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A SOURCE OF ERROR IN THE USE OF SLOPE DETECTION FOR PERTURBATION MEASUREMENTS

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I. INTRODUCTION

It is frequently desirable to measure field intensity distribution in resonant cavities; an important example is the measurement of space-harmonic amplitudes in a periodic structure. In a typical laboratory measurement, the resonant frequency of a cavity is determined as a function of the position of a small metal or dielectric bead. The field distribution itself is then deduced directly from the frequency change. For high precision, it is often desirable to determine each resonant frequency by averaging equal-response frequencies on each side of resonance. When a number of positions of the perturbing bead are involved, this becomes a time-consuming process.

A simpler scheme was advanced by Ayers, Chu and Gallagher[1] which depends on the fact that a graph of the logarithm of the response of a cavity vs frequency is very nearly linear once the cavity response has dropped about 2 db. It is assumed that the perturbation is small enough that the Q and the coupling to the cavity are unchanged. It is the purpose of this note to demonstrate that the effective Q and coupling are changed if the frequency perturbation is much larger than the unperturbed bandwidth

1. W.R. Ayers, E.L. Chu, W.J. Gallagher, "Measurements of interaction impedance in periodic circuits", Internal memorandum, ML Report No. 403, June 1957.

of the cavity. The resulting measurements can then be quite misleading.

II. INTUITION

Consider a cavity consisting of a section of transmission line with short-circuits at each end. Two weakly coupled loops allow transmission measurements. A perturbing bead located in the cavity will load the line with a reactance, thus introducing a phase shift. To restore resonance, the frequency and thus the propagation constant in the transmission line must be changed to counteract this phase shift. On the other hand, any reactance may be represented by an equivalent transformer plus appropriate lengths of transmission line. Such a transformer can cause the excitation at one end of the cavity to be greater than that at the other. For a given excitation at the vicinity of the input coupling loop, then, the total stored energy in the cavity is changed. Thus, the coupling efficiency of the loop is changed and the transmission efficiency of the cavity no longer follows the first-order predictions.

Such an effect has been observed in tests of $2\pi/3$ mode disk-loaded accelerator sections. For a 3-disk cavity, one wavelength long, a frequency perturbation curve of the form of Fig. 1a was observed. It appeared to agree with expectations except for the end-effect where the bead interacts with its image in the end-plate. To eliminate the end-plate errors, a second measurement was made with a 6-disk (two-wavelength) cavity. It was expected that the curve would be identical to two of the original curves pasted together except, of course, for the elimination of end-plate effects on the central hump. Instead, the curve was found to be considerably distorted, as shown in Fig. 1b. It was assumed that the qualitative argument above explained the discrepancy; the measurements were continued using a direct measurement of frequency shift, and subsequent data have all agreed

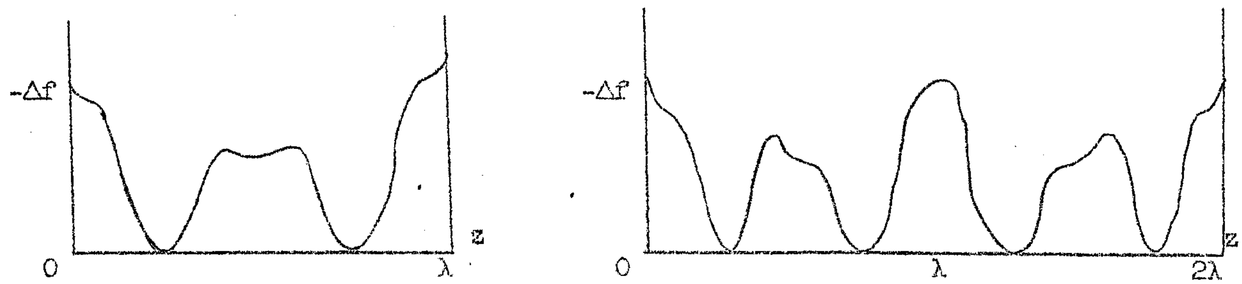


FIG. 1a FIG. 1b

Data from 3- and 6- disk cavities.

with expectations.

But the question remained: is slope-detection always likely to give wrong answers for space-harmonic measurements? May people have fooled themselves by use of this technique? The following analysis was made to determine the limitations of the method.

III. REASONING

Consider a cavity consisting of a section of uniform transmission line symmetrically coupled by ideal transformers of turns-ratio n. Suppose the cavity to have a length L, the transformers arranged to produce a voltage maximum at each end of the cavity, and the propagation constant at resonance to be $\gamma = \alpha + j\beta$. A small change of the operating frequency will produce little change of the attenuation constant α but will change the phase constant to $\beta + \epsilon$. If a small perturbing bead with normalized admittance jB is now introduced at a distance ℓ_1 from one end of the cavity, the equivalent circuit of Fig. 2 completely describes the cavity[2]. The effect of the bead may be completely determined by computing the overall transmission or scattering coefficient s_{16}.

One method of analysis of this circuit proceeds as follows: first, construct a partial scattering matrix S' which describes the properties of the coupling transformers and the bead as separate circuit elements; then, construct a transfer matrix T which connects appropriate terminals of the separate elements. The overall scattering matrix S may then be written and, with a little algebra, s_{16} may be computed.

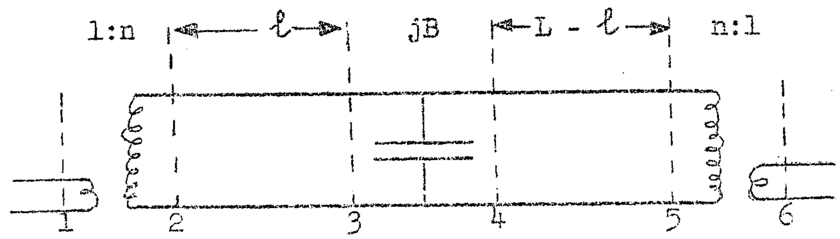


FIG. 2--Equivalent circuit of cavity

2. This paper is presented without regard for space-harmonics in the cavity. It will be shown in a forthcoming paper that the effect of space-harmonics or of transverse motions of the bead can be represented by a variation of the normalization of the bead susceptance.

$$S' = \begin{pmatrix} \frac{1-n^2}{n^2+1} & \frac{2n}{n^2+1} & 0 & 0 & 0 & 0 \\ \frac{2n}{n^2+1} & \frac{n^2-1}{n^2+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-jB}{2+jB} & \frac{2}{2+jB} & 0 & 0 \\ 0 & 0 & \frac{2}{2+jB} & \frac{-jB}{2+jB} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{n^2-1}{n^2+1} & \frac{2n}{n^2+1} \\ 0 & 0 & 0 & 0 & \frac{2n}{n^2+1} & \frac{1-n^2}{n^2+1} \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\gamma\ell} & 0 & 0 & 0 \\ 0 & e^{-\gamma\ell} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\gamma(L-\ell)} & 0 \\ 0 & 0 & 0 & e^{-\gamma(L-\ell)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S = (1 - S'T)^{-1}S' \qquad (1)$$

The algebra is considerably simplified if all quantities are expressed in terms of $\Delta = \alpha L$, a small quantity. For $1/n^2$ substitute $k\Delta$, for ϵ/α write δ, for B write $b\Delta$. Note that $\cos\beta\ell$ represents the electric field distribution of the unperturbed mode. For an undercoupled cavity in which $K \ll 1$, one finds, to first-order in Δ,

$$s_{16}^2 = \frac{k(1 - \Delta/2)}{1 + (\delta + b \cos^2 \beta\ell)^2 + \frac{1}{2} \Delta b^2 \cos^3 \beta\ell \left[4(1 - 2\ell/L) \sin \beta\ell - \cos \beta\ell\right]} \quad (2)$$

The first part of the denominator will be recognized as the normal change of response of a cavity as the frequency variable δ and the amount of perturbation $bE^2 = b \cos^2 \beta\ell$ are varied. The quantity Δ equals the deviation from resonance of the frequency variable δ at the half-power points; k is the usual coupling coefficient. It is the last term in the denominator which leads to the distortion of the response. It appears to be small since it is of order Δ, but the term is also proportional to b^2. From the first term, it will be recognized that b is the ratio of the maximum frequency perturbation introduced by the bead to the half-bandwidth of the cavity. If the bead is large enough to shift the resonant frequency several bandwidths, the factor b^2 may well be large enough to make the factor in $1 - 2\ell/L$ significant.

IV. COMPARISON

In the particular experimental cavity described above, the voltage attenuation is relatively large so that $\Delta = \alpha L \approx 1/20$. The cavity bandwidth was 343 kc at 2856 Mc; the Q was about 8300. If the perturbation is nominally large, say $b = 3$, and the frequency is chosen so that the transmission is already about 2 db down from resonance with no perturbation, the maximum change of transmission is 10 db and the correction for coupling changes in the accelerator cavity is never over two per cent. The experimental data in Fig. 1 was taken with a larger bead which detuned the cavity resonance 1.62 Mc, about nine times the half-bandwidth. Thus $b = 9$, $b^2 = 81$, and the last term in the denominator of Eq. (2) is no longer very small. It now adds as much as 25 per cent to the signal on one side of a maximum and subtracts the same amount on the other side. The correction does indeed account for the observed results.

V. CONCLUSION

In a short cavity or one with very low losses, with αL a small quantity, one need not worry about the influence of a perturbing bead on the coupling to the cavity so long as the maximum perturbation of the resonant frequency is of the order of the bandwidth of the cavity. With a long cavity of many wavelengths, or with a perturbation amounting to several cavity bandwidths, the effective coupling to the cavity can be drastically changed and makes the slope-detection scheme a poor method of taking the required data. A method which measures the resonant frequency directly is not affected by this phenomenon.

In working out the algebra it becomes evident that it is the redistribution of losses within the cavity that causes this sometimes large second-order effect. Since, however, the losses at any point are proportional to the stored energy, the intuitive argument of Section II is quite valid.

In the case of perturbation by dielectric rods, the stored energy and losses remain uniform over the length of the cavity. For a given field strength at the coupling loop, the losses will remain about the same. But the total stored energy in the cavity is decreased, thus leading again to a change of coupling if the system is undercoupled as in the case treated above. Thus the same effect would exist, subject to the same conditions: no difficulty would be experienced if the maximum frequency perturbation is of the order of one or two cavity bandwidths.

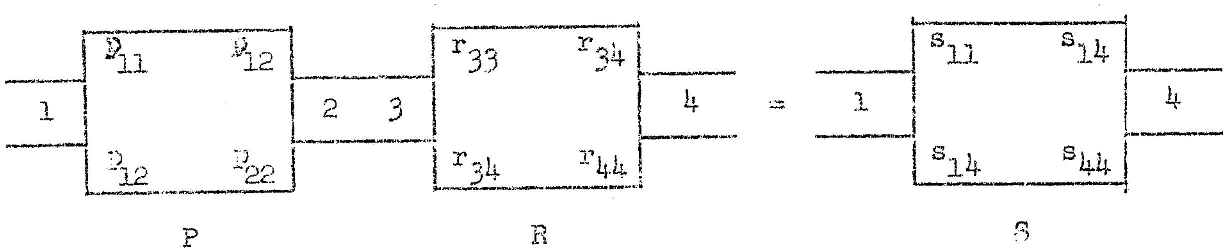
VI. ACKNOWLEDGEMENTS

The author wishes to thank Mme. R. Hirel who made the experimental measurements and who proposed the intuitive argument of Section II.

APPENDIX

Transfer properties of a circuit are occasionally difficult to write down even when the properties of its components are known. In microwave circuits, it is frequently useful to determine the scattering matrix for a circuit composed of two components connected together.

Consider the simple case of two components P with terminals 1 and 2 and R with terminals 3 and 4. Suppose we connect 2 and 3 and determine the overall scattering matrix S between terminals 1 and 4. Then S_{11} will include a direct reflection p_{11}



and a reflection r_{33} modified by the round-trip through P_1 , p_{12}^2 . This latter reflection must be modified by a factor $1 + p_{22}\, r_{33}$ to account for the portion of the wave that is doubly reflected between P and R. In addition there is a portion of the wave which has been reflected even more times. The total reflection is therefore

$$s_{11} = p_{11} + p_{12}^2\, r_{33}(1 + p_{22}\, r_{33} + p_{22}^2\, r_{33}^2 + \dots)$$

$$= p_{11} + \frac{p_{12}^2\, r_{33}}{1 - p_{22}\, r_{33}}$$

Again, s_{14} is a wave which has traveled through P and R but which must be corrected for multiple reflections between P and R:

$$s_{14} = \frac{p_{12}\, r_{34}}{1 - p_{22}\, r_{33}}$$

In the same manner, s_{44} is also found:

$$s_{44} = r_{44} + \frac{r_{34}^2 \, p_{22}}{1 - r_{33} \, p_{22}}$$

This method of computing scattering coefficients could have been used to determine transmission through the cavity in this paper; but such a method involves computing many intermediate terms and generally gives the answer in a very cumbersome form. Moreover, the reasoning above breaks down when two or more pairs of ports are connected together. Below is given a more general method which will handle any such problem.

Let us write a single scattering matrix S' for a "black box" containing all the components of the system. In this paper, this included the two coupling transformers and the perturbing element. Note that in this scattering matrix [see Eq. (1) in text] there are no connections between the input transformers, terminal 1-2, and the perturbing element, terminals 3-4. Then a second matrix T is constructed which contains only the interconnections. It will consist only of ones and zeros if the terminals of S' are connected directly. In Eq. (1), it contains the properties of the transmission lines. It may be a still more complicated circuit. Note, however, that it contains no information about the connections within S'; in Eq. (1), T has no connection across the perturbing element (terminals 3 and 4).

Now let us represent the waves entering the terminals of S' as a vector A' and the waves emerging as a vector B. Then $B = S'A'$. Now, the waves leaving S' are the waves entering T; the waves entering S' are the waves leaving T plus the waves entering the free terminals (the ones not interconnected). If we represent these waves (the useful inputs) as a vector A, then $A' = TB + A$; combining, we get $B = S'(TB + A) = S'TB + S'A$, or $(1 - S'T)B = S'A$, from which we obtain

$$B = (1 - S'T)^{-1} S'A = SA \qquad \text{Q.E.D.}$$

A property of B that is sometimes useful is that it contains the waves emerging from every port of S' in terms of the waves A entering only the free ports.

A given problem may still be arranged in many ways. In the two-component example given above, we might write

$$S' = \left(\begin{array}{c|cc} P & 0 & 0 \\ & 0 & 0 \\ \hline 0\ 0 & & \\ 0\ 0 & R & \end{array}\right), \quad T = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

as in Fig. 4a.

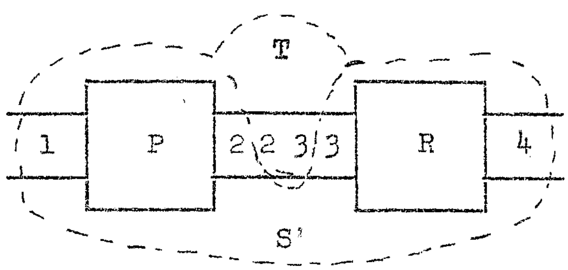


Fig. 4a

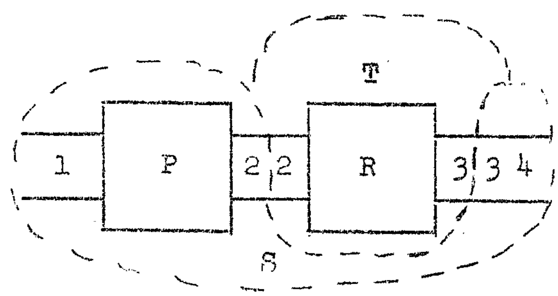


Fig. 4b

or we might include only P and a dummy output terminal in S' and include R in T as in Fig. 4b.

$$S' = \left(\begin{array}{c|cc} P & 0 & 0 \\ & 0 & 0 \\ \hline 0\ 0 & 0 & 1 \\ 0\ 0 & 1 & 0 \end{array}\right), \quad T = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & & & 0 \\ 0 & \multicolumn{2}{c|}{R} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$