E(z) = E_o \left[e^{-Iz} - \frac{ri}{E_o} \left(1 - e^{-Iz} \right) \right] = E_o \left[e^{-Iz} - m \left(1 - e^{-Iz} \right) \right] = E_o\ f(\zeta) \tag{7}$$

In the case of the first section the current is a function of the bunching; the fundamental component varies as $(\sin \omega\tau/2) \big/ (\omega\tau/2)$, where $\omega\tau$ is the phase extension of the bunch. If there is a prebuncher before the first section, $\omega\tau < 1$, and the current varies from 0.96 i to i. If we assume that the field is described by Eq. (7), with $di/dz = 0$, this is equivalent to an increase of about 4% in the beam-loading coefficient m at the beginning of the section $(z < \lambda_o)$.

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*These values correspond to the CFTH Tube TH 2101.

With this assumption, the equations of the motion are

$$\frac{d\gamma}{d\zeta} = - \alpha \, f(\zeta) \sin \delta$$

$$\frac{d\delta}{d\zeta} = 2\pi \left(1 - \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right) \tag{8}$$

and we can obtain the following:

$$\left(\gamma - \sqrt{\gamma^2 - 1}\right) - \left(\gamma_o - \sqrt{\gamma_o^2 - 1}\right) = \frac{-\alpha_o}{2\pi} \int_{\delta_o}^{\delta} f(\zeta) \sin \delta \, d\delta \tag{9}$$

We have seen that the variation of $f(\zeta)$ over a range of a few wave-lengths can contribute to the integral. In this case we can expand the function $f(\zeta)$ and obtain, to first order,

$$f(\zeta) = 1 - I\lambda_o \zeta (1 + m)$$

The asymptotic phase for $\delta_o = 0$ is given by

$$\cos \delta_\infty = 1 - \frac{2\pi}{\alpha_o} \sqrt{\frac{1 - \beta_o}{1 + \beta_o}} - I\lambda_o (1 + m) \int_0^{\Delta_\infty} \zeta \sin \Delta \, d\Delta$$

The asymptotic phase without beam loading is given by

$$\cos \Delta_\infty = 1 - \frac{2\pi}{\alpha_o} \sqrt{\frac{1 - \beta_o}{1 + \beta_o}} \; I\lambda_o \int_0^{\Delta_\infty} \zeta \sin \Delta \, d\Delta$$

With beam loading,

$$\cos \delta_\infty = \cos \Delta_\infty - \mu \sin \Delta_\infty$$

where

$$\mu \sin \Delta_\infty = - I\lambda_o \, m \int_0^{\Delta_\infty} \zeta \sin \Delta \, d\Delta = \frac{I_o \lambda_o m}{2}$$

Thus

$$\mu \cong \frac{- I_o \lambda_o m}{2}$$

As shown in Table II.B.A of the Source Book,

$$\gamma = 0.53 \text{ M}\Omega/\text{cm}$$

$$I = 0.00187 \text{ neper/cm}$$

$$E_o \sim 200 \text{ kv/cm}$$

$$i = 50 \text{ ma}$$

Thus $\mu = - 10^{-3}$ rad $= - 0.06^\circ$, which is completely negligible.

VII. SUMMARY

System	Specifications		Phase Incoherence
Sub-booster	Pulse height deviation from flatness Pulse-to-pulse amplitude jitter	} ± 0.12%	$\pm 1.5^\circ$
Klystron	Pulse height deviation from flatness: ± 0.5% Pulse-to-pulse amplitude jitter: 0.25%		$\pm 2.5^\circ$ $\pm .6^\circ$
Section	Beam loading: 10%		Negligible
			$\pm 4.6^\circ$