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CALCULATION OF PION PHOTOPRODUCTION FROM ELECTRON ACCELERATORS ACCORDING TO THE STATISTICAL MODEL

By

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The pion yield from electron accelerators is of interest mainly to two classes of people. Experimental physicists are eager to find the pion yield high and well-collimated forward so that, for example, the neutrino flux resulting from pi-mu decay will be large enough to be useful in measurements of neutrino-induced processes. A somewhat different viewpoint is held by those who attempt to design shielding for high-energy electron accelerators. Their problem with pions is concerned with the fact that the mu-mesons resulting from pi-mu decay are very difficult to stop in other than very massive shields.[1] The present calculations do nothing to relieve the problems of either of these classes of people.[2]

The pion yields calculated here are based on the following model. The electron beam produces a soft shower on striking a target.

[1] Mu mesons lost about 2 Mev in traversing one gram cm^2 of material. This means that the range of a 20-Bev Muon is about 150 feet in ordinary concrete.

[2] Recent calculations by Drell[3] show that the pion-pair yield is more strongly peaked in the forward direction than the yields calculated here, and probably defines the shielding problem in the forward direction (as well as providing sufficient flux for many interesting experiments).

[3] S. D. Drell, Phys. Rev. Letters 5, 278 (1960). See also: S. D. Drell, "Production of Particle Beams at Very High Energies," Report No. M-200-7, W. W. Hansen Laboratories of Physics, Stanford University.

Some of the photons in the shower are then absorbed by nucleons in the target nuclei. The nucleons then boil off a bunch of pions whose number is calculated according to the statistical model. The yields are transformed into the laboratory frame of reference so that the final results are expressed as the number of pions produced per unit pion energy per solid angle in the laboratory when one electron of stated energy is incident on the target.

I. THE SOFT-SHOWER PHOTON SPECTRUM

The photon spectrum in a soft shower is approximately given by the track-length formula[4]

$$(0.572)\,\frac{E_0}{E_\gamma^2}\,dE_\gamma\ ; \quad (E_\gamma \leq E_0)$$

which is the total path length (in radiation lengths) of photons with energies between E_γ and $E_\gamma + dE_\gamma$ when an electron of energy E_0 is incident on a target many radiations lengths thick.

[4]See for example: B. Rossi, High Energy Particles, Prentice Hall, New York (1952), p. 244.

III. THE STATISTICAL MODEL

Once the photon is absorbed, pion production multiplicity is determined by calculations based on Fermi's statistical model.[5-7] Except in the special case of two particles, the statistical model has never been treated exactly so as to provide an expression in closed form. Hagedorn and his co-workers at CERN have performed the calculation numerically using a fast electronic computer. Their efforts are directed toward an understanding of proton-nucleon collisions for the CERN proton synchrotron, and for this reason we are unable to use their results directly. These results are described in refs. 8-11.

We have used an approximate expression given by Lepore and Stewart[6] in which nucleons are treated non-relativistically and mesons are assumed to have energies in the extreme relativistic range. The formulae are discussed in detail by Milburn[7], and our work is based on Milburn's Eq. (3.23), which is

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[5] E. Fermi, Progr. Theoret. Phys. (Japan) 5, 570 (1950)

[6] J. V. Lepore and R. N. Stewart, Phys. Rev. 94, 1724 (1954)

[7] R. H. Milburn, Rev. Mod. Phys. 27, 1 (1955)

[8] F. Cerulus and R. Hagedorn, N. 2 del Suppl. Nuovo Cimento I 9, 646 (1958); and II 9, 659 (1958)

[9] R. Hagedorn, CERN Report 59-25, July 2, 1959

[10] R. Hagedorn, Nuovo Cimento, 15, 434 (1960)

[11] F. Cerulus and R. Hagedorn, CERN Report 59-3

$$\frac{dV_w}{dW} \approx \left[\frac{M^{3s/2}(2\pi)^{3(s-1)/2}}{(sM)^{3/2}}\right] \times 2^{3n}\pi^{n} \frac{\left[W - sM - n\mu\right]^{3s/2+3n-5/2}}{c^{3(n-1)}\Gamma\left[3(s-1)/2 + 3n\right]} \qquad (3)$$

Here, W is the total energy available in the center of mass, M is the nucleon rest energy, μ is the meson rest energy, s is the number of nucleons, and n is the number of mesons. In our problem $s = 1$, since the nucleon absorbing the incident photon remains to take part in the competition for the energy available in the center of mass. This expression is used in Milburn's Eq. (2.7). The relative probability of n mesons turns out to be

$$S(n,1) = \frac{\Omega^{n}\pi^{n}c^{3}}{(\pi\hbar c)^{3n}\Gamma(3n)}(W - M - n\mu)^{3n-1}$$

$$= \frac{c^3}{M}\left[\frac{8}{3\pi}\left(\frac{M}{\mu}\right)^3\right]^n \frac{(M/W)^n}{\Gamma(3n)}\left[(W/M) - 1 - n(\mu/M)\right]^{3n-1}$$

$$= \frac{c^3}{M} \cdot \frac{(258\ M/W)^n}{\Gamma(3n)}\left[(W/M) - 1 - (0.149)n\right]^{3n-1} \qquad (4)$$

In obtaining the final form of (4), we have taken the "interaction volume" Ω to be

$$\Omega = (2M/W)(4/3)\pi R^3; \quad R = \hbar/(M_\pi c) = 1.4 \times 10^{-13}\text{cm} \qquad (5)$$

Equation (4) must be evaluated for all values of n such that $W > M + n\mu$. The probability p_n of n mesons is then the value of

(4) divided by $\sum_n S(n,1)$. The average number of mesons $\langle n \rangle$ resulting from this calculation is displayed in Fig. 1 as a function of the incident photon energy E_γ.

At this point, a statement must be made concerning the energy-angle distribution of these mesons in the center-of-mass system. We assume the distribution to be isotropic for all multiplicities n. Further, for $n = 1$, the center-of-mass energy of the meson is calculated in the usual way by saying that the meson and nucleon have equal but oppositely directed momenta in the center of mass. For $n > 1$, we assume that the mesons get all the excess energy available in the center of mass and that the nucleon gets none at all. Further, the mesons are assumed to share this energy excess equally so that for a stated value of the incident photon energy E_γ and multiplicity n, the energy of any meson in the center of mass is uniquely determined. This leads to a distinct relation between the meson energy and angle in the laboratory frame of reference.

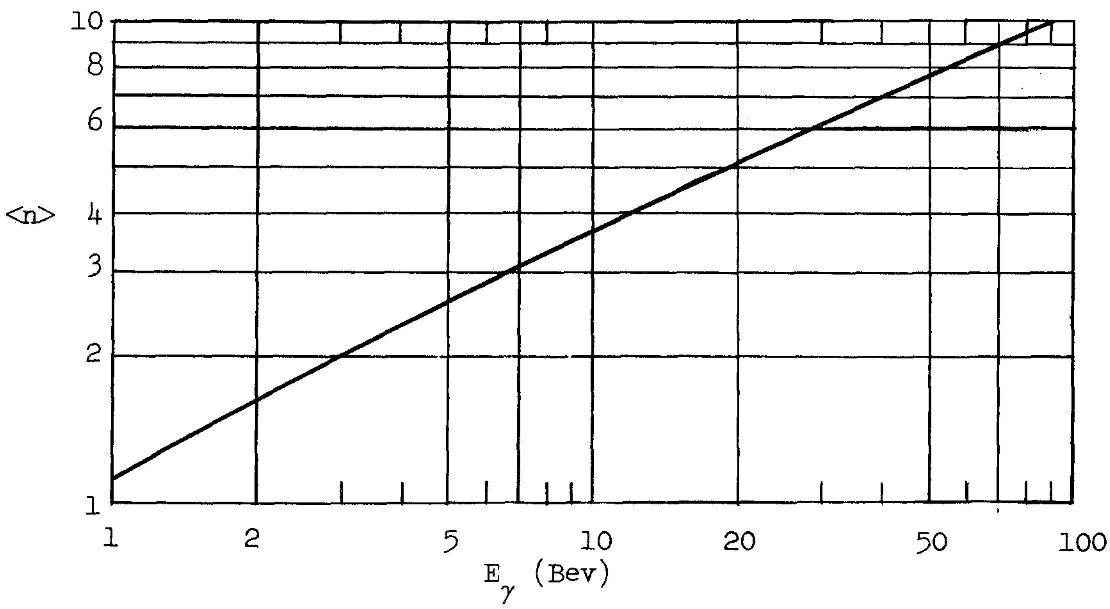


FIG. 1--Calculated pion multiplicity $\langle n \rangle$ vs. E_γ.

IV. LABORATORY YIELD SPECTRUM

The pion yield at any given angle in the laboratory is the sum of the yields due to all multiplicities. Thus if $W(\theta_{lab},E_\pi)$ is the total yield spectrum (pions/electron-steradian-Mev), then

$$W(\theta_{lab},E_\pi) = \sum_n W_n(\theta_{lab},E_\pi) \qquad (6)$$

where E_π is the laboratory kinetic energy of the pions. Now the W_n are calculated as follows*:

$$W_n(\theta_{lab},E_\pi) = \left[\left(\frac{\partial E_\pi}{\partial E_\gamma}\right)_{\theta_{lab}} \left(\frac{\partial \omega}{\partial \omega'}\right)_{E_\gamma}\right]^{-1} U_n(\theta',E_\gamma) \qquad (7)$$

where $\omega = \cos\theta_{lab}$, $\omega' = \cos\theta'$, and $U_n(\theta', E_\gamma)$ is the pion yield spectrum in the center of mass. This is calculated by combining (1), (2) and the probabilities p_n.

$$U_n(\theta',E_\gamma) = \left[\frac{(0.572)E_o}{E_\gamma^2} \cdot \frac{X_o}{\rho}\right] (N\rho)np_n(E_\gamma)\,\frac{\sigma_a(E_\gamma)}{4\pi} \qquad (8)$$

where

*See for example: "Jacobians and Relativistic Kinematics", K. G. Dedrick, Report No. M-226, W. W. Hansen Laboratories of Physics, Stanford University.

$$\left[\frac{(0.572)E_o X_o}{E_\gamma^2 \rho}\right] = \text{track length (cm/Mev)}$$

E_o = primary electron energy (Mev)

X_o = radiation length of target (grams/cm^2)

$N\rho$ = number of target nucleons per cm^3

N = Avogadro's number

ρ = density of target (grams/cm^3)

n = the multiplicity of pion photo-production

$p_n(E_\gamma)$ = the probability of n

$\sigma_a(E_\gamma)/4\pi$ = differential cross-section for photon absorption (cm^2/steradian)

Numerically we have

$$U_n(\theta', E_\gamma) = E_o X_o (3.876 \times 10^{-13}) n \; p_n(E_\gamma) \frac{1 + 2x}{x^4} \tag{9}$$

$$x = E_\gamma / Mc^2$$

V. NUMERICAL RESULTS

Numerical calculations based on the simplified model given here are quite involved, the reason being that for each (E_γ, n) we are led to a distinct relation between E_π and θ_{lab}, and hence also the quantities $\left(\partial E_\pi / \partial E_\gamma\right)_{\theta_{lab}}$ and $\left(\partial \omega / \partial \omega'\right)_{E_\gamma}$. The procedure is to first calculate the $p_n(E_\gamma)$. These are used in (9) to get $U_n(\theta', E_\gamma)/(E_o X_o)$. Next, the kinematics are calculated for all input values of E_γ and n, as well as the derivatives in (7). The $W_n[\theta_{lab}, E_\pi(E_\gamma, n)]$ are then computed. An interpolation program then gives the values of W_n for specific values of E_π so that the yield addition indicated in (6) may be carried out. Finally the yield addition is done for a variety of values of E_o, the primary electron energy. This is complicated by the fact that the photon spectrum given by (1) is cut off at E_o, and gives the result that, for each n, the spectra W_n are cut off at some value of E_π depending on E_o and θ_{lab}. The numerical results to follow show discontinuities because of this feature of the calculation. Naturally, the assumption that the center-of-mass energy is shared equally among the pions (which leads to a unique relation between E_π and θ_{lab} for the given E_γ and n) does nothing to smooth these discontinuities.

The numerical results are meant to be good for a very thick target (many radiation lengths long) of any material. The yield spectra in Figures 2, 3, 4, and 5 give the pion yield for one incident electron of energy E_o. The results in the figures, when multiplied by the radiation length X_o(grams/cm^2), give the numbers of pions per steradian per

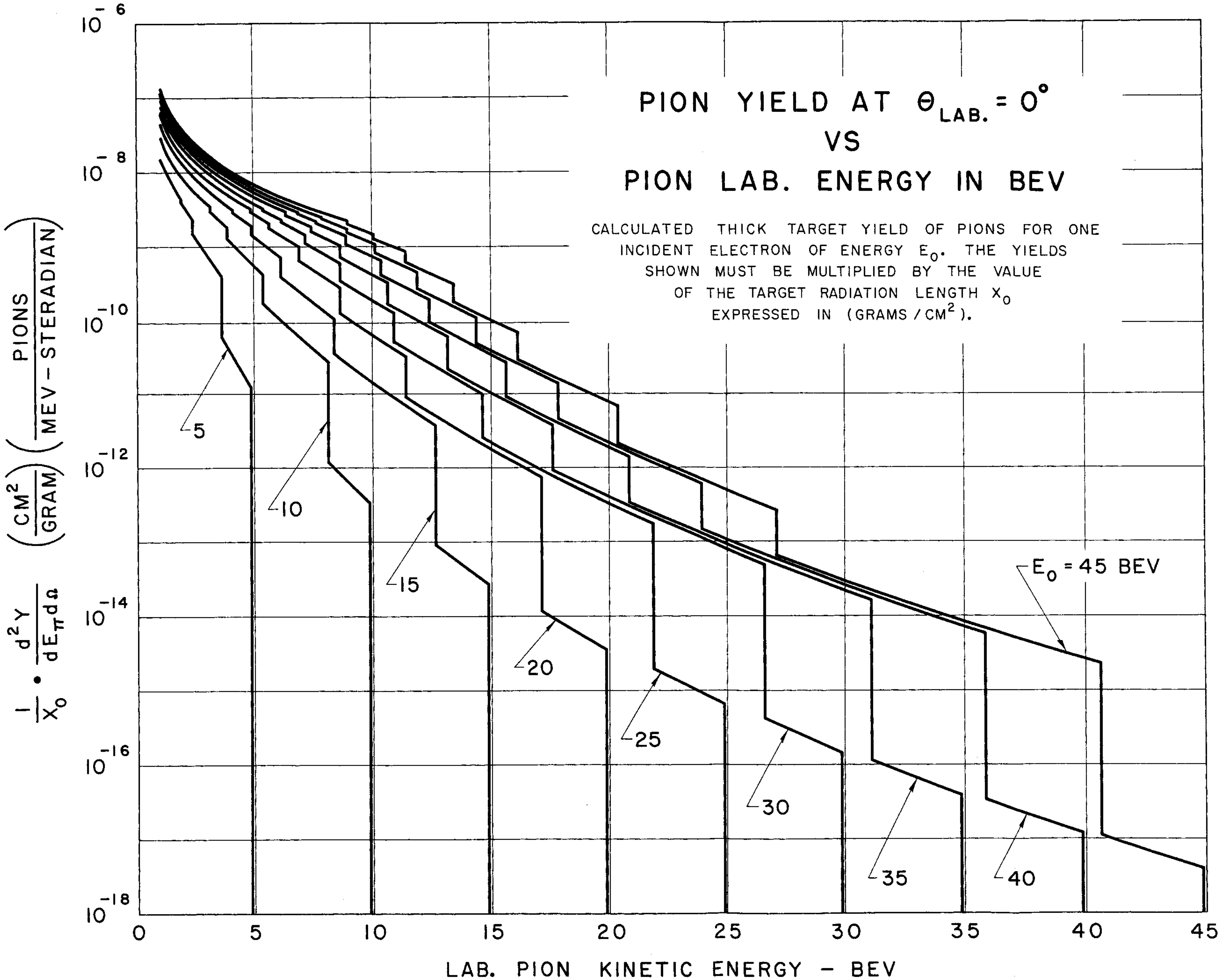
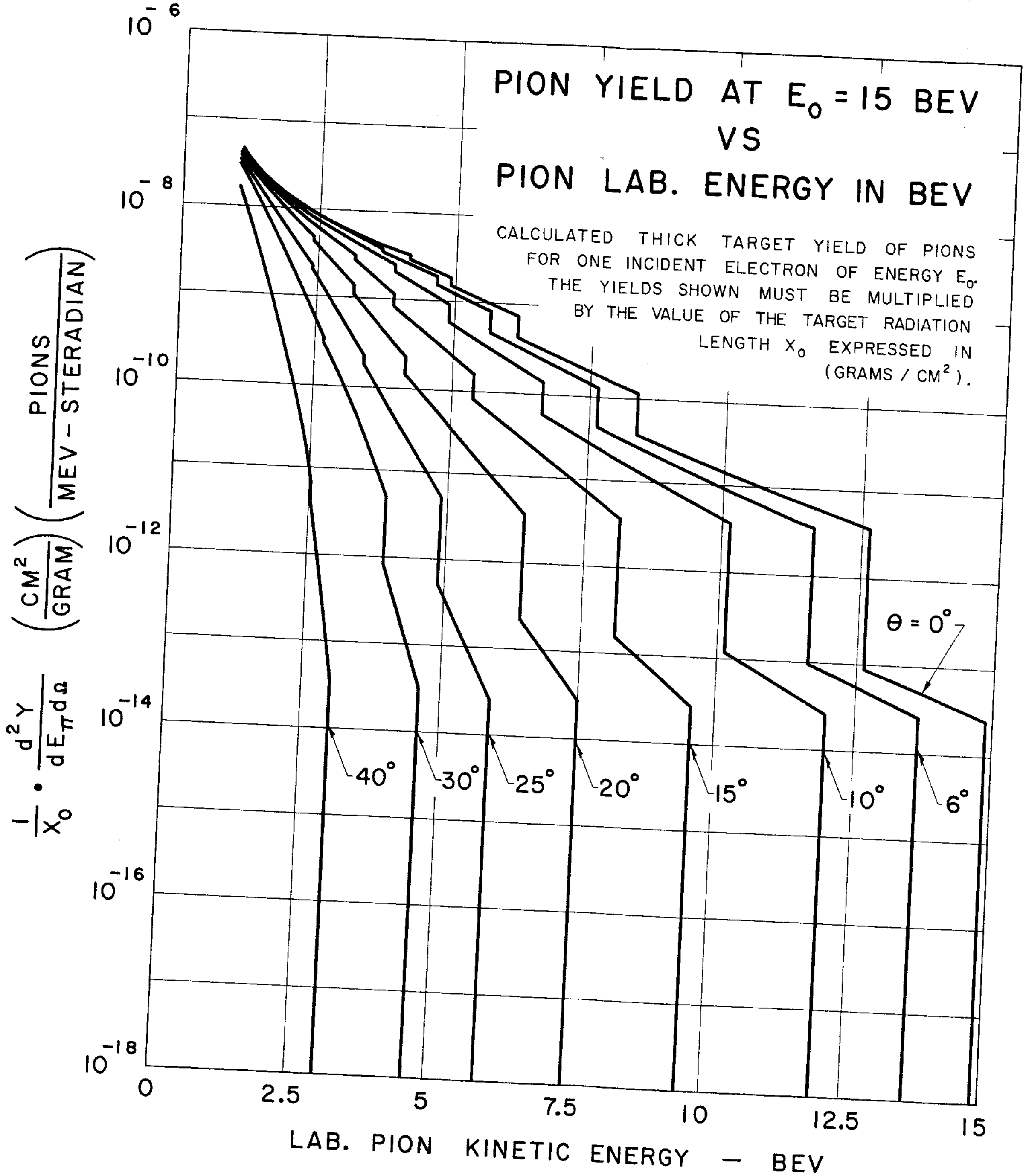

PION YIELD AT $\theta_{LAB.} = 0°$
VS
PION LAB. ENERGY IN BEV
CALCULATED THICK TARGET YIELD OF PIONS FOR ONE INCIDENT ELECTRON OF ENERGY E_0. THE YIELDS SHOWN MUST BE MULTIPLIED BY THE VALUE OF THE TARGET RADIATION LENGTH X_0 EXPRESSED IN (GRAMS / CM^2).
$\frac{1}{X_0} \cdot \frac{d^2Y}{dE_\pi d\Omega}$ $\left(\frac{CM^2}{GRAM}\right)\left(\frac{PIONS}{MEV-STERADIAN}\right)$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
0
5
10
15
20
25
30
35
40
45
$E_0 = 45$ BEV
LAB. PION KINETIC ENERGY - BEV

FIG. 2


PION YIELD AT $E_0 = 15$ BEV
VS
PION LAB. ENERGY IN BEV
CALCULATED THICK TARGET YIELD OF PIONS FOR ONE INCIDENT ELECTRON OF ENERGY E_0. THE YIELDS SHOWN MUST BE MULTIPLIED BY THE VALUE OF THE TARGET RADIATION LENGTH X_0 EXPRESSED IN (GRAMS / CM^2).
$\frac{1}{X_0} \cdot \frac{d^2Y}{dE_\pi d\Omega} \left(\frac{CM^2}{GRAM}\right)\left(\frac{PIONS}{MEV-STERADIAN}\right)$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
0
2.5
5
7.5
10
12.5
15
LAB. PION KINETIC ENERGY — BEV
$\theta = 0°$
40°
30°
25°
20°
15°
10°
6°
FIG. 3

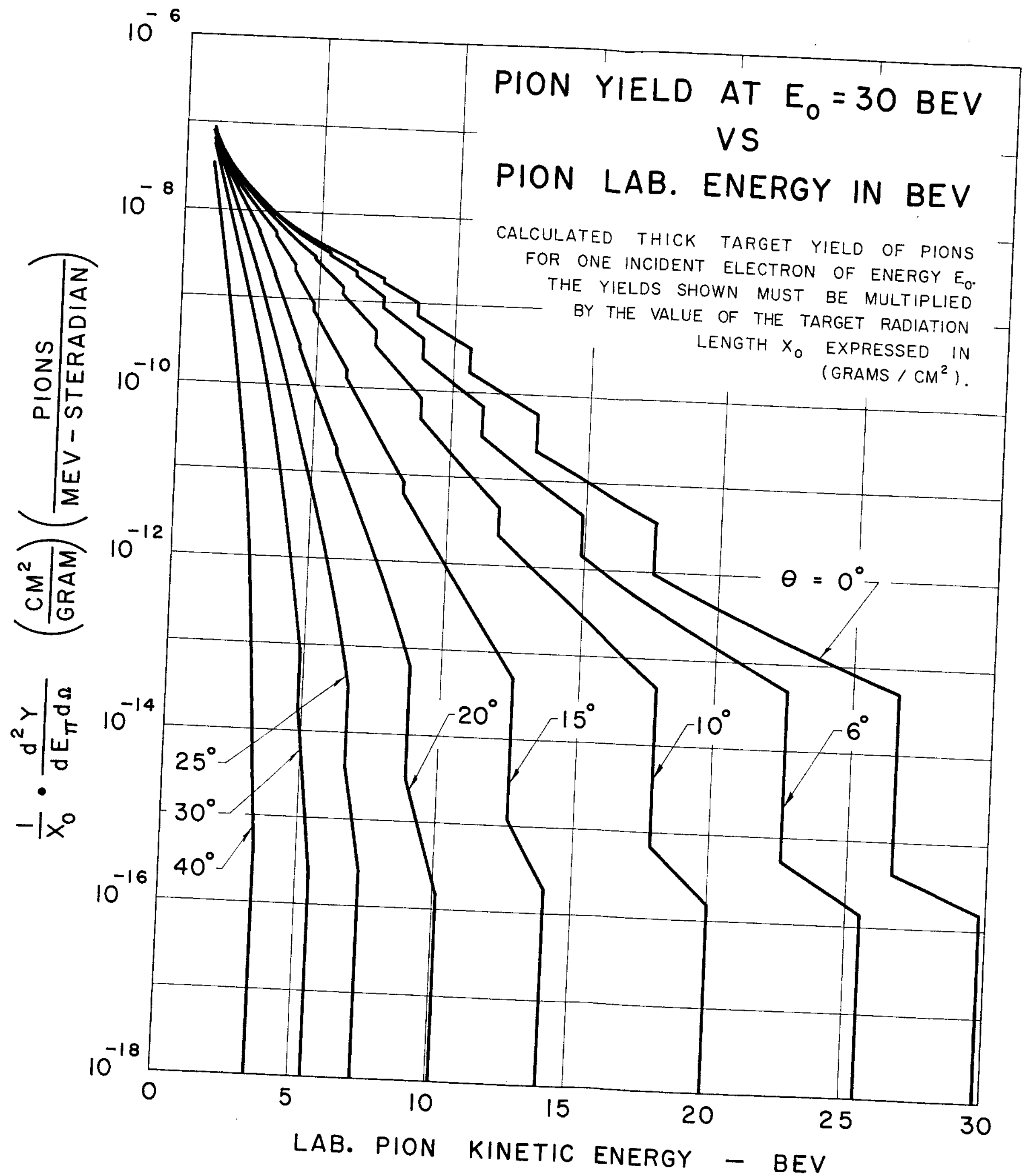

PION YIELD AT $E_0 = 30$ BEV
VS
PION LAB. ENERGY IN BEV
CALCULATED THICK TARGET YIELD OF PIONS FOR ONE INCIDENT ELECTRON OF ENERGY E_0. THE YIELDS SHOWN MUST BE MULTIPLIED BY THE VALUE OF THE TARGET RADIATION LENGTH X_0 EXPRESSED IN (GRAMS / CM^2).
$\frac{1}{X_0} \cdot \frac{d^2 Y}{dE_\pi d\Omega}$ $\left(\frac{CM^2}{GRAM}\right)\left(\frac{PIONS}{MEV-STERADIAN}\right)$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
$\theta = 0°$
6°
10°
15°
20°
25°
30°
40°
0
5
10
15
20
25
30
LAB. PION KINETIC ENERGY — BEV

FIG. 4

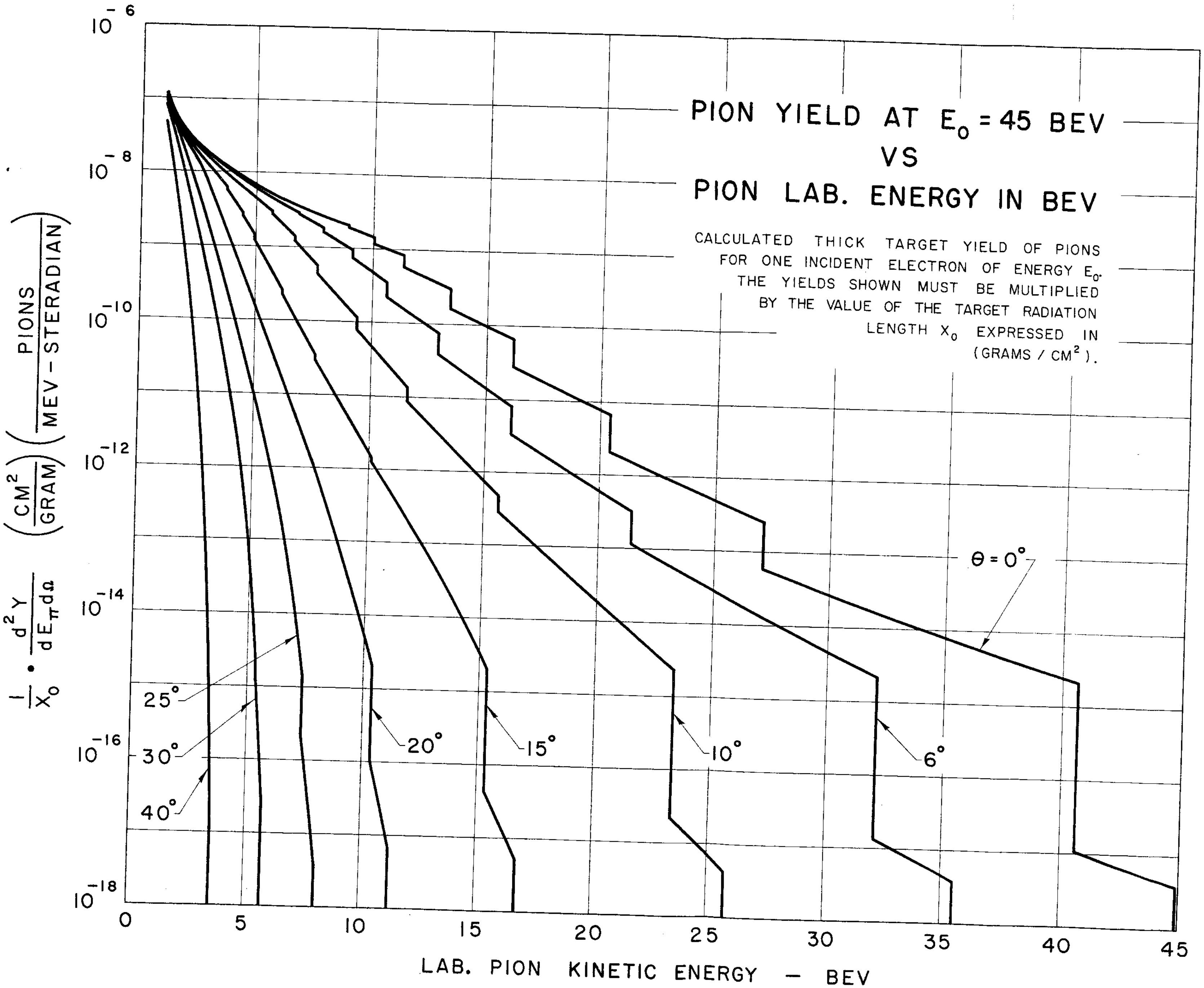

PION YIELD AT E_0 = 45 BEV
VS
PION LAB. ENERGY IN BEV
CALCULATED THICK TARGET YIELD OF PIONS FOR ONE INCIDENT ELECTRON OF ENERGY E_0. THE YIELDS SHOWN MUST BE MULTIPLIED BY THE VALUE OF THE TARGET RADIATION LENGTH X_0 EXPRESSED IN (GRAMS / CM^2).
$\frac{1}{X_0} \cdot \frac{d^2 Y}{dE_\pi d\Omega}$ $\left(\frac{CM^2}{GRAM}\right)\left(\frac{PIONS}{MEV-STERADIAN}\right)$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
0
5
10
15
20
25
30
35
40
45
LAB. PION KINETIC ENERGY — BEV
θ = 0°
6°
10°
15°
20°
25°
30°
40°

FIG. 5