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A Note on Coupler Asymmetries in Long Linear Accelerators

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Note on Coupler Asymmetry in Long Accelerators

Assume the coupler introduces a field perturbation for which the axial electric component is given by

$$e_z = -e_3(x,y,z) \cos (wt + \varphi) \qquad (1)$$

where φ is the phase of the perturbation relative to the accelerating wave. If the (z,x) plane is the plane of symmetry of the coupler we expect that

$$\left(\frac{\partial e_3}{\partial y}\right)_{y=0} = 0$$

but $\frac{\partial e_3}{\partial x} \neq 0$.

The corresponding vector potential is

$$a_z = \frac{e_3(x,y,z)}{k} \sin (wt + \varphi) \qquad (2)$$

where $k = \frac{w}{c} = \frac{2\pi}{\lambda}$

The transverse impulse on an electron traversing the coupler region is, by the well-known deflection theorem*,

$$\Delta p_x \cong -\frac{e}{c}\int_{\text{coupler}} \frac{\partial}{\partial x}(v_z a_z)\frac{dz}{v_z} + \frac{e}{c}(a_2 - a_1)_x$$

$$\cong -\frac{e}{kc} \sin(\theta + \varphi) \int \frac{\partial e_3}{\partial x} dz \qquad (3)$$

where θ is the phase of the electron relative to the accelerating field; transit time is neglected and we assume v_z = const., $v_z \gg v_{x,y}$ and that the integration is over a region where the perturbing field vanishes at and outside the boundaries.

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* W. K. H. Panofsky & W. A. Wenzel, R.S.I. 27, 967.

In dimensionless units,

$$\Delta(\gamma\xi') = -\frac{\sin(\theta+\varphi)}{2\pi}\int \frac{\partial \alpha_3}{\partial \xi}\, d\,\zeta \tag{4}$$

where $\xi = x/\lambda$, $\zeta = z/\lambda$, $(\ )' = \frac{d}{d\zeta}$, $\alpha_3 = \frac{e\,e_3\lambda}{mc^2}$,

and $\gamma = (1-\frac{v^2}{c^2})^{-1/2} \cong$ electron energy in mc^2 units.

Equally oriented couplers. The average impulse per unit length is given by

$$\frac{d(\gamma\xi')}{d\zeta} \cong \frac{\Delta(\gamma\xi')}{\ell} \cong \epsilon\, \sin(\theta+\varphi) \tag{5}$$

where $\ell = L/\lambda$ = number of wavelengths between couplers

and $$\epsilon = -\frac{1}{2\pi\ell}\int_{\text{coupler}} \frac{\partial \alpha_3}{\partial \xi}\, d\,\zeta \tag{5a}$$

Assuming $\frac{d\gamma}{d\zeta} = \alpha\cos\theta \cong$ constant, where

$\alpha = \frac{e\,E_o\lambda}{mc^2}$, E_o = accelerating field amplitude averaged over L, equation (5) integrates to

$$\xi' = \frac{(\gamma\xi')_o}{\gamma} + \frac{\epsilon\sin(\theta+\varphi)}{\alpha\cos\theta} \tag{6a}$$

$$\xi = \xi_o + \frac{(\gamma\xi')_o}{\alpha\cos\theta}\log\frac{\gamma}{\gamma_o} + \frac{\epsilon\sin(\theta+\varphi)}{\alpha\cos\theta}\cdot(\zeta-\zeta_o) \tag{6b}$$

Thus the net effect of the coupler asymmetry is

$$\delta\,\xi' \cong \frac{\epsilon\sin(\theta+\varphi)}{\alpha\cos\theta} \cong \text{const.} \tag{6c}$$

$$\delta\,\xi \cong \frac{\epsilon\sin(\theta+\varphi)}{\alpha\cos\theta}(\zeta-\zeta_o) \tag{6d}$$

If we assume that the perturbing field is confined to a region ΔZ = one disc-spacing, then from the defining equations

$$\epsilon \cong \frac{\Delta Z}{2\pi L}\,\frac{\lambda}{E_o}\,\frac{\partial e_3}{\partial x}\,\alpha \tag{7}$$

so that $$|\delta\,\xi'| \cong \frac{\Delta Z}{2\pi L}\left(\frac{\lambda}{E_o}\frac{\partial e_3}{\partial x}\right)\frac{\sin(\theta+\varphi)}{\cos\theta} \tag{8a}$$

$$|\delta\,\xi| \cong \frac{\Delta Z}{2\pi L}\left(\frac{\lambda}{E_o}\frac{\partial e_3}{\partial x}\right)\left(\frac{\sin(\theta+\varphi)}{\cos\theta}\right)(\zeta-\zeta_o) \tag{8b}$$

Alternating Coupler Orientation. If the couplers are alternately oriented in opposite directions, we average the transverse impulse over two consecutive sections: from Equation (4),

$$\frac{d(\gamma\xi')}{d\zeta} \cong \frac{\Delta(\gamma\xi')_{\zeta,\xi} + \Delta(\gamma\xi')_{\zeta+\ell,-\xi}}{2\ell}$$

$$\cong -\frac{\sin(\theta+\varphi)}{4\pi\ell}\left\{\int\frac{\partial\alpha_3(\xi)}{\partial\xi}d\zeta - \int\frac{\partial\alpha_3(-\xi)}{\partial\xi}d\zeta\right\}$$

$$\cong -\frac{\sin(\theta+\varphi)}{2\pi\ell}\xi\int\frac{\partial^2\alpha_3}{\partial\xi^2}d\zeta$$

$$= \xi\,\epsilon'\sin(\theta+\varphi) \tag{9}$$

where $\epsilon' \equiv \frac{\partial\epsilon}{\partial\xi}$. Terms of order $\epsilon'\xi'\ell$, $\epsilon'''\xi^3$, etc., have been neglected. [ϵ is as defined by Equation (5a)]. It is evident that in this case the effect is small and can be cancelled completely to this order by addition of a weak magnetic focussing field.

"Vertical" Deflection. It has been assumed that $\left(\frac{\partial e_3}{\partial y}\right)_{y=0} = 0$ because the coupler is symmetrical about the (z,x) plane. Since e_3 may have a quadratic term in y there may be a deflection in the y direction which in analogy with Equations (3) through (5) would be given by

$$\frac{d(\gamma\eta')}{d\zeta} \cong \epsilon_2\sin(\theta+\varphi) \tag{10}$$

where $$\epsilon_2 \cong -\frac{\eta}{2\pi\ell}\int_{\text{coupler}}\frac{\partial^2\alpha_3}{\partial\eta^2}d\zeta$$

and $\eta = y/\lambda$.

Again the lowest order term is linear in the deflection and can be cancelled by a weak transverse gradient of magnetic field.

Coupler Asymmetry in the Presence of a Steering Field

Consider the case of equally oriented couplers. Assume that a steering field $B = B(z)$ is present. (Assume $\frac{\partial B}{\partial x} = \frac{\partial B}{\partial y} = 0$ for the moment). Then Equation (5) becomes

$$\frac{d(\gamma \xi')}{d\zeta} \cong \epsilon \sin(\theta + \varphi) + b \qquad (11)$$

where $b = \frac{e\lambda B_y}{mc^2}$

Note that B_y is essentially a component of the de-gaussing field and that the coupler asymmetry is equivalent to an additional vertical component of magnetic field (if $\varphi \neq 0$).

Correct steering will result when the transverse force is essentially cancelled;

$$b \cong -\epsilon_0 \sin\varphi$$

$$\cong -\frac{\Delta z}{2\pi L} \frac{\lambda}{E_0} \frac{\partial e_3}{\partial x} \alpha_0 \qquad (12)$$

where α_0 refers to the average acceleration (i.e., $\frac{\alpha - \alpha_0}{\alpha_0}$ averaged over all electrons and accelerator sections would be essentially $\frac{\Delta p_z}{p_z}$, the relative width of the beam spectrum).

Equation (12) may be re-stated:

$$B_y \cong -\frac{\Delta z}{2\pi L}\left(\frac{\lambda}{E_0}\frac{\partial e_3}{\partial x}\right) E_0 \sin\varphi \qquad (13)$$

Substituting (12) into (11),

$$\frac{d(\gamma \xi')}{d\zeta} \cong \epsilon\left[\sin(\theta+\varphi) - \sin\varphi\right] + \epsilon \frac{\Delta\alpha}{\alpha}\sin(\theta+\varphi) \qquad (14a)$$

or, for small phase angle θ

$$\frac{d(\gamma \xi')}{d\zeta} \cong \epsilon\left[\theta\cos\varphi + \frac{\Delta\alpha}{\alpha}\sin\varphi\right]$$

$$= \epsilon_r \theta + \epsilon_i \frac{\Delta\alpha}{\alpha} \qquad (14b)$$

where $\epsilon_r \equiv$ "real" or in-phase component $= \epsilon\cos\varphi$ and $\epsilon_i \equiv$ "imaginary" or out-of-phase component $= \epsilon\sin\varphi$.

Integrating and keeping only terms dependent on ϵ, we have for the small - θ case

$$\delta \xi' \cong \frac{\epsilon_r}{\alpha}\theta + \frac{\epsilon_i}{\alpha}\frac{\Delta\alpha}{\alpha} \cong \text{constant} \tag{15a}$$

$$\delta \xi \cong \delta \xi' \cdot (\zeta - \zeta_0) \tag{15b}$$

If we assume that θ and $\Delta\alpha$ are essentially random, uncorrelated functions we may estimate that

$$\langle \delta \xi' \rangle \cong \left[\left(\frac{\epsilon_r}{\alpha}\right)^2 \langle \theta^2 \rangle + \left(\frac{\epsilon_i}{\alpha}\right)^2 \left\langle \left(\frac{\Delta\alpha}{\alpha}\right)^2 \right\rangle \right]^{1/2} \tag{15c}$$

$$\langle \delta \xi \rangle \cong \langle \delta \xi' \rangle (\zeta - \zeta_0) \tag{15d}$$

where now $\langle \delta \xi' \rangle$ and $\langle \delta \xi \rangle$ are interpreted as beam divergence and spread respectively, $\langle \theta^2 \rangle$ is the mean square phase spread, and $\left\langle \left(\frac{\Delta\alpha}{\alpha}\right)^2 \right\rangle$ is mean square deviation of the acceleration parameter $\left[\left\langle \left(\frac{\Delta\alpha}{\alpha}\right)^2 \right\rangle^{1/2} \approx \frac{\Delta p_z}{p_z} = \text{beam spectrum width} \right]$.

Numerical Estimates. W. Gallagher* has made a dielectric-bead measurement of the field distribution across a typical coupler of the S-band, $\frac{2\pi}{3}$ mode type. His results can be interpreted as indicating either

$$\frac{\lambda}{E_0}\frac{\partial e_3}{\partial x} \cong +0.6 \tag{16}$$

if we assume $\varphi = 0$, or

$$\left| \frac{\lambda}{E_0}\frac{\partial e_3}{\partial x} \right| \cong 3.0 \tag{17}$$

if we assume $\varphi = \pm \frac{\pi}{2}$.

Thus from Equation (7),

$$\frac{\epsilon_r}{\alpha} \cong \frac{\Delta z}{2\pi L}\left(\frac{\lambda}{E_0}\frac{\partial e_3}{\partial x}\right)_r \cong 1.1 \times 10^{-3} \tag{18}$$

$$\frac{|\epsilon_i|}{\alpha} \cong \frac{\Delta z}{2\pi L}\left|\frac{\lambda}{E_0}\frac{\partial e_3}{\partial x}\right|_i \cong 5.3 \times 10^{-3} \tag{19}$$

where $\frac{L}{\Delta z} \cong 90$ is appropriate to project "M" design.

* Private communication.

However, we have equation (13) which may be stated as

$$B_y = -\frac{\epsilon_i}{\alpha} E_o$$

where B_y is equivalent to an increment in the vertical component of the earth's magnetic field. Under standard operating conditions for Mark III the de-gaussing field is such as to cancel the earth's field to within ~20% so we have an alternate upper limit on $|\epsilon_i|$;

$$\frac{|\epsilon_i|}{\alpha} \lesssim \frac{0.1 \text{ gauss}}{400 \text{ statvolt/cm}} \simeq 2.5 \times 10^{-4} \quad \text{(Mark III)} \tag{20}$$

Assuming that ϵ scales as $\Delta z / L$,

$$\frac{|\epsilon_i|}{\alpha} \lesssim 4 \times 10^{-4} \quad \text{(Project "M")} \tag{21}$$

and

$$\frac{\dot{\epsilon}_a}{\alpha} \lesssim 0.8 \times 10^{-3} \quad \text{(Mark III)} \tag{22}$$

Applying Equations (15) to Mark III we find that the effect of ϵ_i', the out-of-phase component, appears to be negligible;

$$\langle \delta \dot{\xi}' \rangle \lesssim \left[(0.8 \times 10^{-3})^2 (0.3)^2 + (0.4 \times 10^{-3})^2 (2 \times 10^{-2})^2 \right]^{\frac{1}{2}}$$

$$\sim 2.4 \times 10^{-4} \approx \frac{|\epsilon_a|}{\alpha} \langle \theta^2 \rangle^{\frac{1}{2}}$$

and

$$\langle \delta x \rangle \approx \langle \delta \dot{\xi}' \rangle (z - z_o) \quad \text{Mark III} \tag{23}$$

$$\lesssim 2.4 \times 10^{-4} \times 6 \times 10^3 \text{ cm}$$

$$\sim 1.5 \text{ cm}$$

where it has been assumed (somewhat arbitrarily) that $\langle \theta^2 \rangle^{\frac{1}{2}} \approx 0.3$ radian of phase spread and that $\left\langle \left(\frac{\delta \alpha}{\alpha} \right)^2 \right\rangle^{1/2} \approx 2 \times 10^{-2} \approx$ beam spectrum width. The numerical results do not seem to be in serious disagreement with Mark III performance so we may conclude that the estimate (Equation (20)) of ϵ_n is not too large.

If we now assume that estimates (18) and (20) apply to Project "M", we find

$$\left.\begin{aligned} \langle \delta \xi' \rangle &\gtrsim 2.2 \times 10^{-4} \\ \langle \delta x \rangle &\gtrsim 66 \text{ cm} \end{aligned}\right\} \quad \text{"M"} \qquad (24)$$

(taking $\langle \theta^2 \rangle^{1/2} \approx 0.2$ radian and $Z - Z_0 \approx 3 \times 10^5$ cm.) Confining $\langle \delta x \rangle$ to ~ 1 cm would, by Liouville's therocm (i.e., $\langle \delta x \rangle \langle \delta x' \rangle$ conserved in passive focussing elements), increases the angular divergence to $\sim 1.5 \times 10^{-2}$ which is about 100 times too large. Evidently the coupler asymmetry would have to be reduced by $\sim$ a factor of 10.

<u>Numerical Estimates in the Case of Alternating Couplers.</u> Equation (9) may be written

$$\frac{d(\gamma \xi')}{d \zeta} = (\epsilon_r' \sin \theta + \epsilon_i' \cos \theta)\, \xi \qquad (25)$$

where ϵ_r' and ϵ_i' refer to the in-phase and out-of-phase components of $\frac{\partial \epsilon}{\partial \xi}$. From the previous discussion it appears that ϵ_i' is small; assume for the moment that it can be neglected. Equation (7) gives

$$\epsilon_r' = -\frac{\Delta z}{2\pi L}\left(\frac{\lambda^2}{E_0}\,\frac{\partial^2 e_3}{\partial x^2}\right)_r \alpha$$

Gallagher's results indicate

$$\left(\frac{\lambda^2}{E_0}\,\frac{\partial^2 e_3}{\partial x^2}\right)_r \approx +6$$

from which

$$\epsilon_r' \approx -1.1 \times 10^{-2} \alpha \qquad (26)$$

This appears to indicate a rather large transverse force compared to all other known transverse forces (with the exception of externally applied magnetic focussing fields). If we neglect all other transverse forces, Equation (25) integrates to

$$x = A_1 J_0(\sqrt{-KZ}) + A_2 N_0(\sqrt{-KZ}) \qquad (27)$$

where $\qquad K = \dfrac{4\, \epsilon_r' \sin \theta}{\alpha' \lambda}$

For $\epsilon' \sin \theta$ positive the orbits will diverge $\sim$ exponentially and for $\epsilon' \sin \theta$ negative the orbits will undergo damped oscillations.

Using the numerical estimate (26) we get

$$K \approx -4 \times 10^{-3} \sin \theta \text{ cm}^{-1}$$

If we were to run the entire machine with the bunch foreward of the crest* with $\theta_{max} \approx 0.2$, we get typical orbits of the form

$$x \cong x_0 \left(\frac{z_0}{z}\right)^{1/4} \cos (0.8 \times 10^{-4} Z)^{\frac{1}{2}}$$

$$\xi' \cong -(0.45 \times 10^{-2}) X_0 \frac{Z_0^{1/4}}{Z^{3/4}} \sin (0.8 \times 10^{-4} Z)^{\frac{1}{2}}$$

(using asymptotic expressions for the Bessel's functions.)

Thus near the end of the machine, taking $Z_0 \sim 10^4$ cm and $Z \sim 3 \times 10^5$, we get

$$X_{max} \sim (30)^{-\frac{1}{4}} X_0 \cong 0.8 \text{ cm}$$

and $$\xi'_{max} \sim .45 \times 10^{-2} \frac{10}{(30)^{3/4}\, 10^3} X_0$$

$$\cong 0.7 \times 10^{-5} \text{ radian.}$$

where X_0 has been taken as ~ 2 cm so that the numbers refer to peak-to-peak amplitude of the transverse oscillations.

To keep the orbits within bounds without dephasing or external focussing, we would have to require that $\left| K(Z - Z_0) \sin \frac{\theta}{2} \right| \gtrapprox 1$, which gives $K \lessapprox 3 \times 10^{-5}$; a factor of ~ 10 reduction in $\frac{\partial^2 e_z}{\partial x^2}$ is indicated.

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* This would be unacceptable in practice sine for $\theta_{max} \approx 0.2$ there would be a 2% beam spectrum width.