COMPARISON OF THE CONSTANT GRADIENT
and
UNIFORM ACCELERATOR STRUCTURES

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I. Introduction

The constant gradient and the uniform accelerator structures have previously been compared[1,2] from several points of view. In this report, the earlier results will be summarized and the structures will be further compared on the basis of frequency sensitivity and beam loading derivative (dV/di). The general case resulting in constant gradient at any electron beam current was considered in reference 2. In this report, we will confine attention to the particular case where the constant gradient condition is obtained when $i = 0$. This is probably the most useful case in practice.

II. Ratio of Peak to Average Electric Fields

The ratio of peak to average axial electric field strength is unity in the c.g.[3] accelerator while it may be as high as 1.76[4] in the uniform accelerator. Thus, it is clear that the c.g. accelerator can produce electron energies up to 1.76 times as high as an optimized uniform structure when both are operating at the breakdown limit of electric field strength. The relative advantage of the c.g. accelerator in achieving high average gradients without breakdown depends upon the value of the attenuation parameter,[5] $\tau = (\omega t_F/2Q)$. This is shown in

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[1] R.B. Neal, M.L.Report No. 185, Appendix A, February 1953.

[2] R.B. Neal, M.L. Report No. 513, May 1958.

[3] The abbreviation "c.g." will be used for "constant gradient" in the remainder of this report. The sub-script "U" is used in those equations pertaining to the uniform structure.

[4] This magnitude corresponds to a value of the rf attenuation constant $\tau = 1.26$ nepers which gives maximum no-load energy in the uniform accelerator structure.

[5] This parameter has been called $I\ell$ in several previous reports, where I is the rf attenuation in nepers per unit length and ℓ is the length of the accelerator section. The dependence of τ upon filling time t_F is emphasized in this report and in ref. 2 to call attention to the importance of comparing the c.g. and uniform structures at the same value of τ for each. For equal τ, the two structures will have the same filling times, the same stored energies, and the same ratios of input to output rf powers.

Fig. 1 where the ratio of peak to average electric fields $= E_0\ell/V_0$ is plotted versus τ. The equations of these curves are:

$$\left.\frac{E_0\ell}{V_0}\right)_U = \frac{\tau}{1 - e^{-\tau}} \tag{1}$$

$$\left.\frac{E_0\ell}{V_0}\right)_{c.g.} = 1 \tag{2}$$

III. <u>RF POWER DISSIPATION PER UNIT LENGTH</u>

The rf power dissipated per unit length in the walls of the uniform structure is given by:[6]

$$\left.\frac{dP}{dz}\right)_U = -\frac{2\tau P_0}{\ell}\, e^{-2\tau z/\ell} \tag{3}$$

so that the ratio of power loss at the input end to that at the output end of an accelerator section is given by

$$\frac{\left.\frac{dP}{dz}\right)_{z=0}}{\left.\frac{dP}{dz}\right)_{z=\ell}} = e^{2\tau} \tag{4}$$

Thus, this ratio may be as high[4] as 12.4. On the other hand, the power dissipated per unit length in the c.g. accelerator is constant over the entire length of the structure.[2] This means that the temperature rise in c.g. structure can be perfectly compensated by a simple frequency adjustment of the rf power source, thus preventing phase shift between the electrons and the wave. Such an adjustment will usually be imperfect in the uniform structure case leading to phase shift and loss of beam energy. The power loss ratios for the two structures are shown in Fig. 2 vs τ.

[6] This is derived in a number of reports. See, for example, R. B. Neal, JAP <u>29</u>, 1019 (1958)

IV. ELECTRON ENERGY AND BEAM LOADING DERIVATIVE, $\frac{dV}{di}$

For the uniform accelerator structure, the electron energy is given by[6]

$$V_U = \left(2\tau\right)^{1/2} \frac{1-e^{-\tau}}{\tau} \left(P_O \ell r\right)^{1/2} - ir\ell \left(1 - \frac{1-e^{-\tau}}{\tau}\right) \tag{5}$$

Thus, the beam loading derivative is

$$\left.\frac{dV}{di}\right)_U = -r\ell \left(1 - \frac{1-e^{-\tau}}{\tau}\right) \tag{6}$$

Similarly, the electron energy for the constant gradient structure is given by[2]

$$V_{c.g.} = \left(1 - e^{-2\tau}\right)^{1/2} \left(P_O \ell r\right)^{1/2} - \frac{ir\ell}{2} \left(1 - \frac{2\tau e^{-2\tau}}{1-e^{-2\tau}}\right) \tag{7}$$

so that the beam loading derivative is

$$\left.\frac{dV}{di}\right)_{c.g.} = -\frac{r\ell}{2} \left(1 - \frac{2\tau e^{-2\tau}}{1 - e^{-2\tau}}\right) \tag{8}$$

The no-load energies, V_O, for the two structures are plotted versus τ in Fig. 3. The beam loading derivatives are shown in Fig.4. For a given value of τ, the constant gradient structure has a higher no-load energy and a lower beam loading derivative than the uniform structure. Thus, the c.g. structure has greater relative energy advantage in the loaded case than in the unloaded case.

V. MAXIMUM CONVERSION EFFICIENCY

The conversion efficiency η is defined as the ratio of power in the electron beam to the input rf power. Thus,

$$\eta = \frac{Vi}{P_O} \tag{9}$$

For the uniform accelerator the conversion efficiency is given by[6]

$$\eta_U = \tau\left(\frac{2r\ell}{\tau P_0}\right)^{1/2} i\left[\frac{1-e^{-\tau}}{\tau} - \frac{i}{2}\left(\frac{2r\ell}{\tau P_0}\right)^{1/2}\left(1 - \frac{1-e^{-\tau}}{\tau}\right)\right] \tag{10}$$

Differentiating eq. (10) with respect to i, we find the current i which gives maximum η to be

$$i_{\eta_{max}\,(U)} = \left(\frac{\tau P_0}{2r\ell}\right)^{1/2} \frac{1-e^{-\tau}}{\tau - (1-e^{-\tau})} \tag{11}$$

Inserting the value of i from eq. (11) into eq. (10) we obtain for the maximum conversion efficiency:

$$\eta_{max\,(U)} = \frac{1}{2}\,\frac{(1-e^{-\tau})^2}{\tau - (1-e^{-\tau})} \tag{12}$$

Similarly, the conversion efficiency for the c.g. accelerator is given by[2]

$$\eta_{c.g.} = \left(\frac{r\ell}{P_0}\right)^{1/2}\left(1-e^{-2\tau}\right)^{1/2} i\left[1 - \frac{i}{2}\left(\frac{r\ell}{P_0}\right)^{1/2} \frac{(1-e^{-2\tau}) - 2\tau e^{-2\tau}}{(1-e^{-2\tau})^{3/2}}\right] \tag{13}$$

This is maximum when i is given by

$$i_{\eta_{max}(c.g.)} \left(\frac{P_0}{r\ell}\right)^{1/2} \frac{(1-e^{-2\tau})^{3/2}}{(1-e^{-2\tau}) - 2\tau e^{-2\tau}} \tag{14}$$

Inserting eq. (14) in eq. (13) we obtain for the maximum conversion efficiency

$$\eta_{max(c.g.)} = \frac{1}{2}\,\frac{(1-e^{-2\tau})^2}{(1-e^{-2\tau}) - 2\tau e^{-2\tau}} \tag{15}$$

The maximum conversion efficiencies for the two structures and the corresponding values of $i_{\eta_{max}}$ are shown in Fig. 5 vs τ.

VI. ELECTRON ENERGY IN TERMS OF $i_{\eta_{max}}$

From eqs. (5) and (11) and eqs. (7) and (14) we find that the electron energies from both the uniform and the c.g. accelerators can be written

$$V = V_0\left(1 - \frac{1}{2}\,\frac{i}{i_{\eta_{max}}}\right) \tag{16}$$

where V_O is the no-load energy given by the first term on the right in eq. (5) for the uniform accelerator and the first term in the right in eq. (7) for the c.g. accelerator and $i_{\eta_{max}}$ is given by eqs. (11) and (14) for the uniform and c.g. structures, respectively.

From eq. (16) it is clear that in each of the two structures, when the beam loading is sufficient to result in maximum conversion efficiency, the electron energy is reduced to one-half of the no-load energy.

VII. GROUP VELOCITY AND FILLING TIME

The group velocity v_g is a constant over the entire length of the uniform accelerator structure. Thus, the filling time t_F is given by

$$t_{F(U)} = \frac{\ell}{v_g} = \frac{2Q}{\omega} \tau \tag{17}$$

where ℓ is the length of the accelerator section. τ is the rf attenuation over the length of the section and is therefore equal to

$$\tau_U = \frac{1}{2} \ln\left(\frac{P_O}{P_\ell}\right) \tag{18}$$

so that the filling time may also be written

$$t_{F(U)} = \frac{Q}{\omega} \ln\left(\frac{P_O}{P_\ell}\right) \tag{19}$$

In the c.g. accelerator, the group velocity varies linearly over the accelerator length and is given by[2]

$$v_{g(c.g.)} = \frac{\omega\ell}{Q} \frac{1 - \frac{z}{\ell}\left(1 - e^{-2\tau}\right)}{\left(1 - e^{-2\tau}\right)} \tag{20}$$

The incremental time dt required for the rf wave to move through the distance dz is given by

$$dt = \frac{dz}{v_g} \tag{21}$$

Inserting the value of v_g from eq. (20) into eq. (21) and integrating from $z = 0$ to $z = \ell$, we obtain for the filling time

$$t_{F(c.g.)} = \frac{2Q}{\omega} \tau = \frac{Q}{\omega} \ln\left(\frac{P_O}{P_\ell}\right) \tag{22}$$

Thus, the uniform structure and the c.g. structure having the same value of τ have equal filling times.

VIII. STORED ENERGY

The stored energy per unit length in the uniform structure is given by

$$u_{(U)} = \frac{P}{v_g} = \frac{P_0 e^{-2\tau z/\ell}}{v_g} \tag{23}$$

Integrating over the accelerator length ℓ, the total stored energy is

$$U_{(U)} = P_0 t_F \left(\frac{1 - e^{-2\tau}}{2\tau}\right) \tag{24}$$

In the c.g. structure, the stored energy per unit length is

$$u_{(c.g.)} = \frac{P}{v_g} = \frac{P_0\left[1 - \frac{z}{\ell}\left(1 - e^{-2\tau}\right)\right]}{\frac{\omega\ell}{Q}\,\frac{1 - \frac{z}{\ell}\left(1 - e^{-2\tau}\right)}{1 - e^{-2\tau}}}$$

$$= \frac{P_0 Q}{\omega\ell}\left(1 - e^{-2\tau}\right) \tag{25}$$

The stored energy per unit length in the c.g. accelerator is seen in eq. (25) to be constant over the entire length. Thus, the total stored energy is

$$U_{c.g.} = \frac{P_0 Q}{\omega}\left(1 - e^{-2\tau}\right) = P_0 t_F \left(\frac{1 - e^{-2\tau}}{2\tau}\right) \tag{26}$$

which is observed to be identical to the stored energy in the uniform structure with the same value of τ.

IX. FREQUENCY SENSITIVITY

By frequency sensitivity, we refer to the fractional loss of electron energy $\frac{\delta V_0}{V_0}$ caused by a fractional change in frequency $\delta f/f$ of the input radiofrequency power from the frequency giving phase synchronism of the rf wave and the electron beam.

The frequency sensitivity of the uniform structure has been previously calculated[7] and is given by:

$$\left(\frac{\delta V_O}{V_O}\right)_U \cong \left(Q \frac{\delta f}{f}\right)^2 \left[\frac{2\tau\ (\tau + 2)}{e^{\tau} - 1} - 4\right] \tag{27}$$

The frequency sensitivity of the c.g. structure will now be determined using the results of reference 2. The electron energy in the negligible beam loading case is gi... by:

$$V_{O(c.g.)} = \left(1 - e^{-2\tau}\right)^{1/2} \left(P_O \ell r\right)^{1/2} \tag{28}$$

where $\tau = \frac{\omega t_F}{2Q}$ as before, t_F is the filling time of the structure,

P_O is the input rf power, ℓ is the length of the section, and r is the shunt impedance per unit length. The incremental phase shift between the electrons and the wave is given by[7]

$$d\Delta = \frac{2\pi}{\beta_e} \left(\beta_W \frac{c}{v_g} - 1\right) \frac{\delta f}{f} \frac{dz}{\lambda} \tag{29}$$

Inserting the value of c/v_g given in references 1 and 2

$$\frac{c}{v_g} = \frac{Q\lambda}{2\pi\ell} \frac{1 - e^{-2\tau}}{1 - \frac{z}{\ell}\ (1 - e^{-2\tau})} \tag{30}$$

into eq. (29) and integrating between the limits of 0 and z we obtain for the total phase shift (when $\beta_W = \beta_e = 1$ and $c/v_g \gg 1$)

$$\Delta_{c.g.} \cong Q \frac{\delta f}{f} \ \ell n \left[1 - \frac{z}{\ell}\left(1 - e^{-2\tau}\right)\right] \tag{31}$$

The electron energy when there is no phase shift in the structure can be written:

$$V_{c.g.} = \int_0^{\ell} E dz = E\ell \tag{32}$$

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[7] R.B. Neal, M.L. Report No. 185, pp. 55-57, February 195...

Similarly, when there is phase shift, the energy is given by

$$V'_{c.g.} = \int_0^{\ell} E \cos \Delta \; dz \tag{33}$$

which, when Δ remains sufficiently small, is closely equal to

$$V'_{c.g.} = E\int_0^{\ell} \left(1 - \frac{\Delta^2}{2}\right) dz \tag{34}$$

Thus, from eqs. (32) and (34) we may write

$$\left.\frac{\delta V_0}{V_0}\right)_{c.g.} = -\frac{1}{2\ell} \int_0^{\ell} \Delta^2 \; dz \tag{35}$$

Inserting the value of Δ from eq. (31) into (35) and integrating, we obtain

$$\left.\frac{\delta V_0}{V_0}\right)_{c.g.} = \left(Q \frac{\delta f}{f}\right)^2 \left[\frac{2\tau e^{-2\tau} \; (\tau + 1)}{1 - e^{-2\tau}} - 1\right] \tag{36}$$

For purposes of comparison, the quantities

$$\left.\frac{\delta V_0}{V_0}\right)_{U} \quad \text{and} \quad \left.\frac{\delta V_0}{V_0}\right)_{c.g.}$$

are plotted in Fig. 6. We note that the constant gradient structure has less frequency sensitivity than the uniform structure over the entire range of τ.

X. EXAMPLES.

To illustrate the topics discussed in this report, we have considered specific examples of the two structures whose parameters are quite close to those of the proposed Stanford Project M accelerator (StageI). A comparison of the various derived characteristics is shown in Table I.

XI. CONCLUSIONS

The constant gradient accelerator is equal to or superior to the uniform accelerator in all the characteristics considered in this report.

Its relative superiority depends upon the value of τ (the normalized filling time parameter), increasing as τ increases.

Table I [1]

COMPARISON OF CONSTANT GRADIENT AND UNIFORM

ACCELERATOR STRUCTURES AT $\tau = 0.6$

Characteristic	Const. Grad.	Uniform	Ratio $\left(\frac{C.G.}{U.}\right)$
$\frac{E_0 \ell}{V_0}\left(\frac{\text{Peak Elec. Field}}{\text{Aver. " "}}\right)$	1.0000	1.3298	0.752
$\left.\frac{dP}{dz}\right)_{z=0} \Big/ \left.\frac{dP}{dz}\right)_{z=\ell}$	1.0000	3.3201	0.301
V_0 (no-load energy)	11.772 Bev	11.600 Bev	1.015
$-\,dV/di$	33.242 $\frac{\text{Bev}}{\text{amp.}}$	34.155 $\frac{\text{Bev}}{\text{amp.}}$	0.973
V (at i = 50 ma)	10.110 Bev	9.892 Bev	1.022
η_{max} (Max. Beam Conversion Efficiency)	0.724	0.684	1.058
$i_{\eta_{max}}$	177.1 ma	169.8 ma	1.043
v_g/c (Normalized Group Velocity)	.0201 → .0061	0.0117	1.718 → 0.521
t_F (Filling Time)	0.872 μsec	0.872 μsec	1.000
U (Stored Energy)	731.2 joules	731.2 joules	1.000
$\Delta_{z=\ell}$ (For δf = 0.1 Mc/sec.)	0.546 rad.	0.546 rad.	1.000
$\frac{\delta V_0}{V_0}$ (For δf = 0.1 Mc/sec.)	0.0357	0.0424	0.842

[1] Assumed Parameters:

$\tau = 0.6$

L = 9600 Ft. (293,000 cm.)

P_{ot} = 1440 Mw

No. of sections = 960

Length per Section,

ℓ = 10 Ft. (305 cm.)

$r = 0.47 \times 10^6$ ohms/cm.

i = 50 milliamperes

Q = 13,000

$\omega = 1.79 \times 10^{10}$ rad./sec.

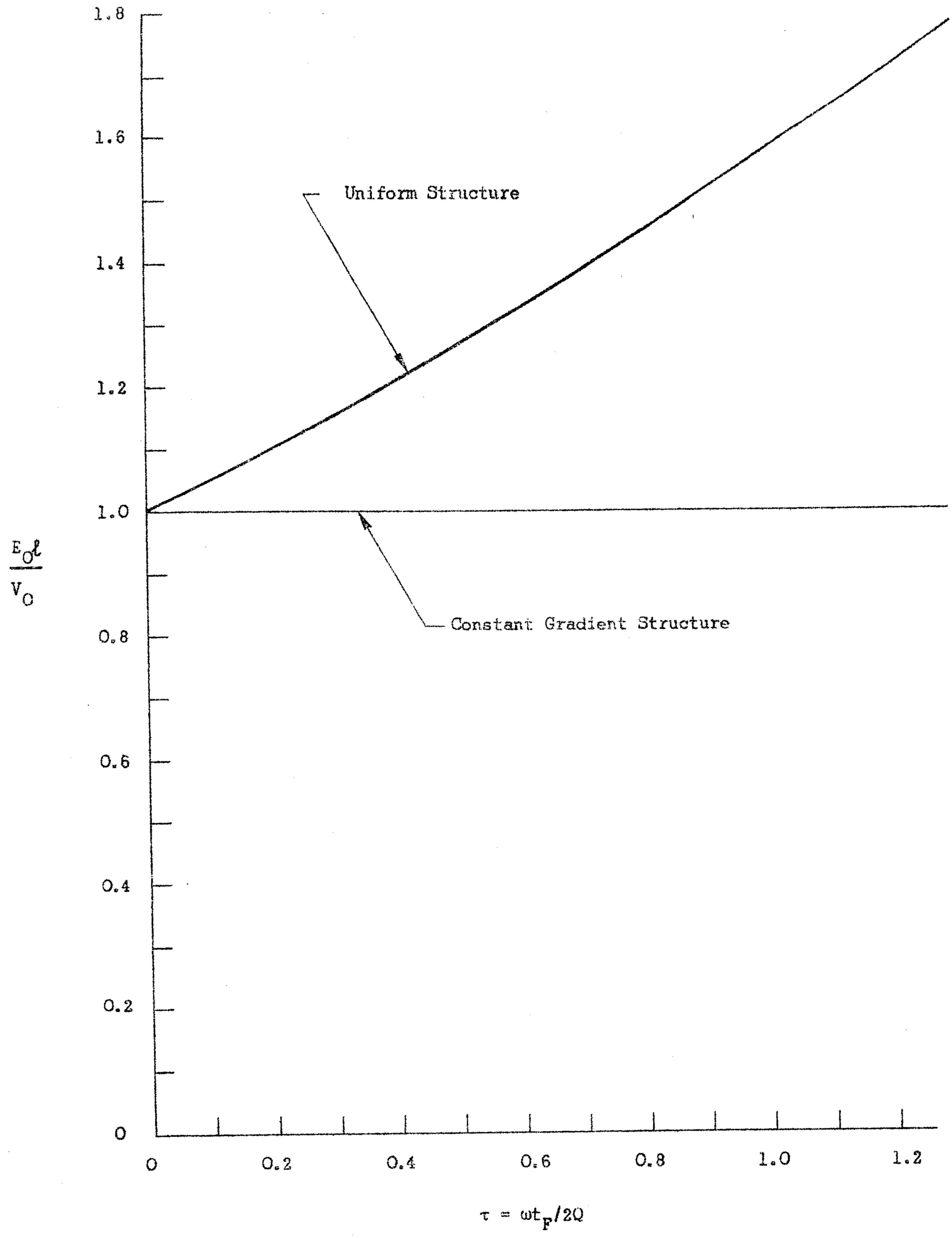

Fig. 1 Ratios of Peak to Average Axial Electric Field Strengths in Uniform and Constant Gradient Accelerator Structures vs. τ

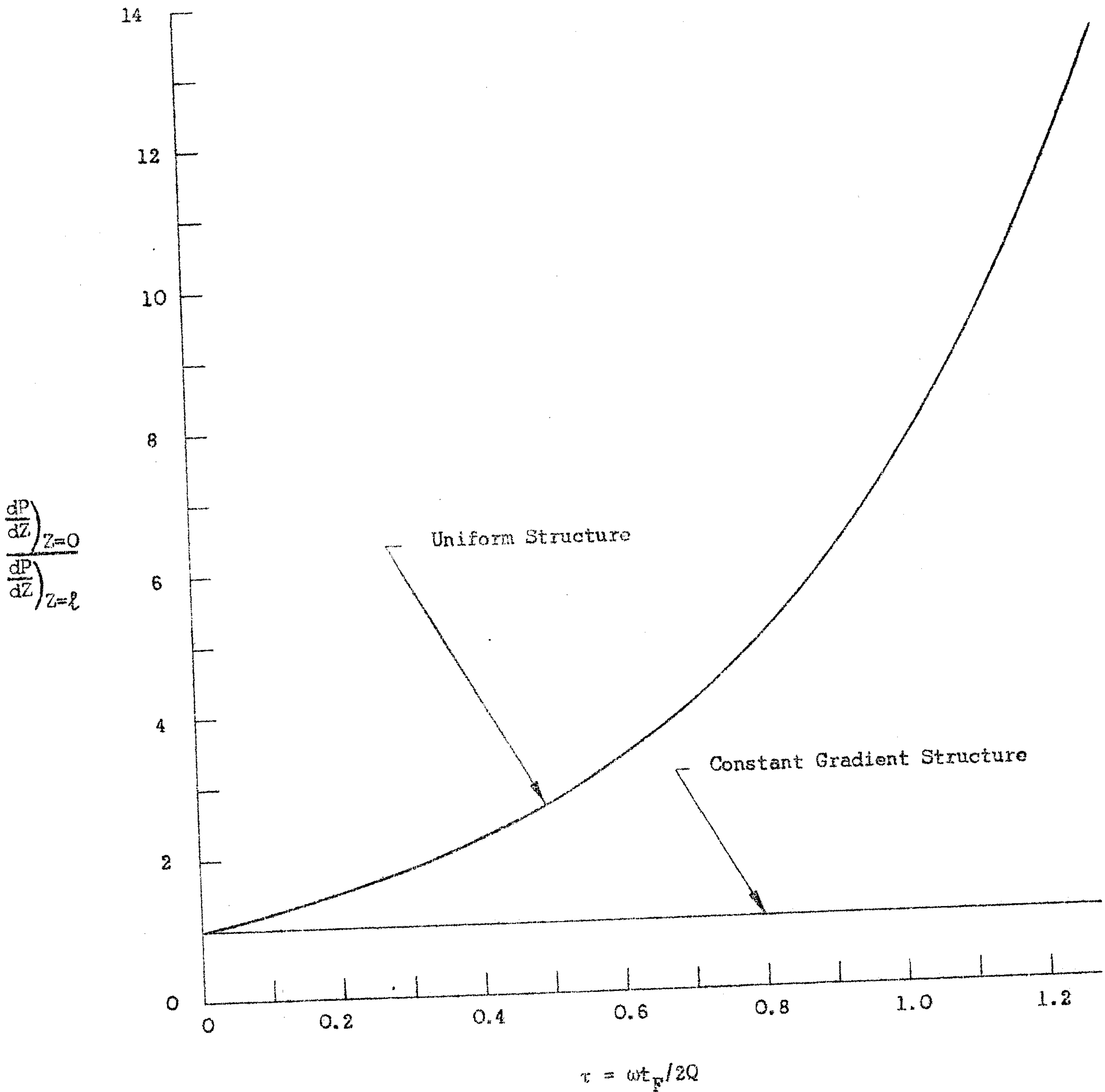

Fig. 2 Ratio of Power Losses at Input and Output Ends of Accelerator Section for Uniform and Constant Gradient Structures vs. τ

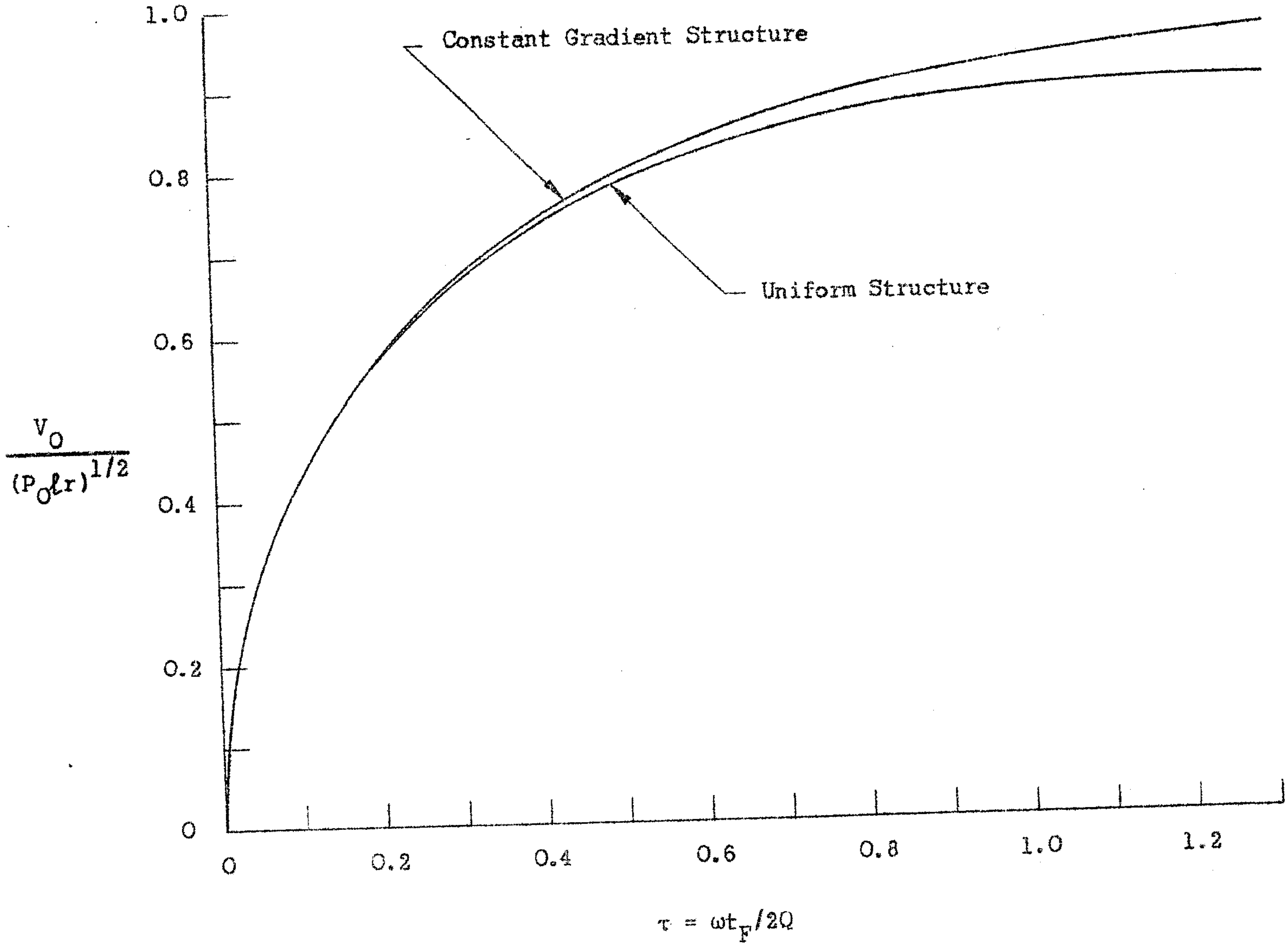

Fig. 3 Unloaded Beam Energies for Uniform and Constant Gradient Accelerator Structures vs. τ

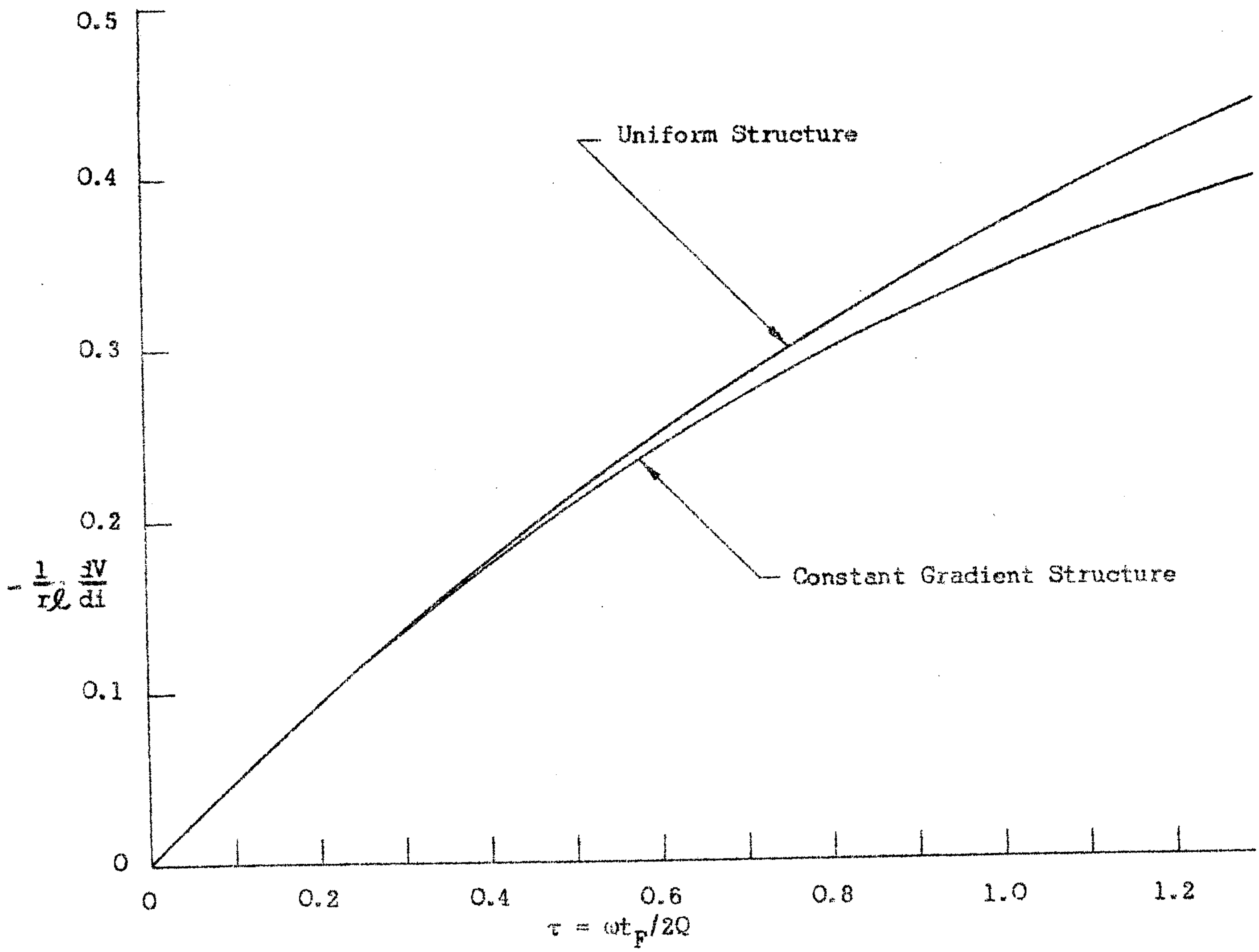

Fig.4 Beam Loading Derivatives for Uniform and Constant Gradient Accelerator Structures vs. τ

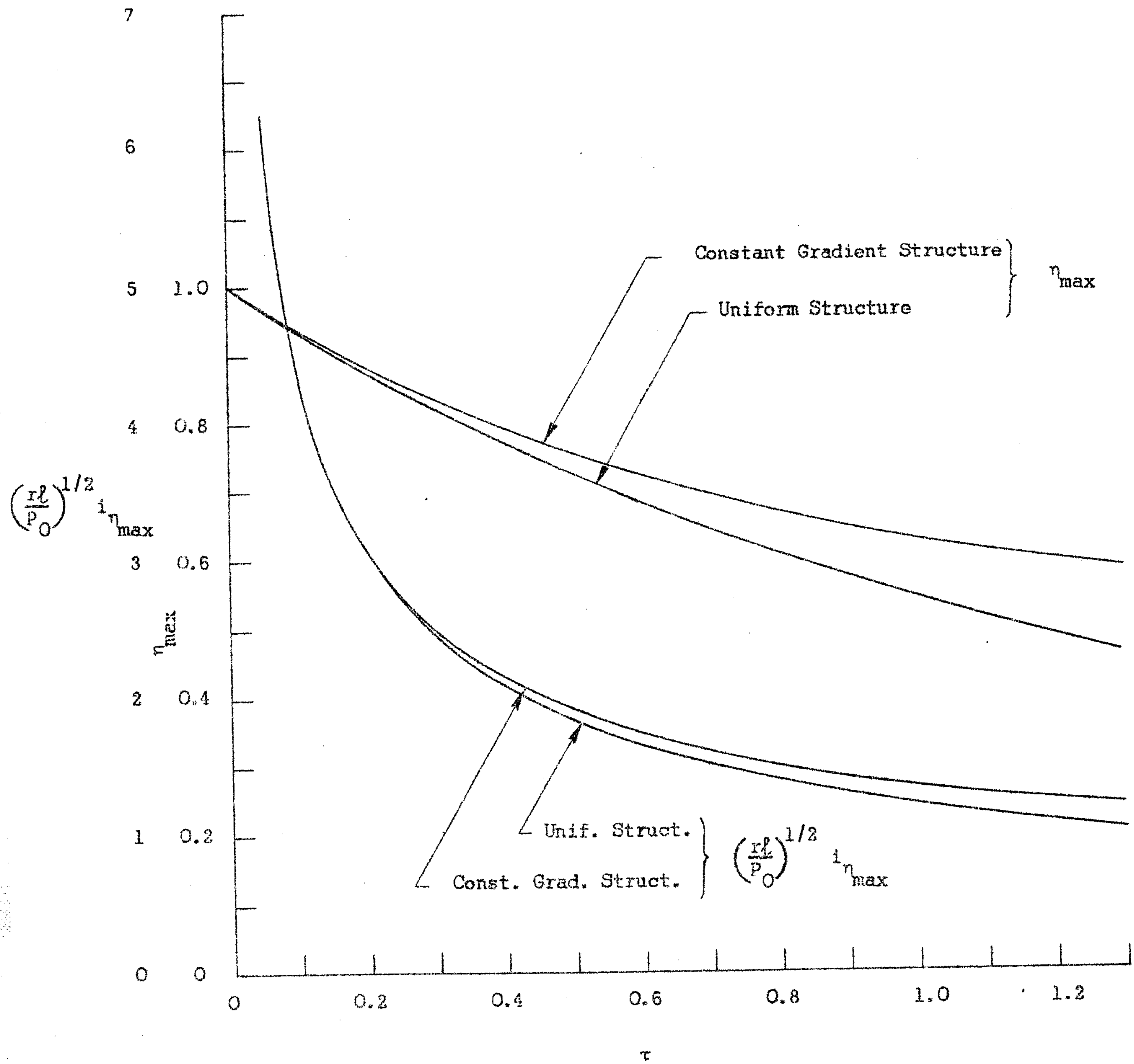

Fig.5 Maximum Beam Conversion Efficiencies and Corresponding Values of Peak Beam Current For Uniform And Constant Gradient Accelerators vs. τ

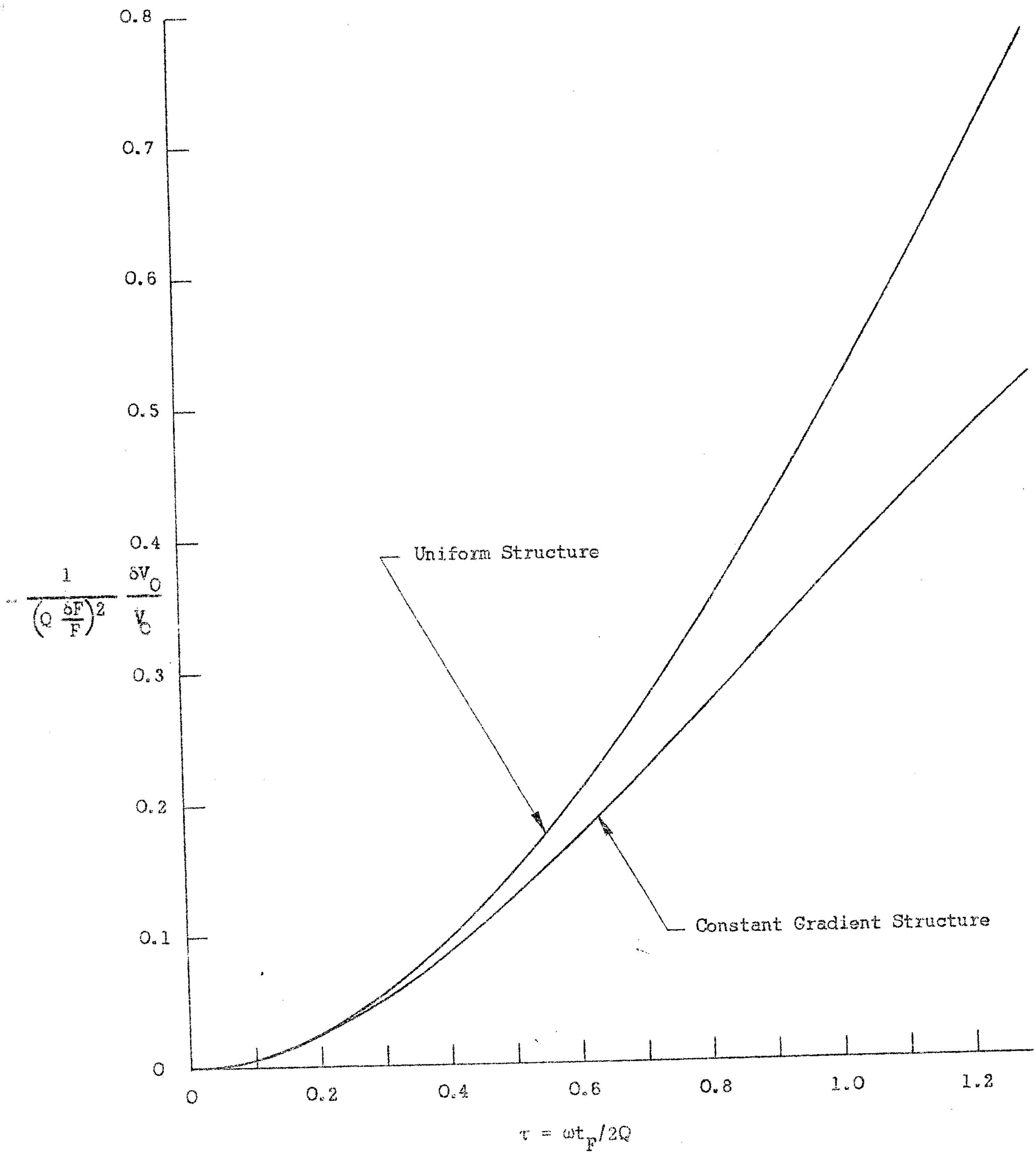

Fig. 6 Frequency Sensitivities of Uniform and Constant Gradient Accelerator Structures vs. τ