TRANSIENT BEAM LOADING CALCULATIONS FOR LINEAR ELECTRON ACCELERATORS PART I: UNIFORM STRUCTURES; PHASE MODULATION

By
R. H. Helm

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PROJECT M
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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I. UNIFORM STRUCTURES; PHASE MODULATION

A. INTRODUCTION

This report contains a summary of results on calculations of the transient beam loading in axially uniform linear electron accelerators. The effects of transient phase modulations in the rf field and in the beam current are included. Formulae and numerical results are given for several special examples.

B. FORMULATION

B.1 "Ideal" Conditions.

The equations describing transient beam loading usually are developed for the ideal situation in which it is assumed that all the electrons ride on the accelerating crests of the rf wave; i.e., one assumes perfect bunching, perfect phasing and of course no phase modulations. The following assumptions also are made:

a. the accelerator is uniform in the axial direction;

b. the beam current is constant along the axis; that is, no current is lost along the accelerator;

c. electron velocity is constant and equal to the guide phase velocity;

d. the guide group velocity, $d\omega/dk$, is assumed constant over a wide enough frequency range to contain the side-bands corresponding to adiabatic modulations of the electromagnetic field;

e. the rf properties of the accelerator guide (Q, shunt impedance, group velocity, phase velocity) are assumed to be unaffected by the presence of the beam;

f. electron transit time is neglected, i.e., electron velocity $\gg$ group velocity.

Under the above assumptions the following equation[1,2] describes conservation of energy at any point in the accelerator:

$$\frac{\partial U}{\partial t} + \frac{\partial P}{\partial z} + \frac{\omega}{Q} U = S \qquad \text{(B.1.1a)}$$

1. See references at end of report.

or

$$\frac{1}{v}\frac{\partial P}{\partial t} + \frac{\partial P}{\partial z} + 2IP = S \qquad \text{(B.1.1b)}$$

where

t is time;

z is distance along the accelerator axis;

ω is rf angular frequency;

v is group velocity;

$Q = \dfrac{2\pi \times \text{(stored energy)}}{\text{(energy lost per cycle)}}$;

$P(z,t)$ is power flowing in the z direction;

$U(z,t) = P(z,t)/v$ = stored energy per unit length in the electromagnetic field;

$S(z,t)$ is power generated per unit length;

$2I = \omega/Qv$ is power attenuation coefficient.

The quantities U, P and S are <u>time averaged</u> over an rf cycle and consequently do not contain rapidly varying rf phase terms.

The following relations apply:[2]

$$U/v = P = \frac{E^2(z,t)}{2Ir} \qquad \text{(B.1.2)}$$

$$S = - E(z,t)i(t) \qquad \text{(B.1.3)}$$

where $E(z,t)$ is the (slowly-varying*) amplitude of the rf wave, r is shunt impedance per unit length, and $i(t)$ is the effective beam current.

Equation (B.1.2) may be considered as defining the shunt impedance r. The <u>effective current</u> appearing in Eq. (B.1.3) would be given, for example by

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*The criteria that any function $f(z,t)$ be "slowly varying" will be $(\partial f/\partial t) \ll (\omega/2\pi)f$ and $(\partial f/\partial z) \ll f/\lambda$.

$$i(t) = \frac{\sin \Delta/2}{\Delta/2}\, i_0(t) \tag{B.1.4}$$

if the current were in the form of rectangular bunches of phase spread Δ, and $i_0(t)$ were the dc component. It will be assumed for the present that Δ is vanishingly small, so that $i(t) = i_0(t)$.

Insertion of Eqs. (B.1.2) and (B.1.3) into (B.1.1) yields the usual expression[3]

$$\left(\frac{\partial}{\partial z} + \frac{1}{v} \frac{\partial}{\partial t} + I \right) E(z,t) = - rI\; i(t) \tag{B.1.5}$$

which describes transient beam-loading under the "ideal" conditions listed above.

As Leiss[3] has pointed out, Eq. (B.1.5) is solved most readily by making a Laplace transformation with respect to t. In Leiss's notation,

$$\left(\frac{\partial}{\partial z} + I + s/v \right) E(z,s) = - rI\; i(s) \tag{B.1.6}$$

where for any function $f(t)$ the Laplace transform is denoted as

$$f(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad . \tag{B.1.7}$$

The solution of (B.1.6) is

$$E(z,s) = E(0,s)\, e^{-(I+s/v)z} - \frac{rI\; i(s)}{I + s/v} \left[1 - e^{-(I+s/v)z} \right] \tag{B.1.8}$$

where the boundary conditions are

$$E(0,t) = \text{applied field at } z = 0$$

and no power is reflected from the end of the accelerator; the initial condition $E(z,t) = 0$ for $t \leq 0$ also is implied.

The voltage gained by an electron in a section of accelerator of length L will be*

$$V(t) = \int_0^L E(z,t)\, dz \qquad (B.1.9)$$

or, by integration of (B.1.8),

$$\frac{V(s)}{L} = \frac{E(0,s)\left[1 - e^{-(A+sT)}\right]}{A + sT} - \frac{rA\, i(s)}{A + sT}\left[1 - \frac{1 - e^{-(A+sT)}}{A + sT}\right] \qquad (B.1.10)$$

where

$$A = IL$$

and

$$T = L/v = \text{filling time} \quad .$$

Equations (B.1.5) and (B.1.9) also can be solved in terms of integral equations, either by direct physical reasoning,[4] or by other standard methods. The solutions in integral form are

$$E(z,t) = E(0,t - z/v)\, e^{-Iz} - Ir \int_0^z i(t - z'/v)\, e^{-iz'} dz' \qquad (B.1.11)$$

$$V(t) = \int_0^L E(0,t - z/v)\, e^{-Iz}\, dz - Ir \int_0^L dz \int_0^z i(t - z'/v)\, e^{-Iz'} dz$$

(B.1.12)

*Transit time of the electrons is neglected; see condition (f), above.

B.2. Beam Loading with Phase Modulation

2.1 Formulation of the differential equations.

If phase modulation is present in the electromagnetic wave, one can represent the wave by two components in quadrature:

$$E_z(z,t) = E(z,t) \cos [\theta(z,t) + \delta(z,t)]$$

$$= E_1(z,t) \cos \theta(z,t) - E_2(z,t) \sin \theta(z,t) \qquad (B.2.1)$$

where $E(z,t)$ is the amplitude at z,t; $\theta(z,t) = (z/v_{p1} - t)\omega_1$; ω_1 is a constant reference frequency; $v_{p1} = v_p(\omega_1)$ is the phase velocity for $\omega = \omega_1$; and $\delta(z,t)$ is the (slowly varying) phase angle relative to the phase of a reference signal at frequency ω_1.

Evidently E_1, E_2 are defined by

$$\left.\begin{aligned} E_1(z,t) &= E(z,t) \cos \delta(z,t) \\ E_2(z,t) &= E(z,t) \sin \delta(z,t) \end{aligned}\right\} \qquad (B.2.2)$$

The boundary condition at $z = 0$ is

$$\left.\begin{aligned} E_1(0,t) &= E(0,t) \cos \delta(0,t) \\ E_2(0,t) &= E(0,t) \sin \delta(0,t) \end{aligned}\right\} \qquad (B.2.3)$$

where $E(0,t)$ and $\delta(0,t)$ are the (slowly varying) amplitude and phase of the applied field at $z = 0$.

Since E_1 and E_2 represent vector components which are in time quadrature, their interaction with the beam current should be described individually by Eq. (B.1.5):

$$\left.\begin{aligned} \left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + I\right) E_1(z,t) &= - Ir\, i_1(z,t) \\ \left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + I\right) E_2(z,t) &= - Ir\, i_2(z,t) \end{aligned}\right\} \qquad (B.2.4)$$

where i_1, i_2 are the components of the current which are locally in phase with E_1 and E_2, respectively.

To see how i_1, i_2 should be defined, we write the phase angle of the effective current as

$$\theta_b(z,t) = \left(\frac{z}{v_e} - t\right)\omega_1 + \delta_b(t)$$

$$= \left(\frac{z}{v_{p1}} - t\right)\omega_1 - \left(\frac{1}{v_{p1}} - \frac{1}{v_e}\right)\omega_1 z + \delta_b(t)$$

$$= \theta(z,t) - \epsilon z + \delta_b(t) \qquad \text{(B.2.5)}$$

where v_e is the electron velocity, $\delta_b(t)$ is the (slowly varying) phase of the beam current, and

$$\epsilon = \left(\frac{1}{v_{p1}} - \frac{1}{v_e}\right)\omega_1 \quad . \qquad \text{(B.2.6a)}$$

Using the identity $(d/d\omega)(\omega/v_p) = 1/v = (\text{group velocity})^{-1}$ and expanding ω_1/v_{p1} in Taylor series about ω_0,

$$\epsilon \cong \left(\frac{1}{v} - \frac{1}{v_e}\right)(\omega_1 - \omega_0)$$

$$\cong \frac{\omega_1 - \omega_0}{v} \qquad \text{(B.2.6b)}$$

where ω_0 is the frequency for which $v_p(\omega_0) = v_e$.

Now it is evident from Eq. (B.2.5) that the current can be represented as

$$i_z(z,t) = i(t) \cos\left[\theta(z,t) + \delta_b(t) - \epsilon z\right]$$

$$= i_1(z,t) \cos\theta(z,t) - i_2(z,t) \sin\theta(z,t)$$

where

$$i_1(z,t) = i(t) \cos [\delta_b(t) - \epsilon z]$$
$$i_2(z,t) = i(t) \sin [\delta_b(t) - \epsilon z] \qquad (B.2.7)$$

and $i(t)$ is the amplitude of the effective beam current. Since i_1 is in phase with E_1 and i_2 is in phase with E_2, Eq. (B.2.7) defines the proper forms of i_1 and i_2 to be used in Eqs. (B.2.4).

Equations (B.2.4) and (B.2.7) may be combined in a single complex equation*

$$\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t} + I - j\epsilon\right) \tilde{E}(z,t) = -Ir\ \tilde{i}(t) \qquad (B.2.8)$$

where

$$\tilde{E}(z,t) = [E_1(z,t) + jE_2(z,t)]\ e^{j\epsilon z} \qquad (B.2.9)$$

$$\tilde{i}(t) = i(t)\ e^{j\delta_b(t)} \qquad (B.2.10)$$

and the boundary condition at $z = 0$ is

$$\tilde{E}(0,t) = E(0,t)\ e^{j\delta(0,t)} \qquad (B.2.11)$$

which is equivalent to Eqs. (B.2.3).

The voltage gain of an electron in passing through a section of length L is

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*The equivalence of (B.2.8) and (B.2.4) is readily shown by substituting (B.2.9) and (B.2.10) in (B.2.8).

$$V(t) = \int_0^L E(z,t) \cos [\theta_b(z,t) - \theta(z,t) - \delta(z,t)]\, dz$$

$$= \int_0^L E(z,t) \cos [\delta_b - \epsilon z - \delta(z,t)]\, dz$$

$$= \int_0^L E_1(z,t) \cos [\delta_b(t) - \epsilon z] + E_2(z,t) \sin [\delta_b(t) - \epsilon z]\, dz$$

(B.2.12)

Hence, if we define a complex voltage gain by

$$\tilde{V}(t) = \int_0^L \tilde{E}(z,t)\, dz \qquad \text{(B.2.13)}$$

then the real voltage gain is given by

$$V(t) = \text{Re}\left[\tilde{V}(t)\, e^{-j\delta_b(t)}\right] \qquad \text{(B.2.14)}$$

which is equivalent to (B.2.12). (Re [ ] is understood to mean "Real part of [ ]"; Im [ ] means "Imaginary part of [ ]".)

Equation (B.2.8) contains the steady-state formulation of Loew[5] if we note that ϵ is equivalent to Loew's δ, that Re $(\tilde{E})$, Im $(\tilde{E})$ respectively are Loew's E_1 and E_2, and that Loew treats the case of $\delta_b = 0$ and $\delta(0,t)$ = constant.

The modulated phases, $\delta(0,t)$ and $\delta_b(t)$ may of course be described in terms of equivalent frequency modulations,

$$\delta(0,t) = \delta(0,0) - \int_0^t \left[\omega(t') - \omega_1\right] dt' \qquad \text{(B.2.15)}$$

$$\delta_b(t) = \delta_b(0) - \int_0^t \left[\omega_b(t') - \omega_1\right] dt' \qquad \text{(B.2.16)}$$

where $\omega(t)$ and $\omega_b(t)$ are the instantaneous frequencies of the applied field and the current, respectively.

2.2 Solutions of the differential equations.

Since Eq. (B.2.8) is formally identical to Eq. (B.1.5), it has solutions which are formally the same. Thus in the Laplace formulation, we have corresponding to (B.1.8) and (B.1.10)

$$\tilde{E}(z,s) = \tilde{E}(0,s)\, e^{-(\tilde{I}+s/v)z} - \frac{rI\, \tilde{i}(s)}{\tilde{I} + s/v}\left[1 - e^{-(\tilde{I}+s/v)z}\right] \quad (B.2.17)$$

and

$$\frac{\tilde{V}(s)}{L} = \frac{\tilde{E}(0,s)\left[1 - e^{-(\tilde{A}+sT)}\right]}{\tilde{A} + sT} - \frac{rA\, \tilde{i}(s)}{\tilde{A} + sT}\left[1 - \frac{1 - e^{-(\tilde{A}+sT)}}{\tilde{A} + sT}\right] \quad (B.2.18)$$

where

$$T = L/v,$$

$$\tilde{I} = I - j\epsilon \cong I - j\,\frac{\omega_1 - \omega_0}{v}$$

$$\tilde{A} = \tilde{I}L = A - j\epsilon L \cong A - j(\omega_1 - \omega_0)T$$

and the Laplace transforms of $\tilde{E}(0,t)$ and $\tilde{i}(t)$ are defined by

$$\tilde{E}(0,s) = \int_0^\infty E(0,t)\, e^{j\delta(0,t) - st}\, dt \quad (B.2.19)$$

$$\tilde{i}(s) = \int_0^\infty i(t)\, e^{j\delta_b(t) - st}\, dt \quad (B.2.20)$$

The integral solutions, corresponding to Eqs. (B.1.11), (B.1.12), are

$$\tilde{E}(z,t) = \tilde{E}(0,t - z/v)\, e^{-\tilde{I}z} - Ir\int_0^z \tilde{i}(t - z'/v)\, e^{-\tilde{I}z'}\, dz' \quad (B.2.21)$$

and

$$\tilde{V}(t) = \int_0^L \tilde{E}(0,t - z/v)\, e^{-\tilde{I}z}\, dz - Ir \int_0^L dz \int_0^z \tilde{i}(t - z'/v)\, e^{-\tilde{I}z'}\, dz'$$

(B.2.22)

It may be recalled that ω_1, the reference frequency, is arbitrary subject only to the restriction that the change in ϵz over a wave length be small--i.e., that

$$\epsilon \cong \frac{\omega_1 - \omega_0}{v} \ll \lambda^{-1} \quad .$$

We are free to choose $\omega_1 = \omega_0$ or $\epsilon = 0$, for example, in which case

$$\tilde{I} = I, \qquad \tilde{A} = A, \qquad \tilde{E}(z,t) = E_1(z,t) + jE_2(z,t)$$

with considerable simplification in Eqs. (B.2.17), (B.2.18), (B.2.21) and (B.2.22), and with no loss in generality. A choice of ω_1 other than ω_0 may be useful in some problems (e.g., comparison of the present formulation with that of Loew; see p. 8 of this report).

C. TYPICAL EXAMPLES

C.1. System of Units Convenient for Accelerator Systems.

One usually will be interested in the relative variation of $V(t)$, the beam energy, for different applied rf envelopes $E(0,t)$ and beam currents $i(t)$. It will be convenient to change to the following system of units:

t in units of the filling time L/v;

z in units of L, the length of a section;

V in units of average voltage gain/unit length (i.e., V/L in the notation of Sec. B becomes V in the present section).

Further, V will be expressed as the superposition of two terms,

$$V = V_E + V_b \qquad \text{(C.1.1)}$$

where V_E depends on the applied field, and V_b depends only on the beam current.* Thus Eq. (B.2.18) becomes

$$\tilde{V}_E(s) = \frac{\tilde{E}(0,s)\left[1 - e^{-(\tilde{A}+s)}\right]}{\tilde{A} + s} \qquad (C.1.2)$$

$$\tilde{V}_b(s) = \frac{-rA\,\tilde{i}(s)}{\tilde{A} + s}\left[1 - \frac{1 - e^{-(\tilde{A}+s)}}{\tilde{A} + s}\right] \qquad (C.1.3)$$

Equation (B.2.22) becomes

$$\tilde{V}_E(t) = \int_0^1 \tilde{E}(0,t - z)\, e^{-\tilde{A}z}\, dz \qquad (C.1.4)$$

$$\tilde{V}_b(t) = -rA \int_0^1 dz \int_0^z \tilde{i}(t - z')\, e^{-\tilde{A}z'}\, dz' \qquad (C.1.5)$$

The definitions of $\tilde{V}$, $\tilde{E}$, and $\tilde{i}$ are the same as in Sec. B. Of course all the quantities in these equations become real in the synchronous case, giving the results of Sec. B.1.

C.2. Examples

Some typical examples now may be given:

2.1** Step-function applied field, no phase modulation

We adopt the following definitions:

$$E(0,t) = E_0\, u(t - t_E)$$

$$\delta(0,t) = 0, \qquad \delta_b(t) = 0$$

*That is, V_E is voltage gain for $i(t) \to 0$, and V_b is voltage gain for $E(0,t) \to 0$.

**Examples 2.1 and 2.2 are given by Leiss, Ref. 3.

where $u(t - t_E)$ is the unit step function,

$$u(t - t_E) = \begin{cases} 0, & t \leq t_E \\ 1, & t \geq t_E \end{cases}$$

Then

$$E(0,s) = E_0 \frac{e^{-st_E}}{s}$$

$$V_E(s) = \frac{E_0 e^{-st_E}\left(1 - e^{-(A+s)}\right)}{s(A + s)} \qquad \text{[by Eq. (C.1.2)]}$$

Application of standard inverse Laplace transforms gives

$$V_E(t) = \frac{E_0}{A}\left\{\left[1 - e^{-A(t-t_E)}\right] u(t - t_E) - \left[e^{-A} - e^{-A(t-t_E)}\right] u(t - t_E - 1)\right\}$$

or

$$V_E(t + t_E) = \frac{E_0}{A}\left\{\left(1 - e^{-At}\right) u(t) - \left(e^{-A} - e^{-At}\right) u(t - 1)\right\} \tag{C.2.1}$$

Equation (C.2.1) is plotted in Fig. 1 with $A = 0.57$ (corresponding to a proposed Project M design).

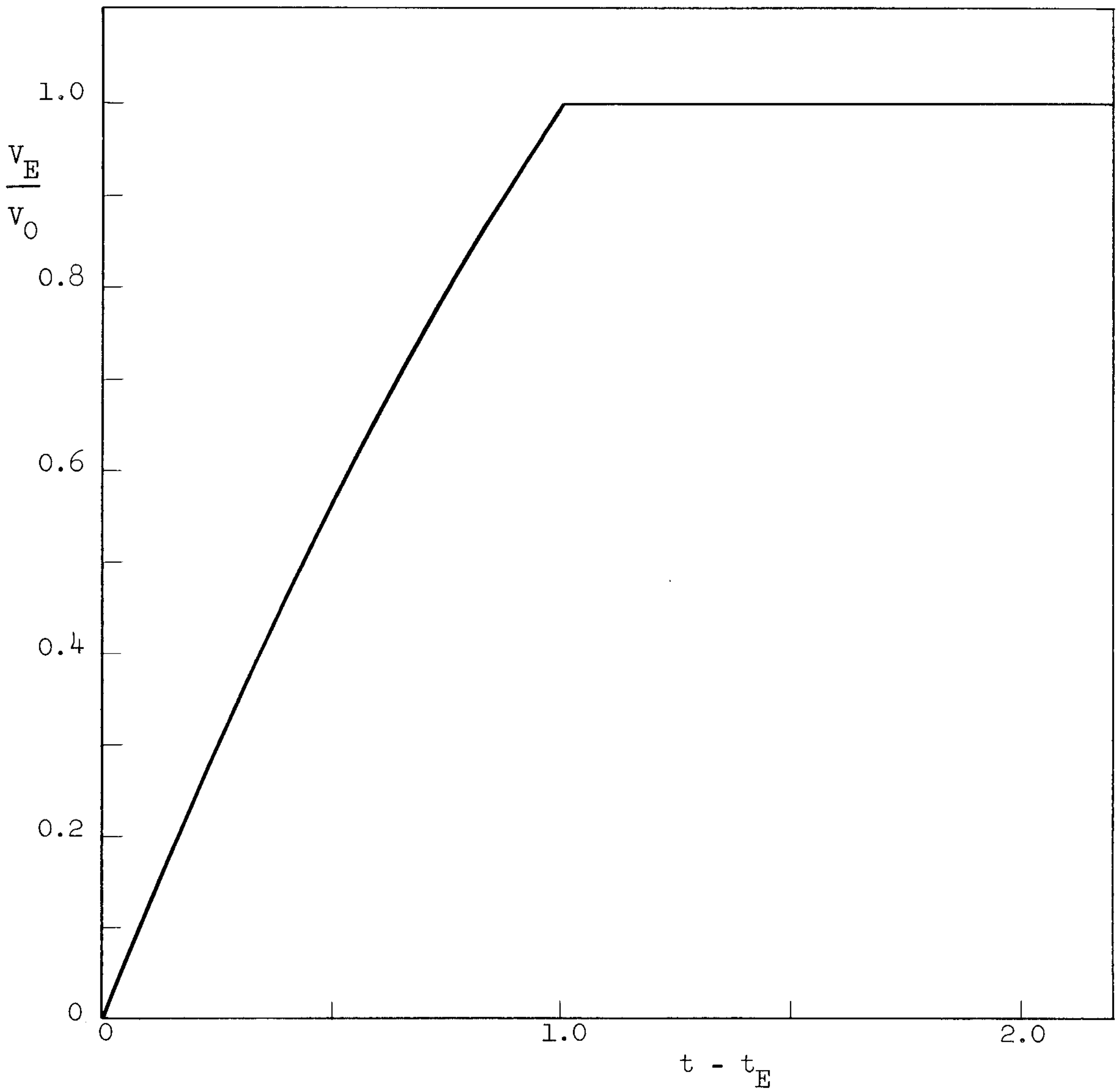

FIG. 1--Relative voltage gain for step-function applied field (Equation C.2.1, Sec. C.2.1). Project M parameters (A = 0.57) are assumed.

$$V_O = \text{steady-state voltage gain} = \frac{E_O(1 - e^{-A})}{A} .$$

2.2* Step-function current, no phase modulation.

$$i(t) = i_0 \, u(t - t_b), \qquad \delta_b(t) = 0$$

$$i(s) = i_0 \frac{e^{-st_b}}{s}$$

$$V_b(s) = - \frac{Ari_0 \, e^{-st_b}}{s(A + s)} \left[1 - \frac{1 - e^{-(A+s)}}{A + s} \right] \qquad \text{[Eq. (C.1.3)]}$$

$$V_b(t + t_b) = - \frac{ri_0}{A} \left\{ \left[(At - A + 1) \, e^{-At} - e^{-At} - 1 + A \right] u(t) - \left[(At - A + 1) \, e^{-At} - e^{-A} \right] u(t - 1) \right\} \qquad (C.2.2)$$

Equation (C.2.2) is plotted in Fig. 2 with $A = 0.57$.

2.3 Exponentially rising E field, no phase modulation.

$$E(0,t) = E_0 \left[1 - e^{-\alpha(t - t_E)} \right] u(t - t_E)$$

$$\delta(0,t) = \delta_b(t) = 0$$

$$E(0,s) = E_0 \, e^{-st_E} \left(\frac{1}{s} - \frac{1}{s + \alpha} \right)$$

*Examples 2.1 and 2.2 are given by Leiss, Ref. 3.

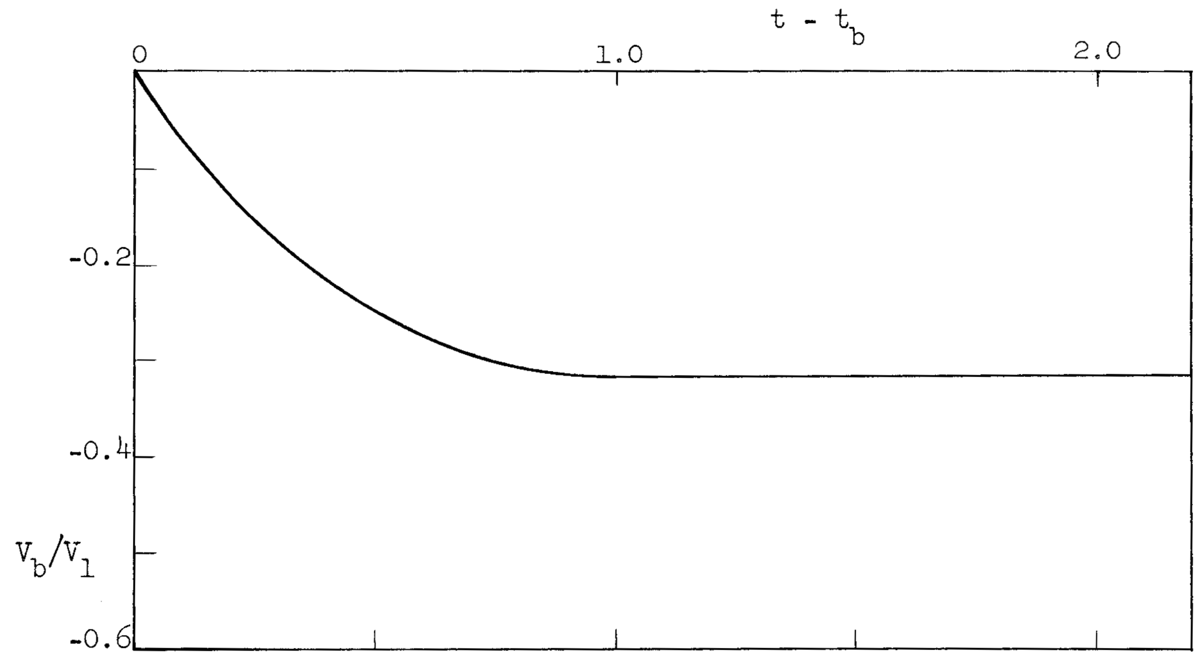

FIG. 2--Voltage gain induced by step-function beam current (Equation C.2.2, Sec. C.2.2). V_1 is the steady-state voltage gain which would result from an applied field of magnitude $i_0 r$; i.e.,

$$V_1 = \frac{i_0 r(1 - e^{-A})}{A}.$$

$$V_E(t + t_E) = \frac{E_0}{A} \left\{ \left[1 - e^{-At} - \frac{A}{\alpha - A} \left(e^{-At} - e^{-\alpha t} \right) \right] u(t) \right.$$

$$\left. - \left[e^{-A} - e^{-At} - \frac{A}{\alpha - A} \left(e^{-At} - e^{-A-\alpha(t-1)} \right) \right] u(t - 1) \right\} \quad (C.2.3)$$

Equation (C.2.3) is plotted in Fig. 3 with $A = 0.57$ and $\alpha = 8.3$ (i.e., rise time = T/α = 0.1 μsec for Project M design with T = 0.83 μsec.).

2.4 Exponentially rising beam current, no phase modulation.

$$i(t) = i_0 \left(1 - e^{-\beta(t - t_b)} \right) u(t - t_b)$$

$$\delta_b(t) = 0$$

$$i(s) = i_0 \, e^{-st_b} \left(\frac{1}{s} - \frac{1}{s + \beta} \right)$$

$$V_b(s) = - \frac{Ar \, i_0}{s + A} \, e^{-st_b} \left(\frac{1}{s} - \frac{1}{s + \beta} \right) \left(1 - \frac{1 - e^{-(A+s)}}{A + s} \right)$$

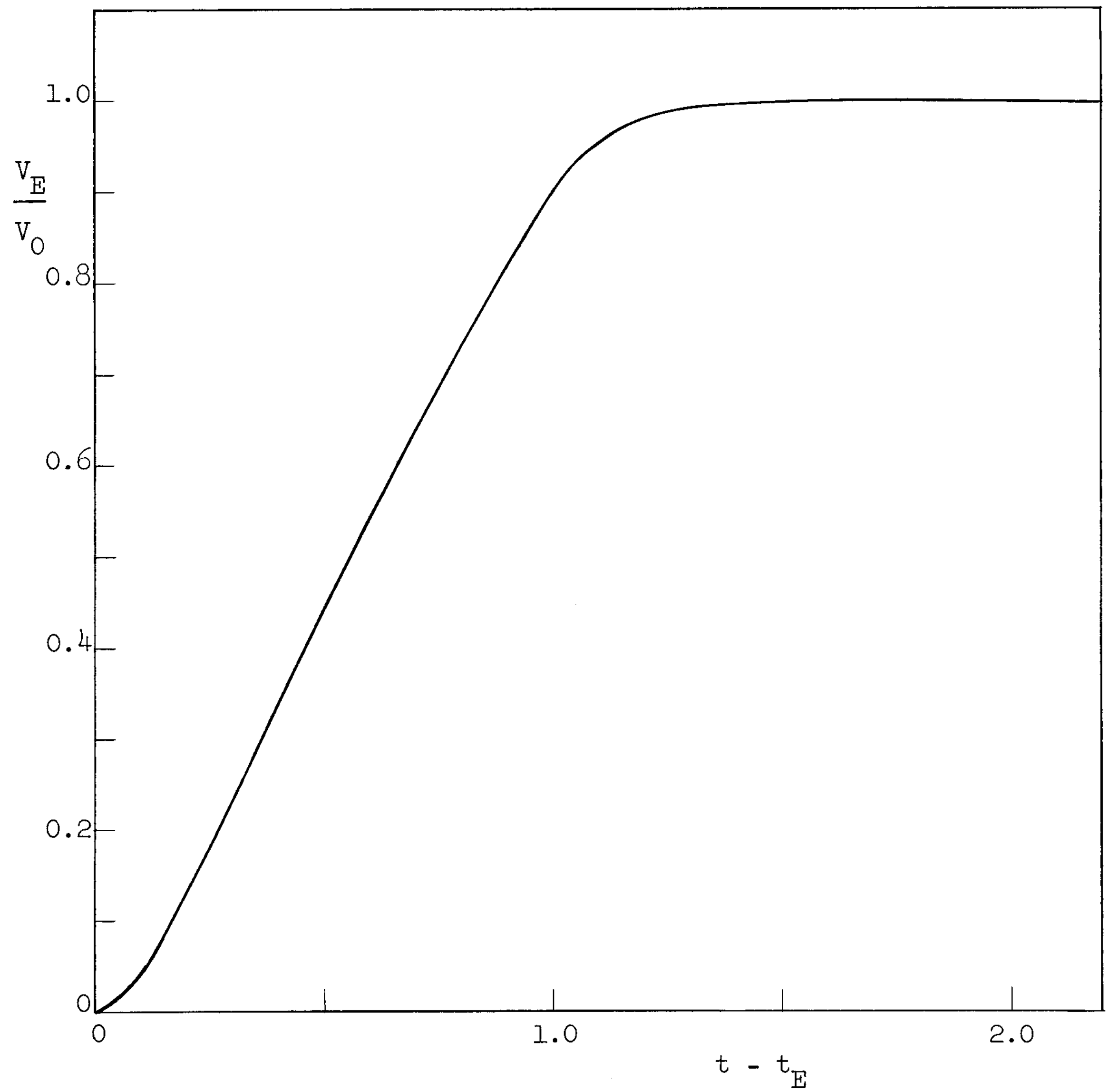

FIG. 3--Voltage gain for applied field with exponentially-damped rise (Equation C.2.3, Sec. C.2.3).

$$E(0,t + t_E) = E_0(1 - e^{-\alpha t})\, u(t);$$

$$\alpha = 8.3 \qquad (\text{rise time} = T/\alpha = 0.1\ \mu\text{sec}).$$

$$V_b(t + t_b) = -Ar\, i_0 \left\{ \left[\frac{(At - A + 1)\, e^{-At} - 1 + A}{A^2} \right.\right.$$

$$\left. + \frac{[(\beta - A)\, t - \beta + A - 1]\, e^{-At} + (\beta - A + 1)\, e^{-\beta t}}{(\beta - A)^2} \right] u(t)$$

$$- \left[\frac{(At - A + 1)\, e^{-At} + e^{-At} - e^{-A}}{A^2} \right.$$

$$\left.\left. + \frac{[(\beta - A)\, t - \beta + A - 1]\, e^{-At} + e^{-A-\beta(t-1)}}{(\beta - A)^2} \right] u(t - 1) \right\} \qquad \text{(C.2.4)}$$

Equation (C.2.4) is plotted in Fig. 4 with $A = 0.57$ and $\beta = 8.3$.

2.5. Step function applied field, constant frequency error.

$$E(0,t) = E_0\, u(t - t_E)$$

$$\delta(0,t) = \delta_0 - (t - t_E)\, \Omega \qquad \text{[by Eq. (B.2.15)]}$$

where $\Omega = T\delta\omega$ = cumulative phase error in one filling time. Choosing $\omega_1 = \omega_0$, so that $\tilde{A}$ is real,

$$\tilde{E}(0,s) = E_0\, e^{-st_E}\, \frac{e^{j\delta_0}}{s + j\Omega}$$

$$\tilde{V}_E(s) = \frac{E_0\, e^{-st_E}\, e^{j\delta_0} \left[1 - e^{-(A+s)}\right]}{(A + s)(s + j\Omega)}$$

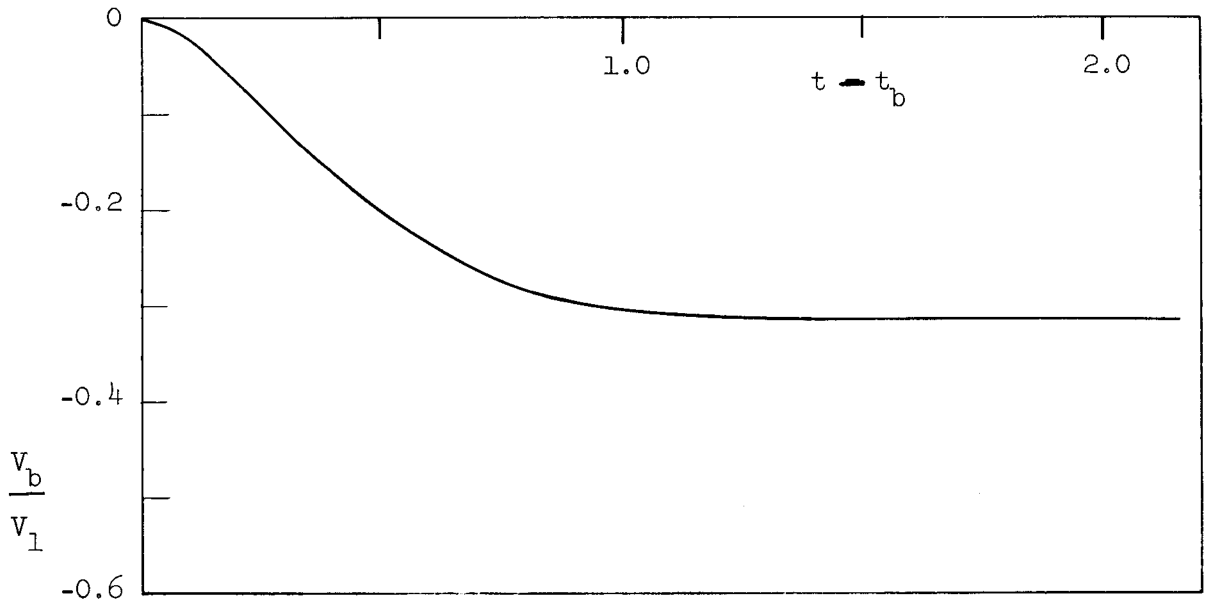

FIG. 4--Voltage gain induced by beam current with exponentially-damped rise. (Equation C.2.4, Sec. C.2.4).

$$i(t + t_b) = i_0(1 - e^{-\beta t})\ u(t)$$

$\beta = 8.3$ (rise time = $T/\beta = 0.1\ \mu$sec)

$$\tilde{V}_E(t + t_E) = \frac{E_0 \, e^{j\delta_0}}{A - j\Omega} \left\{ \left(e^{-j\Omega t} - e^{-At} \right) u(t) \right.$$

$$\left. - \left[e^{-A-j\Omega(t-1)} - e^{-At} \right] u(t-1) \right\} \qquad (C.2.5)$$

The real V_E will depend on how the beam is phase modulated. For example:

5a) Beam at correct frequency. For convenience, define δ_b such that

$$\delta_b(t) = \text{constant} = \delta_0 + \Delta + \arctan(\Omega/A)$$

where Δ is arbitrary; then

$$\tilde{V}_E(t + t_E) = \text{Re}\left[\tilde{V}(t + t_E) \, e^{-j\delta_b} \right] \qquad \text{[by Eq. (B.2.14)]}$$

$$= \frac{E_0}{\sqrt{A^2 + \Omega^2}} \left\{ \left[\cos(\Omega t + \Delta) - e^{-At} \cos \Delta \right] u(t) \right.$$

$$\left. - \left[e^{-A} \cos(\Omega t - \Omega + \Delta) - e^{-At} \cos \Delta \right] u(t-1) \right\} \qquad (C.2.5a)$$

5b) Beam with same frequency error as applied field. In this case we define δ_b such that

$$\delta_b(t + t_E) = -\Omega t + \delta_0 + \Delta + \arctan(\Omega/A) \quad .$$

Then

$$V_E(t + t_E) = \frac{E_0}{\sqrt{A^2 + \Omega^2}} \left\{ \left[\cos\Delta - e^{-At}\cos(\Omega t - \Delta)\right] u(t) \right.$$

$$\left. - \left[e^{-A}\cos(\Omega - \Delta) - e^{-At}\cos(\Omega t - \Delta)\right] u(t-1) \right\} \quad \text{(C.2.5b)}^*$$

Equations (C.2.5a,b) are plotted in Figs. 5 with $A = 0.57$, $\Omega = \pi/4$, and several values of Δ. This value of Ω corresponds to a frequency error of

$$\frac{\delta\omega}{\omega_0} = \frac{\Omega}{\omega_0 T} = \frac{\pi/4}{2\pi \cdot 2860 \cdot 0.83} = 5.3 \times 10^{-5}$$

or

$$\delta f = 0.15 \text{ Mc/sec.}$$

Figures 5 shows that there is an optimum choice of the relative phase Δ for a particular frequency error.

2.6. Step function current, constant frequency error.

We choose $\omega_1 = \omega_0$.

$$i(t) = i_0\, u(t - t_b)$$

$$\delta_b(t) = \Delta_0 - \Omega(t - t_b)$$

(where $\Omega = T\delta\omega$ as in Example 5)

---

*Equation (C.2.5b) reduces to Eq. (3.33) of Ref. 2 (except for terms of order v_g/v_e) if we set $\delta_b(t) = \delta(0,t)$, $t \geq 1$, and use small-angle approximations.

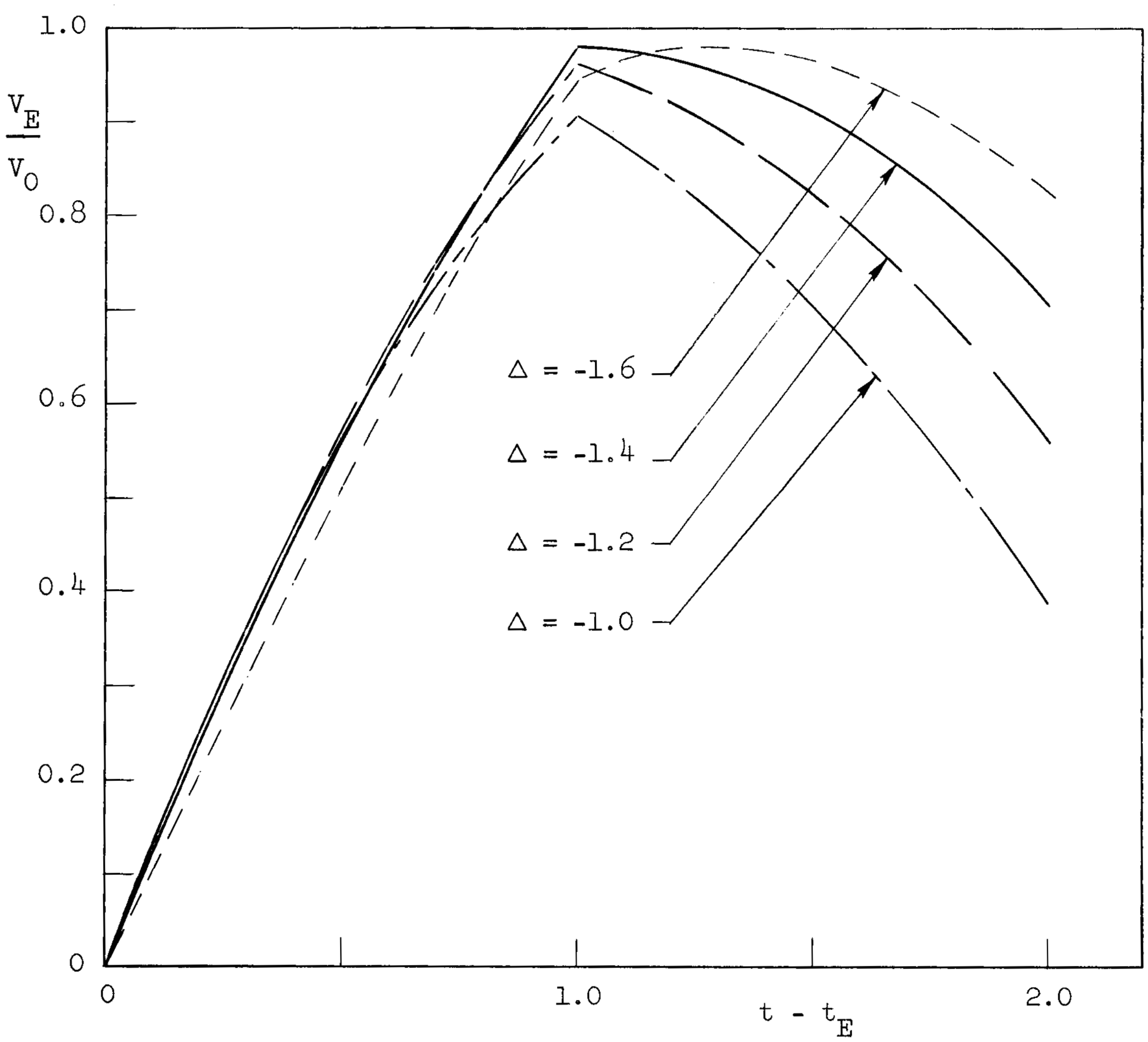

FIG. 5a--Voltage gain for step-function applied field with constant frequency error; beam bunched at correct frequency. (Equation C.2.5a, Sec. C.2.5).

$$\delta(0, t + t_E) = -\ \Omega t$$

$$\delta_b(t) = \Delta + \arctan\ (\Omega/A)$$

$$\Omega = T\delta\omega = \pi/4; \quad A = 0.57.$$

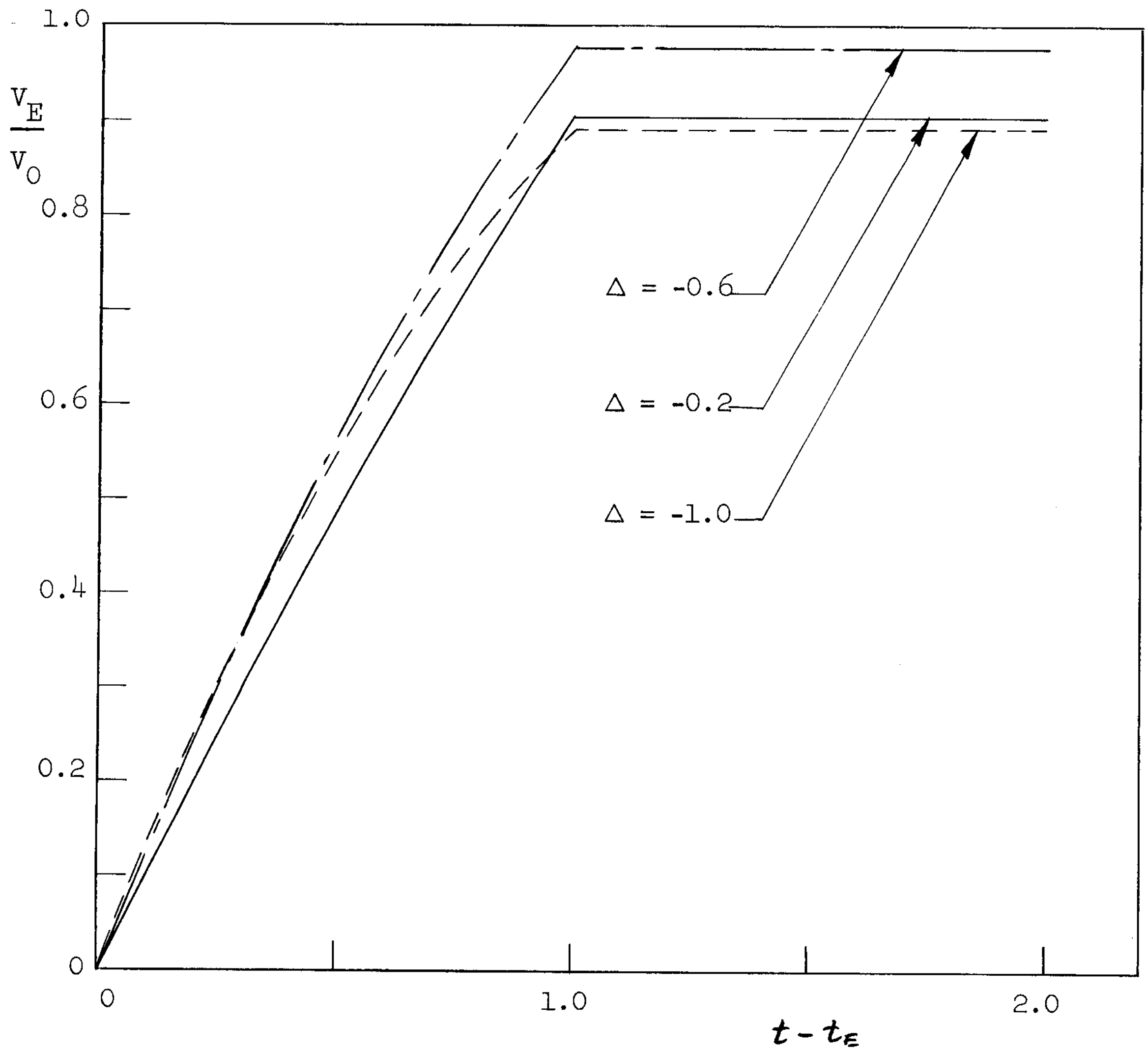

FIG. 5b--Voltage gain for step-function applied field with field and beam both at same incorrect frequency. (Equation C.2.5b, Sec. C.2.5).

$$\delta(0, t + t_E) = -\Omega t$$

$$\delta(t + t_E) = -\Omega t + \Delta + \arctan(\Omega/A)$$

$$\Omega = \pi/4, \quad A = 0.57.$$

$$\tilde{i}(s) = i_0 \, e^{-st_b} \frac{e^{j\Delta_0}}{s + j\Omega}$$

$$\tilde{V}_b(s) = - \frac{Ar \, i_0 \, e^{-st_b + j\Delta_0}}{(s + A)(s + j\Omega)} \left[1 - \frac{1 - e^{-(A+s)}}{A + s} \right]$$

$$\tilde{V}_b(t + t_b) = - \frac{Ar \, i_0 \, e^{j\Delta_0}}{(A - J\Omega)^2} \left\{ \left[\left[(A - j\Omega)\, t - A + j\Omega + 1 \right] e^{-At} \right. \right.$$

$$\left. + (A - j\Omega - 1)\, e^{-j\Omega t} \right] u(t)$$

$$\left. - \left[\left[(A - j\Omega)\, t - A + j\Omega + 1 \right] e^{-At} - e^{-A-j\Omega(t-1)} \right] u(t - 1) \right\}$$

$$V_b(t + t_b) = \text{Re} \left[\tilde{V}_b(t + t_b) \, e^{-j\delta_b(t+t_b)} \right]$$

$$= - \frac{Ar \, i_0}{\rho^2} \left\{ \left[\rho(t - 1)\, e^{-At} \cos(\Omega t + \eta) \right. \right.$$

$$\left. + e^{-At} \cos(\Omega t + 2\eta) + \rho \cos \eta - \cos 2\eta \right] u(t)$$

$$- \left[\rho(t - 1)\, e^{-At} \cos(\Omega t + \eta) + e^{-At} \cos(\Omega t + 2\eta) \right.$$

$$\left. \left. - e^{-A} \cos(\Omega + 2\eta) \right] u(t - 1) \right\} \qquad (C.2.6)$$

where

$$\rho = \sqrt{A^2 + \Omega^2}$$

$$\eta = \arctan (\Omega/A)$$

Equation (C.2.6) is plotted in Fig. 6 with $A = 0.57$ and $\Omega = \pi/4$.

2.7. More general forms of $\tilde{E}(0,t)$, $\tilde{i}(t)$.

As can be seen from the results of Examples 2.5 and 2.6, the analytic solution of the problem becomes excessively tedious except in the very simplest cases. Consequently the problem has been coded for a high-speed digital computer, allowing quite arbitrary forms for the input functions. A typical example is shown in Fig. 7.

2.8. Superposition of solutions.

The linear nature of the equations allows linear superposition of solutions. In general,

$$\tilde{V}(t) = \sum_{m=1}^{M} \tilde{V}_{Em}(t) + \sum_{n=1}^{N} \tilde{V}_{bn}(t) \qquad (C.2.7)$$

where $\tilde{V}_{Em}$, $\tilde{V}_{bn}$ are the voltage gains corresponding to an applied field $\tilde{E}_m(0,t)$ and current $\tilde{i}_n(t)$, respectively. The various beam components would have real voltage gains given by

$$V_n(t) = \mathrm{Re} \left[\tilde{V}(t)\ e^{-j\delta_{bn}(t)} \right] \qquad (C.2.8)$$

As an example of the usefulness of this formulation, consider the case of a beam current uniformly distributed within bunches of finite phase width Δ (at the <u>correct</u> frequency ω_0) so that the current distribution as a function of phase is given by

$$i(\delta,t)\ d\delta = i_0(t)\ \frac{d\delta}{\Delta}\ e^{j\delta} \left[u(\delta + \Delta/2) - u(\delta - \Delta/2) \right] \quad .$$

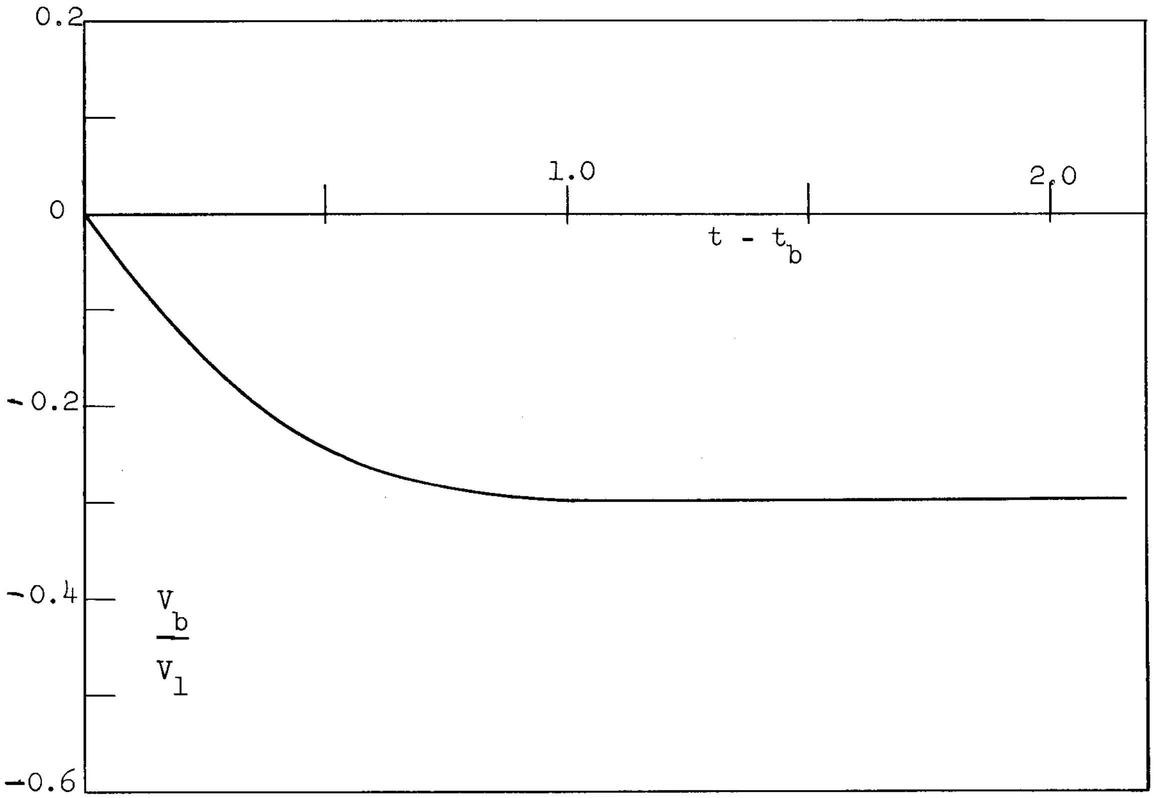

FIG. 6--Voltage gain induced by step-function beam current at incorrect frequency. (Equation C.2.6, Sec. C.2.6).

$$\delta_b(t + t_b) = -\Omega t$$

$$\Omega = \pi/4 \ .$$

Note close similarity to Fig. 2, which corresponds to $\Omega = 0$.

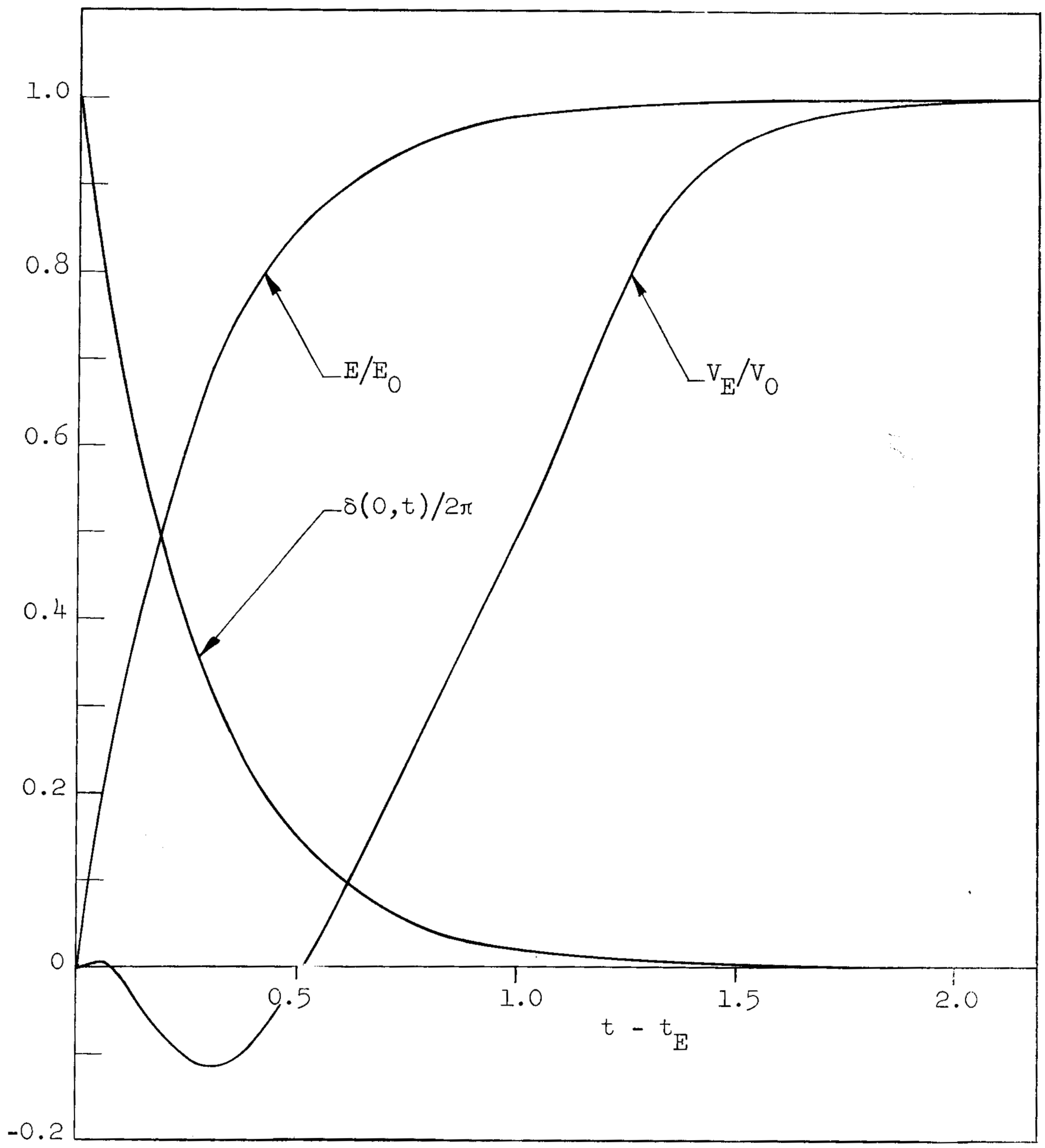

FIG. 7--Voltage gain for an example of simultaneous amplitude and phase modulation.

$$E(0,t + t_E) = E_0(1 - e^{-\alpha t})\, u(t)$$

$$\delta(0,t + t_E) = \delta_0 e^{-\alpha t}$$

$$\delta_b = 0.$$

$\delta_0 = 2\pi$, $\alpha = 3.8$ (rise time = 0.22 μsec)

This example corresponds qualitatively to the case of a pulse with finite rise time applied to the klystron amplifier.

The corresponding complex voltage gain is

$$\tilde{V}_b(\delta,t)\ d\delta = V_{0b}(t)\ \frac{e^{j\delta}\ d\delta}{\Delta} \left[u(\delta + \Delta/2) - u(\delta - \Delta/2) \right]$$

where $V_{0b}(t)$ is the solution corresponding to $\delta = 0$; the total beam-loading component of voltage gain is

$$\tilde{V}_b(t) = \int_{-\Delta/2}^{\Delta/2} \tilde{V}_b(\delta,t)\ d\delta = V_{0b}(t)\ \frac{\sin \Delta/2}{\Delta/2} \quad .$$

The real beam-loading component then is simply

$$\tilde{V}_b(\delta,t) = \mathrm{Re} \left[\tilde{V}_b(t)\ e^{-j\delta} \right] = V_{0b}(t)\ \frac{\sin \Delta/2}{\Delta/2} \cos \delta \quad . \qquad (C.2.9)$$

Other examples of superposition are shown in Figs. 8, where results of some of the previous examples are superposed.

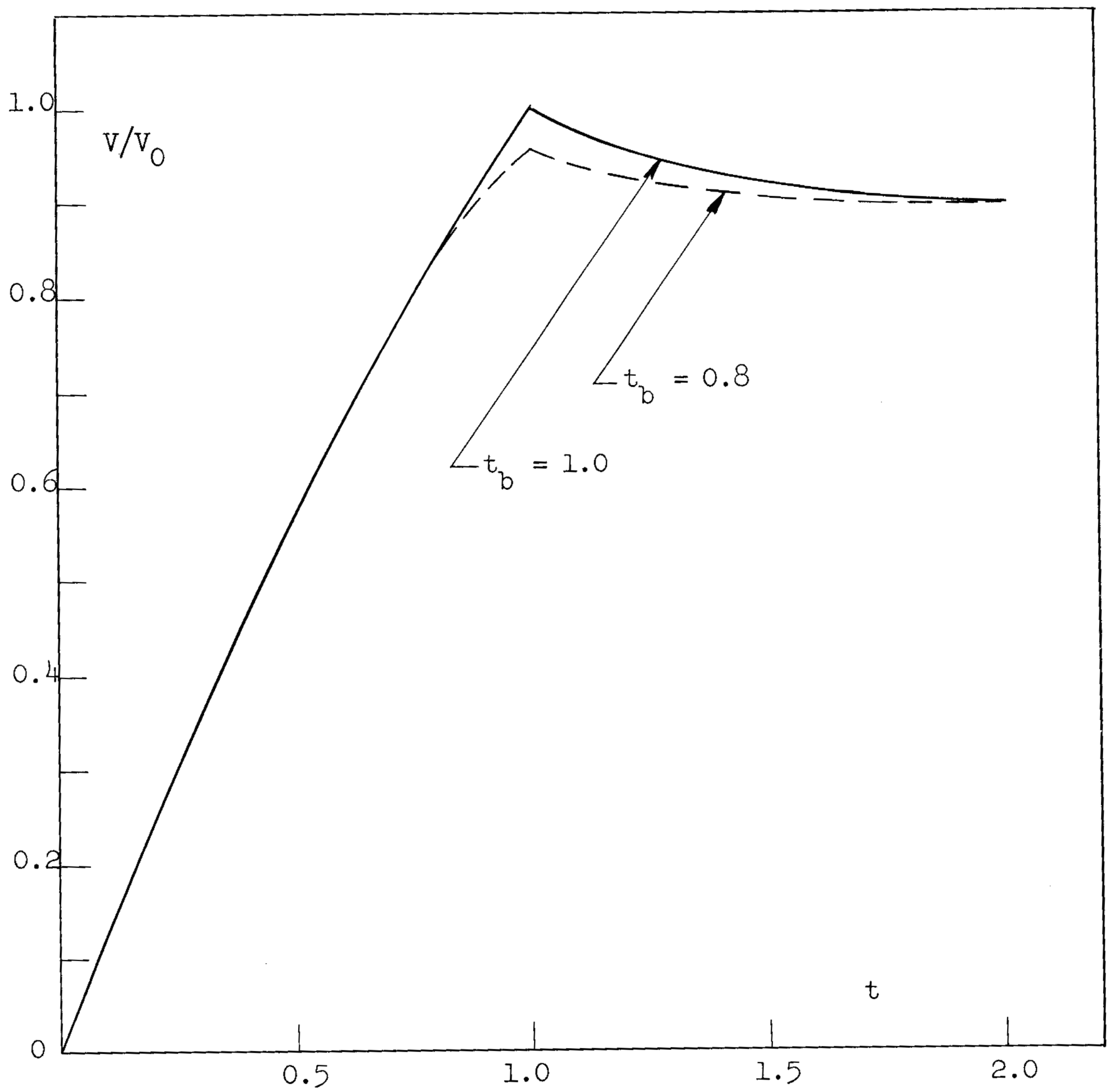

FIG. 8a--Superposition of Fig. 1 and Fig. 2 with $m = \frac{ri_0}{E_0} = 0.321$ (10% beam-loading with Project M parameters). The case of $t_b = 1.0$ corresponds to injection after the accelerator sections are filled with rf power, while $t_b = 0.8$ is an example of early injection. Note that in this case (Project M parameters, 10% beam-loading) it is not possible to reduce the transient energy variation very much by early injection alone.

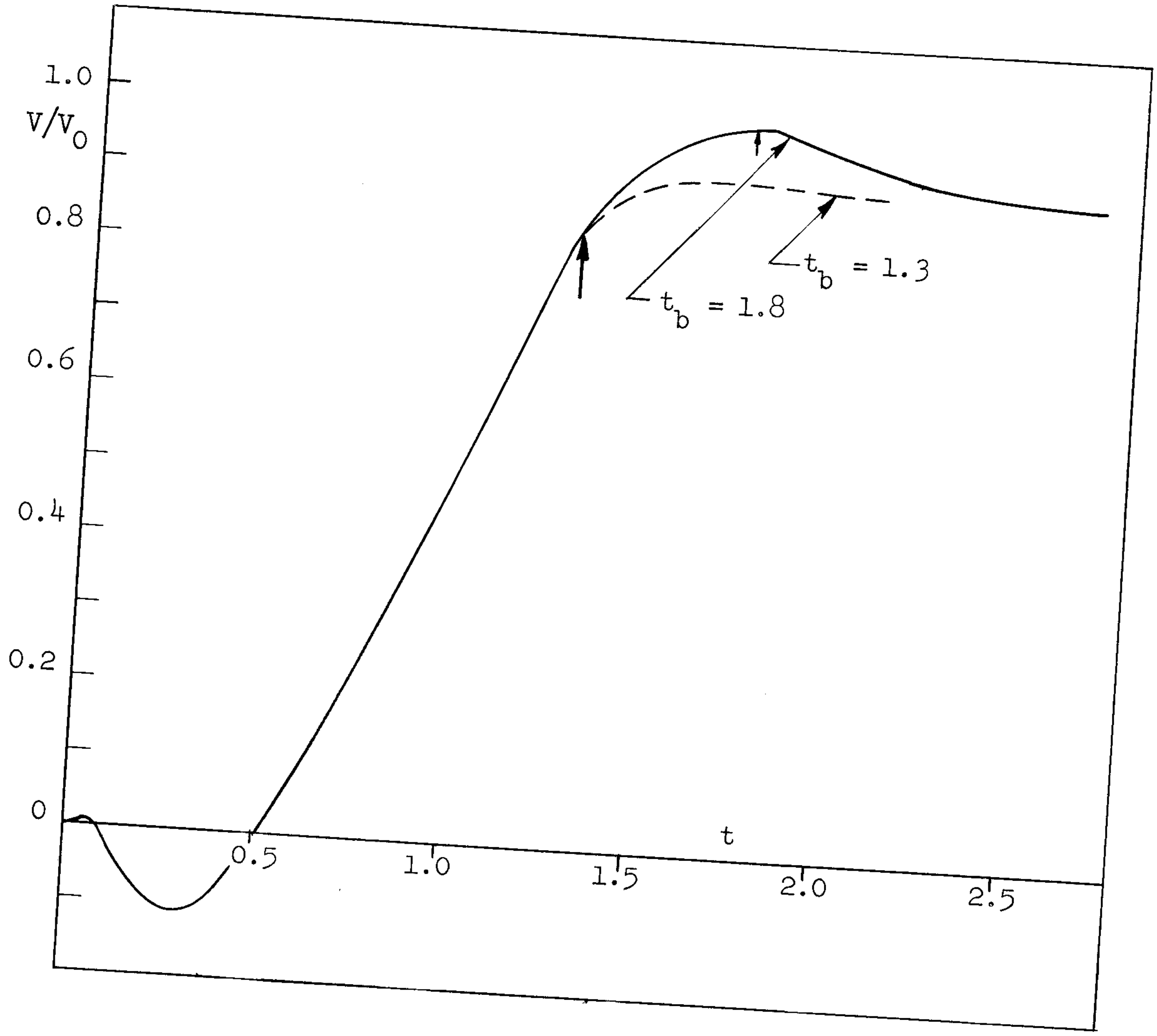

FIG. 8b--Superposition of Fig. 2 and Fig. 7 for 10% beam-loading. The case of $t_b = 1.8$ is for injection after the accelerator sections are effectively filled; the case of $t_b = 1.3$ corresponds to early injection. Note that by taking advantage of the "natural" shape of the rf pulse, the transient energy spread can be reduced considerably; i.e., for $t_b = 1.3$ the energy is within 1% of the steady-state value for about 85% of the transient period.

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