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JACOBIANS AND RELATIVISTIC KINEMATICS

By

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It is often necessary to calculate the energy-angle distribution of particles in one Lorentz frame of reference from knowledge of the distribution in some other frame. This problem occurs in the interpretation of experiments where an energy-angle distribution is measured in the laboratory frame of reference and it is desired to calculate the distribution in the center of mass. Problems of this type are discussed in Section I.

A distinct but related problem is the subject of Section II, which is concerned with reactions where two reaction product particles are produced and hence specification of the bombarding energy gives rise to a definite laboratory energy-angle relationship for the reaction products. In this problem we are interested in calculating the laboratory yield spectrum of particles produced when the bombarding particles are distributed over a range of energies. An example of this problem is that of calculating the laboratory yield spectrum of single photopions given knowledge of the incident photon spectrum and the cross section for pion photo-production as a function of photon energy.

In both of these problems, Jacobians of the Lorentz transformations occur as multiplicative factors in transforming particle yields between coordinate systems. In the description of the second problem mentioned above, where one is concerned with a distribution of bombarding energies, it is useful to group all possible reactions into classes having Jacobians and angle transformations of a given form. This classification scheme is the subject of Section III.

I. THE FIRST PROBLEM

The total energy, momentum and angle in the laboratory frame of reference will be denoted by the quantities E, p, and θ respectively. The situation in the moving frame of reference, which in this study is the center-of-mass frame, is described by the corresponding primed quantities E', p', and θ'. The velocity of the center-of-mass frame with respect to the laboratory in units of the velocity of light is denoted by β, and Γ is written for $(1 - \beta^2)^{-\frac{1}{2}}$. The Lorentz transformation may be written[1]

$$E = \Gamma(E' + \beta P'\omega') \tag{1a}$$

$$P\omega = \Gamma(P'\omega' + \beta E') \tag{1b}$$

$$P \sin \theta = P' \sin \theta' \tag{2}$$

$$E' = \Gamma(E - \beta P\omega) \tag{3a}$$

$$P'\omega' = \Gamma(P\omega - \beta E) \tag{3b}$$

$$E^2 - P^2 = Mc^2 = E'^2 - P'^2 \tag{4}$$

where

$$\omega = \cos \theta; \ \omega' = \cos \theta'$$

$$P = pc; \ P' = p'c$$

$W'(E',\omega')$ denotes the energy-angle distribution of particles in the moving frame of reference, and $W(E,\omega)$ is the distribution in the laboratory. Conservation of particles requires that

$$W'(E',\omega')dE' \, d\omega' = W(E,\omega)dE \, d\omega \tag{5}$$

where (E',ω') and (E,ω) are related through the Lorentz transformations (1) or (3). We then write

$$dE' \, d\omega' = \frac{\partial(E',\omega')}{\partial(E,\omega)} dE \, d\omega \tag{6}$$

where $\partial(E',\omega')/\partial(E,\omega)$ is the Jacobian of the Lorentz transformation.

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[1] See for example: "The Kinematics of High-Energy Collisions," by K. Dedrick, ML Report No. 574, February 1959, W. W. Hansen Laboratories of Physics, Stanford University, Stanford, California.

Substitution of (6) into (5) yields

$$W(E,\omega) = \frac{\partial(E',\omega')}{\partial(E,\omega)} W'(E',\omega') \tag{7}$$

Once this Jacobian has been calculated, other useful Jacobians may be obtained from it:

$$\frac{\partial(P',\omega')}{\partial(E,\omega)} = \frac{\partial(P',\omega')}{\partial(E',\omega')} \cdot \frac{\partial(E',\omega')}{\partial(E,\omega)} = \frac{E'}{P'} \cdot \frac{\partial(E',\omega')}{\partial(E,\omega)} \tag{8}$$

$$\frac{\partial(E',\omega')}{\partial(P,\omega)} = \frac{\partial(E,\omega)}{\partial(P,\omega)} \cdot \frac{\partial(E',\omega')}{\partial(E,\omega)} = \frac{P}{E} \cdot \frac{\partial(E',\omega')}{\partial(E,\omega)} \tag{9}$$

$$\frac{\partial(P',\omega')}{\partial(P,\omega)} = \frac{\partial(P',\omega')}{\partial(E',\omega')} \cdot \frac{\partial(E',\omega')}{\partial(P,\omega)} = \frac{E'P}{EP'} \cdot \frac{\partial(E',\omega')}{\partial(E,\omega)} \tag{10}$$

The Jacobian $\partial(E',\omega')/\partial(E,\omega)$ has a particularly simple form and is given by evaluating the determinant

$$\frac{\partial(E',\omega')}{\partial(E,\omega)} = \begin{vmatrix} \left(\frac{\partial E'}{\partial E}\right)_\omega & \left(\frac{\partial \omega'}{\partial E}\right)_\omega \\ \left(\frac{\partial E'}{\partial \omega}\right)_E & \left(\frac{\partial \omega'}{\partial \omega}\right)_E \end{vmatrix} \tag{11}$$

From Eq. (3) we find

$$\left(\frac{\partial E'}{\partial E}\right)_\omega = \Gamma\left(1 - \beta\omega \frac{E}{P}\right)$$

$$\left(\frac{\partial \omega'}{\partial E}\right)_\omega = \frac{\Gamma}{P'}\left(\omega \frac{E}{P} - \beta - \omega' \frac{E'}{P'} + \beta\omega\omega' \frac{EE'}{PP'}\right)$$

$$\left(\frac{\partial E'}{\partial \omega}\right)_E = - \Gamma\beta P$$

$$\left(\frac{\partial \omega'}{\partial \omega}\right)_E = \frac{\Gamma P}{P'}\left(1 + \beta\omega' \frac{E'}{P'}\right)$$

so that the Jacobian of Eq. (11) is

$$\frac{\partial(E',\omega')}{\partial(E,\omega)} = \frac{P}{P'} = \frac{\sin\theta'}{\sin\theta} \tag{12}$$

This last form is obtained using Eq. (2). Substitution of (12) into (8), (9), and (10) gives[2]

$$\frac{\partial(P',\omega')}{\partial(E,\omega)} = \frac{E'}{P'} \cdot \frac{\sin\theta'}{\sin\theta} \tag{13}$$

$$\frac{\partial(E',\omega')}{\partial(P,\omega)} = \frac{P}{E} \cdot \frac{\sin\theta'}{\sin\theta} \tag{14}$$

$$\frac{\partial(P',\omega')}{\partial(P,\omega)} = \frac{E'}{E} \cdot \frac{\sin^2\theta'}{\sin^2\theta} \tag{15}$$

It is important to note that in Eq. (5) we wrote $dE'\,d\omega'$ and $dE\,d\omega$ rather than $|dE'd\omega'|$ and $|dE\,d\omega|$ as suggested by the usual particle conservation statement. If the Lorentz tranformation reveals that more than one point in the E'-ω' plane corresponds to a given point in the $E - \omega$ plane, the particle conservation statement properly reads

$$W(E,\omega) = \sum_i W'(E_i',\omega_i') \left| \frac{\partial(E_i',\omega_i')}{\partial(E,\omega)} \right| \tag{16}$$

where the summation is over all points (E_i', ω_i') in the E'-ω' plane which correspond to the point (E,ω) in the E-ω plane. It is possible to make meaningful measurements only if the relationship between (E,ω) and (E',ω') is single-valued. Consequently, criteria for the existence of a multivalued relationship must be developed.

The Lorentz tranformation equations may be used to construct curves in the E-ω plane of constant E' for several values of E'. If any of these curves touch or cross one another, then it is clear that more than one point in the E'-ω' plane corresponds to a point in the E-ω plane. The same is true for curves of constant ω'. In fact, if these constant E' (or ω') curves are to cross at all, the curves $E' =$ constant and $E' + dE' =$ constant will cross at some point in the diagram. Let $E(E',\omega')$ and $\omega(E',\omega')$ be determined by the Lorentz transformation. Arbitrary small changes in E' and ω' are then made and we ask under what circumstances we are led to the same values of E and ω.

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2. Eq. (12) was previously obtained by Panofsky (private communication) and Eq. (14) is given by Behr and Hagdorn, CERN Report No. 60-20, May 1960.

$$E(E' + dE', \omega' + d\omega') = E(E',\omega') + \left(\frac{\partial E}{\partial E'}\right)_{\omega'} dE' + \left(\frac{\partial E}{\partial \omega'}\right)_{E'} d\omega'$$

$$\omega(E' + dE', \omega' + d\omega') = \omega(E',\omega') + \left(\frac{\partial \omega}{\partial E'}\right)_{\omega'} dE' + \left(\frac{\partial \omega}{\partial \omega'}\right)_{E'} d\omega' \tag{17}$$

On setting the left-hand sides of these expressions equal to $E(E',\omega')$ and $\omega(E',\omega')$, respectively, we obtain

$$0 = \left(\frac{\partial E}{\partial E'}\right)_{\omega'} dE' + \left(\frac{\partial E}{\partial \omega'}\right)_{E'} d\omega'$$

$$0 = \left(\frac{\partial \omega}{\partial E'}\right)_{\omega'} dE' + \left(\frac{\partial \omega}{\partial \omega'}\right)_{E'} d\omega' \tag{18}$$

A solution to these equations exists for arbitrary dE' and $d\omega'$ only if the determinant of the coefficients vanishes. This determinant is the Jacobian $\partial(E,\omega)/\partial(E',\omega')$. Similarly, more than one point in the E-ω plane corresponds to a point in the E'-ω' plane if $\partial(E',\omega')/\partial(E,\omega)$ vanishes. The points at which a Jacobian either vanishes or is infinite will be called critical points, and imply a multivalued relationship between (E,ω) and (E',ω').

As an example, the Lorentz transformation for $\beta = \sqrt{3/2}$ is plotted in Fig. 1 and shows how points in the E'-θ' plane transform to the E-θ plane. Kinetic energies rather than total energies are shown in units of the rest energy Mc^2. At the point $(E_{kc}/Mc^2 = 1, \theta = 0)$, the Jacobian $\partial(E,\omega)/\partial(E',\omega')$ vanishes. As may be seen from Fig. 1, this point correspons to $E'_k = 0$ and all values of θ'. According to Eq. (12), $\partial(E,\omega)/\partial(E',\omega')$ is zero since P' is zero at this point. Other critical points form a locus and are located at $E_k/Mc^2 = 0$ for all values of θ. In this case, $\partial(E',\omega')/\partial(E,\omega)$ is zero in that P vanishes.

This discussion shows that Eq. (7) is valid in all regions of the E-ω plane except at the critical points mentioned in the preceding paragraph, and that except at these points the mapping of the E'-ω' plane onto the E-ω plane is a one-to-one mapping. The practical significance of this result is that particle yields measured in the laboratory at some specific values of (E,ω) correspond to the yields at specific values of (E',ω') in the center-of-mass system.

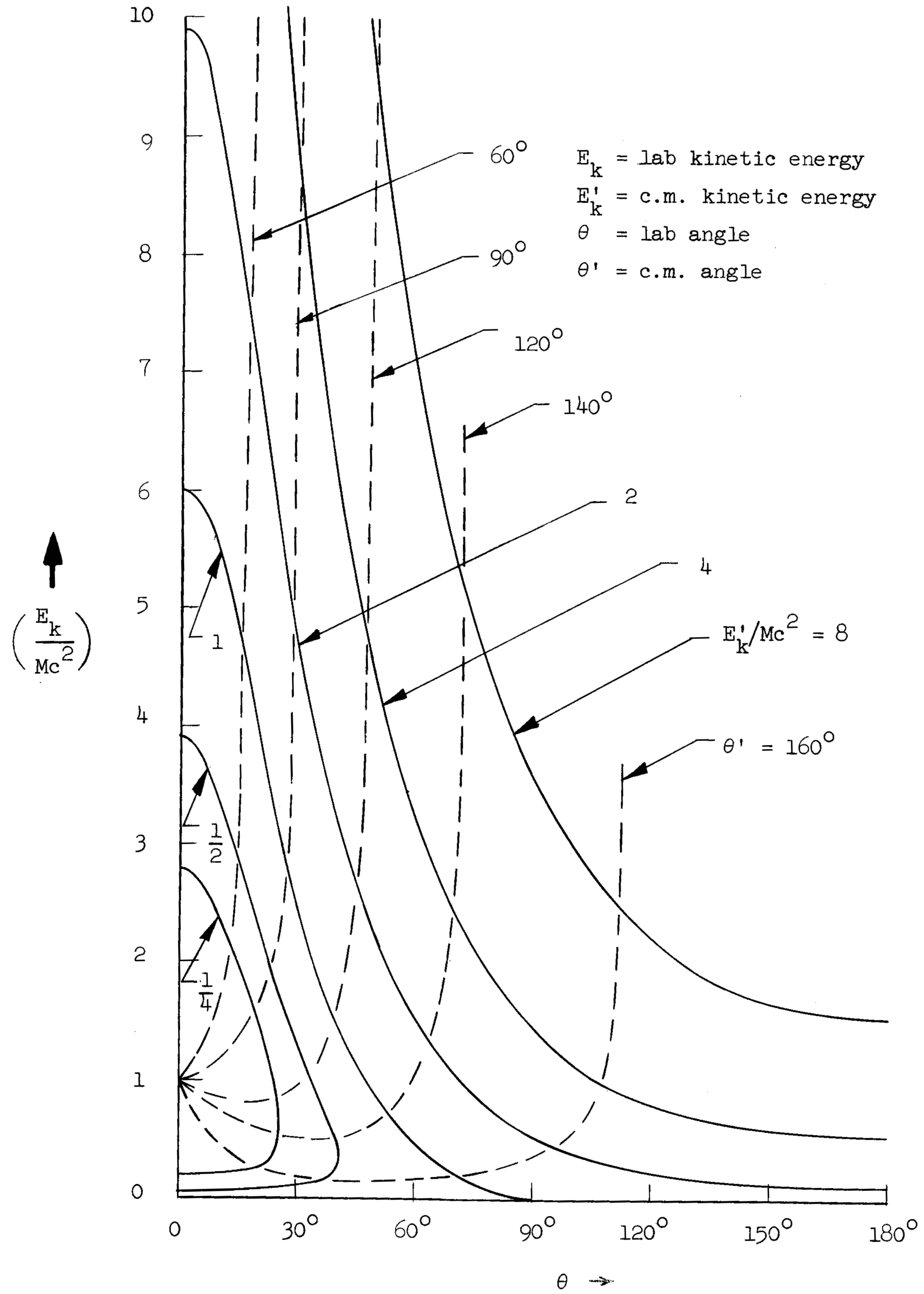

FIG. 1--Lorentz transformation for $\beta = \sqrt{3}/2$

In the next section, considerations of this nature will be seen to have more serious consequences in certain cases.

II. THE SECOND PROBLEM

Here we consider reactions of the type

$$P_1 + P_2 \rightarrow P_3 + P_4$$

where P_1 is the bombarding particle with rest energy M_1, P_2 is the Target ~~largest~~ particle with rest energy M_2, and P_3 and P_4 are the reaction products having rest energies M_3 and M_4, respectively. The kinetic energy of the bombarding particle P_1 is not chosen fixed as in Section I; we are concerned with a distribution of bombarding energies.

The center-of-mass motion depends on the energy of the bombarding particle, and for a given value of the bombarding energy, the energy of a reaction product particle in the center of mass is uniquely determined by some function of the bombarding energy. In this way, the laboratory energies and angles of the reaction products are determined by a function of the bombarding energy E_0 and the center-of-mass angle θ'. For a distribution of bombarding energies, there exists a distribution function $U(E_0, \cos\theta')$ which is the number of particles produced in the center of mass in the differential solid angle $\sin\theta'\, d\phi'\, d\theta'$ when bombarding energies between E_0 and $E_0 + dE_0$ are causing the reaction. The spectrum of either of the two reaction products in the laboratory may be described by a distribution function $W(E, \cos\theta)$. Particle conservation requires*

$$U(E_0,\omega')dE_0 d\omega' = W(E,\omega)dE\, d\omega \tag{19}$$

where:

E_0 = bombarding energy

E = laboratory energy of reaction product

$$\omega' = \cos\theta'; \; \omega = \cos\theta.$$

The Jacobian most readily calculated is

$$dE\, d\omega = \frac{\partial(E,\omega)}{\partial(E_0,\omega')}\, dE_0\, d\omega' \tag{20}$$

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*Eq. (19) is valid if the mapping from (E,ω) to (E_0,ω') is a one-to-one mapping. See the discussion of Eq. (16).

so that (19) becomes

$$W(E,\omega) = \left[\frac{\partial(E,\omega)}{\partial(E_0,\omega')}\right]^{-1} U(E_0,\omega') \tag{21}$$

In this problem, β and E' may be written as functions of E_0 alone. The Jacobian is

$$\frac{\partial(E,\omega)}{\partial(E_0,\omega')} = \begin{vmatrix} \left(\frac{\partial E}{\partial E_0}\right)_{\omega'} & \left(\frac{\partial \omega}{\partial E_0}\right)_{\omega'} \\ \left(\frac{\partial E}{\partial \omega'}\right)_{E_0} & \left(\frac{\partial \omega}{\partial \omega'}\right)_{E_0} \end{vmatrix} \tag{22}$$

This Jacobian can also be written

$$\frac{\partial(E,\omega)}{\partial(E_0,\omega')} = \left(\frac{\partial E}{\partial E_0}\right)_{\omega} \left(\frac{\partial \omega}{\partial \omega'}\right)_{E_0} \tag{23}$$

This latter expression is perhaps easier to understand physically since $(\partial\omega/\partial\omega')_{E_0}$ is the familiar solid-angle transformation factor and account is taken of the distribution of E_0 through the factor $(\partial E/\partial E_0)_{\omega}$. Evaluation of the Jacobian of Eq. (22) yields*

$$\frac{\partial(E,\omega)}{\partial(E_0,\omega')} = \frac{P'}{P}\left[\frac{dE'}{dE_0} + \Gamma^2 P'\omega' \frac{d\beta}{dE_0}\right] \tag{24}$$

This expression is valid for both reaction product particles provided proper subscripts are employed. Equation (24) for the reaction product with rest energy M_4 is then

$$\frac{\partial(E_4,\omega_4)}{\partial(E_0,\omega_4')} = \frac{P_4'}{P_4}\left[\frac{dE_4'}{dE_0} + \Gamma^2 P_4'\omega_4' \frac{d\beta}{dE_0}\right] \tag{25}$$

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*Variations of this expression may be constructed in the manner used in deriving (8), (9), and (10).

or

$$\frac{\partial(E_4,\omega_4)}{\partial(E_0,\omega_4')} = \frac{\Gamma^2 P_4'^2}{P_4}\frac{d\beta}{dE_0}\left[\frac{1}{\Gamma^2 P_4'}\left(\frac{dE_4'}{d\beta}\right) + \omega_4'\right] \tag{26}$$

In addition to the critical points at $P_4' = 0$ and $P_4 = 0$, there exists the possibility that the quantity in the square brackets in Eq. (26) is zero for certain values of ω_4'. We now write

$$\frac{1}{\Gamma^2 P_4'}\left(\frac{dE_4'}{d\beta}\right) + \omega_4' = f + \omega_4' \tag{27}$$

If the magnitude of f is less than unity, there exists a locus of critical points for the Jacobian corresponding to a range of ω_4'. It is important, then, to know how f varies with bombarding energy in all possible reactions. This is the subject for discussion in the remainder of this section.

The quantity f is most conveniently discussed when expressed as a function of E_0', the total energy available in the center of mass. In terms of E_0', f is given by

$$f(E_0') = \frac{1}{\Gamma^2 P_4'}\left(\frac{dE_4'}{d\beta}\right) = \frac{\left(E_0'^2 + \delta_2\epsilon_2\right)}{\left(E_0'^2 + \delta_1\epsilon_1\right)} \cdot \sqrt{\frac{\left(E_0'^2 - \delta_1^2\right)\left(E_0'^2 - \epsilon_1^2\right)}{\left(E_0'^2 - \delta_2^2\right)\left(E_0'^2 - \epsilon_2^2\right)}}\ ,\quad (M_2 \neq 0) \tag{28}$$

where*:

$E_0' = E_3' + E_4'$ = total energy available in the center of mass.

$\epsilon_1 = M_1 + M_2;\ \delta_1 = M_1 - M_2$

$\epsilon_2 = M_3 + M_4;\ \delta_2 = M_3 - M_4$

Now E_0' is never less than ϵ_1, and in endo-ergic reactions ϵ_2 is greater than ϵ_1, so that a reaction can proceed only if $E_0' > \epsilon_2$. Hence we are

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*This expression is not valid for $M_2 = 0$, in which case f is identically zero since E_4' is independent of β and hence $(dE_4'/d\beta)$ vanishes. See also Eq. (37).

interested in the behavior of (28) for values of E_0' greater than either ϵ_1 or ϵ_2, depending on which is the greater. Further, since $|\delta_1| \leqq \epsilon_1$ and $|\delta_2| \leqq \epsilon_2$, we see that f as given by (28) is positive for values of E_0' in the region of interest, and becomes unity as E_0' approaches infinity. The values of E_0' leading to $f^2 - 1 = 0$ are readily calculated and may be examined to see if indeed they fall in the region of interest. The details of this examination are too lengthy to be included here, but they show that in only two cases are there values of E_0' in the region of interest satisfying $f^2 - 1 = 0$. These cases are:

<u>Case 1</u>: $M_1 + M_2 > M_3 + M_4$ and $M_3 > M_1$ (implies $M_2 > M_4$)

<u>Case 2</u>: $M_3 + M_4 > M_1 + M_2$ and $M_1 > M_3$ (implies $M_4 > M_2$)

That is, unless the criteria for these cases are satisfied, either there are no critical points in the Jacobian due to the factor given by (27) or the Jacobian is critical for certain ω_4' for all E_0' in the region of interest. This latter condition obtains if $f^2 - 1 < 0$ for all E_0' in the region of interest. The value of $E_0'^2$ satisfying $f^2 - 1 = 0$ for both Cases 1 and 2 is

$$M_1 M_3 + \frac{M_2 M_4}{M_3 - M_1}\left[M_3\left(\frac{M_2}{M_4}\right) - M_1\left(\frac{M_4}{M_2}\right)\right] \tag{29}$$

and the bombarding energy E_0 giving rise to the above value of $E_0'^2$ is

$$E_b = \frac{M_1}{2M_2(M_3 - M_1)}\left[(M_3 - M_1 - M_2)^2 - M_4^2\right] \tag{30}$$

For values of E_0 greater than E_b, we find that $f^2 - 1$ is positive for Case 1 and negative for Case 2. Similarly, for E_0 less than E_b, it may be seen that $f^2 - 1$ is negative for Case 1 and positive for Case 2. Thus the Jacobian has critical points for Case 1 when the bombarding energy is less than the value given by (30), and for Case 2 when the bombarding energy is greater than the value given by (30).

III. GENERAL FEATURES OF THE LORENTZ TRANSFORMATIONS

In problems of the type discussed in Section II the characteristics of the energy-angle plots for product particles can be separated into certain classes. This classification is based on two characteristics of the transformations, viz: (a) the relation between the center-of-mass angle and the laboratory angle of a reaction product particle; (b) the behavior of the Jacobian as given by (22) or (26).

The Jacobian has been discussed in the previous section in terms of $f(E_0')$ given by Eq. (28). In a similar manner, the center-of-mass to laboratory angle transformation can be characterized by a number γ which may be expressed as a function of E_0'. The angle transformation is

$$\tan \theta_i = \frac{\sqrt{1 - \beta^2} \sin \theta_i'}{\gamma_i + \cos \theta_i'} \tag{31}$$

and is obtained by dividing Eq. (2) by Eq. (1b) and using the definition

$$\gamma_i = \frac{\beta E_i'}{P_i'} = \frac{\beta}{\beta_i'} \tag{32}$$

The subscript i denotes which of the product particles is being considered. It is shown in reference 1 that for $\gamma_i < 1$ the angle transformation gives a single-valued relation between θ_i and θ_i', and for $\gamma_i > 1$ there are usually two values of θ_i' corresponding to a given value of θ_i.

By using energy and momentum conservation in the center-of-mass system it can be shown that

$$\gamma(E_0') = \frac{\left(E_0'^2 - \delta_2\epsilon_2\right)}{\left(E_0'^2 - \delta_1\epsilon_1\right)} \sqrt{\frac{\left(E_0'^2 - \delta_1^2\right)\left(E_0'^2 - \epsilon_1^2\right)}{\left(E_0'^2 - \delta_2^2\right)\left(E_0'^2 - \epsilon_2^2\right)}}\ , \ (M_2 \neq 0) \tag{33}$$

where (33) refers to γ_4 (i.e., i = 4); the quantities δ_1, δ_2, ϵ_1, and ϵ_2 are defined below Eq. (28); and as before E_0 is the total energy available in the center-of-mass. The general features of (33) are similar to those of $f(E_0')$ of Eq. (28) and are described immediately below Eq. (28), with the exception that $\gamma^2 - 1$ is zero in the region of interest in the following special cases:

<u>Case 3</u>: $M_1 + M_2 > M_3 + M_4$ and $M_4 > M_2$ (implies $M_1 > M_3$)

<u>Case 4</u>: $M_3 + M_4 > M_1 + M_2$ and $M_2 > M_4$ (implies $M_3 > M_1$)

The value of $E_0'^2$ satisfying $\gamma^2 - 1 = 0$ for both Cases 3 and 4 is

$$M_2 M_4 + \frac{M_1 M_3}{M_4 - M_2} \left[M_4 \left(\frac{M_1}{M_3} \right) - M_2 \left(\frac{M_3}{M_1} \right) \right] \qquad (34)$$

The bombarding energy E_0 giving rise to the above value of $E_0'^2$ is

$$E_a = \frac{1}{2(M_4 - M_2)} \left[(M_1 + M_2 - M_4)^2 - M_3^2 \right] \qquad (35)$$

For values of E_0 greater than E_a, we find that $\gamma^2 - 1$ is positive for Case 3 and negative for Case 4. Similarly, for E_0 less than E_a, it may be seen that $\gamma^2 - 1$ is negative for Case 3 and positive for Case 4.

The general features of these Lorentz tranformations for $M_2 \neq 0$ are shown in Fig. 2-4. The behavior of γ and f is shown as a function of the bombarding energy E_0, and from these graphs one may sketch the mapping of the transformations also shown in these figures. The heavy solid lines on these latter plots correspond to large values of E_0, and the light lines correspondingly denote small values of E_0. In endo-ergic reactions (See Fig. 4) the bombarding-energy threshold is given by

$$E_{th} = \frac{1}{2M_2} \left[(M_3 + M_4)^2 - (M_1 + M_2)^2 \right] \qquad (36)$$

The energies E_a and E_b referred to in Figs. 3 and 4 are given by (35) and (30), respectively.

In the event that $M_2 = 0$, such as occurs in decay-in-flight reactions, the formulae for f and γ of (28) and (33) are meaningless and special relations must be developed. These are:

$$\text{For } M_2 = 0 \begin{cases} f = 0 \\ \gamma = \dfrac{M_1^2 - \delta_2\epsilon_2}{\sqrt{(M_1^2 - \delta_2^2)(M_1^2 - \epsilon_2^2)}} \cdot \dfrac{\sqrt{E_0^2 + 2M_1E_0}}{E_0 + M_1} \end{cases} \tag{37}$$

In the limit $E_0 \to \infty$, it is easy to show that γ is greater than or equal to unity, so that the angle relationship becomes double-valued for $E_0 > E_c$, where

$$E_c = \frac{1}{2M_4}\left[(M_1 - M_4)^2 - M_3^2\right] \tag{38}$$

The mapping of the transformation and the quantities f and γ are sketched in Fig. 5.

The graphs in Figs. 2-5 are sketches which have been drawn to point out various features of the transformation equations. In some cases the details shown in the mappings of the transformations may be found to be of little consequence in the design of experiments. As an example, consider the reaction

$$\gamma + p \to n + \pi^+$$

where we choose

$$M_1 = 0$$

$$M_2 = 938.20 \text{ Mev}$$

$$M_3 = 939.49 \text{ Mev}$$

$$M_4 = 139.63 \text{ Mev}$$

In this reaction, $M_1 + M_2 < M_3 + M_4$ and $M_2 > M_4$ so that conditions for the last case shown in Fig. 4 are satisfied. Equations (35) and (36) yield in this example

$$E_a = 153.35 \text{ Mev}$$

$$E_{th} = 151.50 \text{ Mev}$$

This shows that the angle relation for the pi-meson is double-valued only for incident photon energies between these values of E_a and E_{th}.

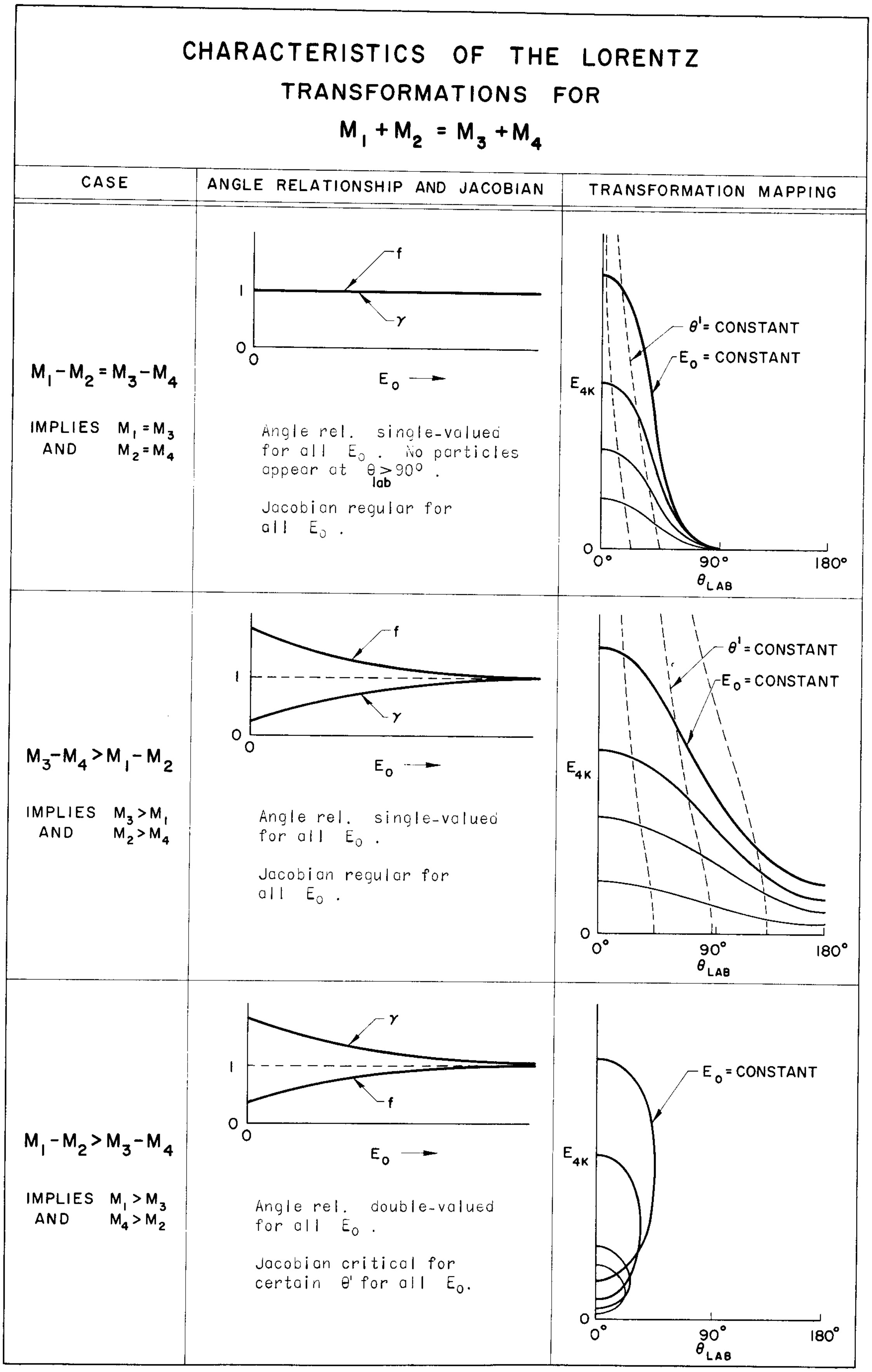

CHARACTERISTICS OF THE LORENTZ TRANSFORMATIONS FOR
$M_1 + M_2 = M_3 + M_4$
CASE
ANGLE RELATIONSHIP AND JACOBIAN
TRANSFORMATION MAPPING
$M_1 - M_2 = M_3 - M_4$
IMPLIES $M_1 = M_3$ AND $M_2 = M_4$
f
γ
1
0
0
E_0
Angle rel. single-valued for all E_0. No particles appear at $\theta_{lab} > 90°$.
Jacobian regular for all E_0.
θ' = CONSTANT
E_0 = CONSTANT
E_{4K}
0°
90°
180°
θ_{LAB}
$M_3 - M_4 > M_1 - M_2$
IMPLIES $M_3 > M_1$ AND $M_2 > M_4$
Angle rel. single-valued for all E_0.
Jacobian regular for all E_0.
$M_1 - M_2 > M_3 - M_4$
IMPLIES $M_1 > M_3$ AND $M_4 > M_2$
Angle rel. double-valued for all E_0.
Jacobian critical for certain θ' for all E_0.

Figure 2

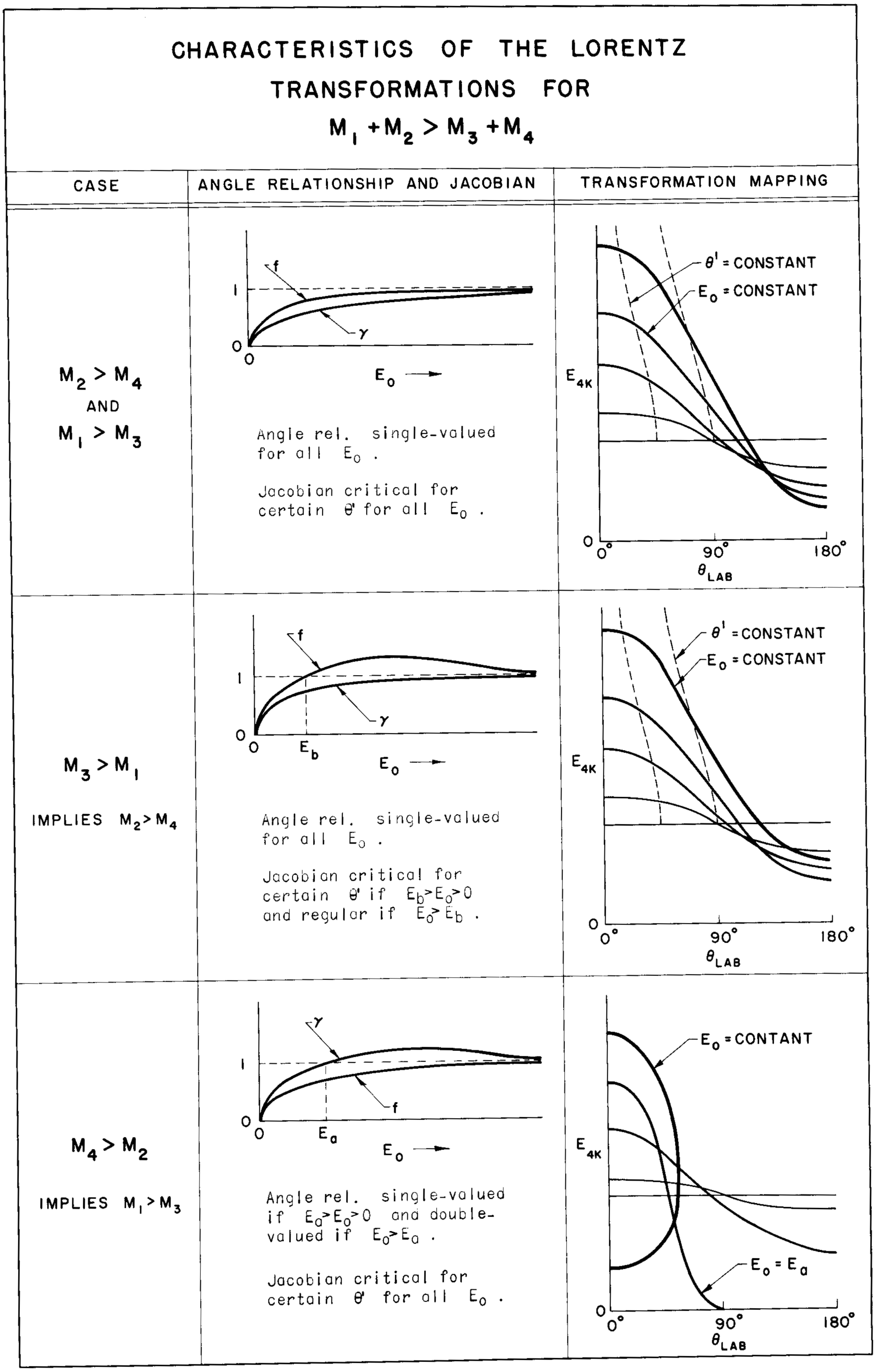

CHARACTERISTICS OF THE LORENTZ TRANSFORMATIONS FOR
$M_1 + M_2 > M_3 + M_4$
CASE
ANGLE RELATIONSHIP AND JACOBIAN
TRANSFORMATION MAPPING
$M_2 > M_4$ AND $M_1 > M_3$
f
γ
1
0
0
E_0
Angle rel. single-valued for all E_0.
Jacobian critical for certain θ' for all E_0.
θ' = CONSTANT
E_0 = CONSTANT
E_{4K}
0
0°
90°
180°
θ_{LAB}
$M_3 > M_1$
IMPLIES $M_2 > M_4$
f
γ
1
0
0
E_b
E_0
Angle rel. single-valued for all E_0.
Jacobian critical for certain θ' if $E_b > E_0 > 0$ and regular if $E_0 > E_b$.
θ' = CONSTANT
E_0 = CONSTANT
E_{4K}
0
0°
90°
180°
θ_{LAB}
$M_4 > M_2$
IMPLIES $M_1 > M_3$
γ
f
1
0
0
E_a
E_0
Angle rel. single-valued if $E_a > E_0 > 0$ and double-valued if $E_0 > E_a$.
Jacobian critical for certain θ' for all E_0.
E_0 = CONTANT
$E_0 = E_a$
E_{4K}
0
0°
90°
180°
θ_{LAB}

Figure 3

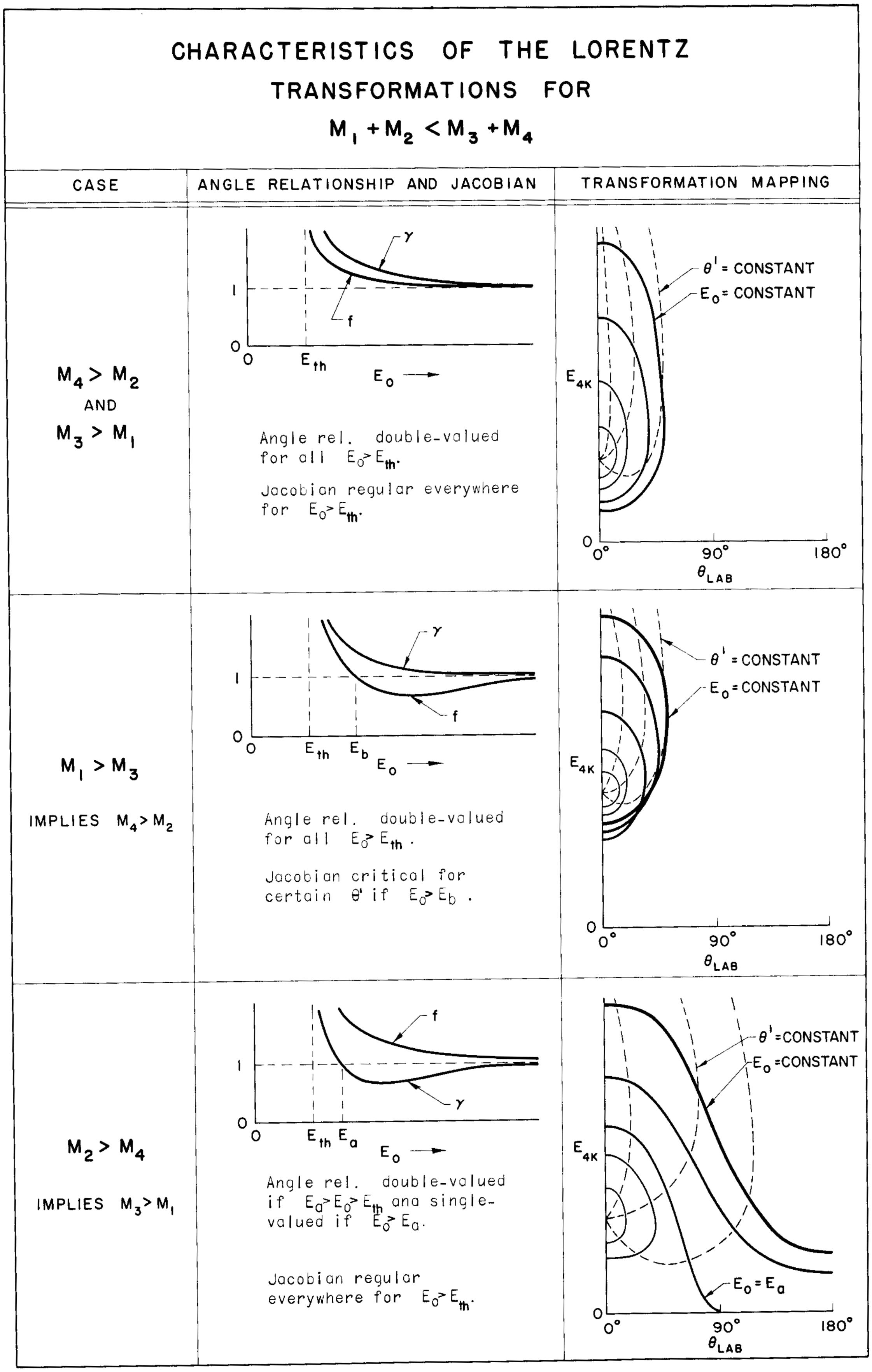

CHARACTERISTICS OF THE LORENTZ TRANSFORMATIONS FOR
$M_1 + M_2 < M_3 + M_4$
CASE
ANGLE RELATIONSHIP AND JACOBIAN
TRANSFORMATION MAPPING
$M_4 > M_2$ AND $M_3 > M_1$
γ
f
1
0
0
E_{th}
E_0 →
Angle rel. double-valued for all $E_0 > E_{th}$.
Jacobian regular everywhere for $E_0 > E_{th}$.
θ' = CONSTANT
E_0 = CONSTANT
E_{4K}
0°
90°
180°
θ_{LAB}
$M_1 > M_3$
IMPLIES $M_4 > M_2$
E_b
Angle rel. double-valued for all $E_0 > E_{th}$.
Jacobian critical for certain θ' if $E_0 > E_b$.
$M_2 > M_4$
IMPLIES $M_3 > M_1$
E_a
Angle rel. double-valued if $E_a > E_0 > E_{th}$ and single-valued if $E_0 > E_a$.
Jacobian regular everywhere for $E_0 > E_{th}$.
$E_0 = E_a$

Figure 4

CHARACTERISTICS OF THE LORENTZ TRANSFORMATIONS FOR $M_2 = 0$

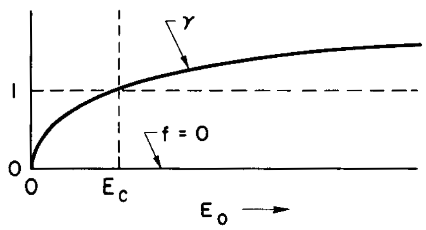
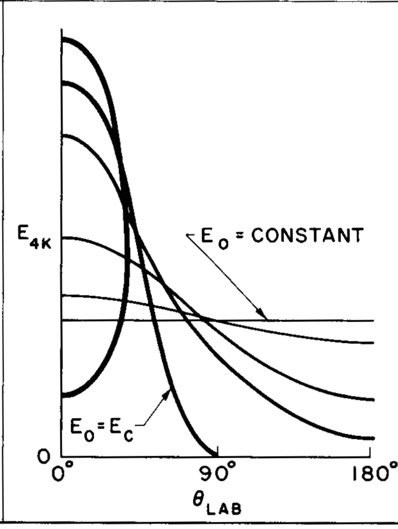
CASE	ANGLE RELATIONSHIP AND JACOBIAN	TRANSFORMATION MAPPING
$M_1 > M_3 + M_4$ (DECAY IN FLIGHT)	 Angle rel. single-valued if $E_C > E_0 > 0$ and double-valued if $E_0 > E_C$. Jacobian critical for $\theta' = 90^\circ$ at all values of E_0 .	

Figure 5

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