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# Impedances for Higher-Order Gauss-Laguerre Modes in the LEAP Cell

Abstract. Numerical calculations of the longitudinal monopole wakefield impedances for the LEAP cell are presented.

The transverse electric field distribution for a free-space mode originally generated in an axisymmetric cavity (i.e. a laser cavity) can be decomposed into the eigenmodes of the cavity, called Gauss-Laguerre modes, given by[1]:

$$E_{\perp}(r,\phi,z) = A_{\sqrt{\frac{2}{\pi}}} \sqrt{\frac{p!}{(p+m)!}} \frac{1}{w(z)} \left[ \frac{\sqrt{2}r}{w(z)} \right]^m L_{pm} \left( \frac{2r^2}{w^2(z)} \right) \times \exp\left( -\frac{r^2}{w^2(z)} - jkz - j\frac{\pi rr^2}{\lambda R(z)} - j(2p+m+1)\phi_o(z) \right) \exp(jm\phi)$$

$$(1)$$

with

$$w(z) = w_o \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$

$$R(z) = z + \frac{z_R^2}{z}$$

$$\phi_o(z) = \tan^{-1} \left(\frac{z}{z_R}\right)$$
(2a-c)

as the waist size, wave front radius of curvature, and phase, and the  $exp(j\omega t)$  time dependence is assumed. Note that the above formula differs with that published in [1] by a factor of  $r^{m/2}$ . These eigenfunctions are normalized such that:

$$\int_{0}^{2\pi\infty} \int_{0}^{\infty} |E_{\perp}(r,\phi,z)|^2 r dr d\phi = 1,$$
(3)

and a comparison with the orthogonality integral for the Laguerre functions (e.g. [2]) shows that equation (1) has indeed the correct form.

Consequently, the normalization constant should be:

$$A = \sqrt{2PZ_o} , \qquad (4)$$

where  $Z_o$  is the free space impedance, and P is the peak power flow. For a Gaussian pulse, this is just

$$P = \frac{E_o}{\sqrt{2\pi\sigma_t}},\tag{5}$$

where  $E_o$  is the total pulse energy, and  $\sigma_t$  is the rms pulse length.

Evaluating the longitudinal field component is done using the divergence-free condition, which for light linearly polarized in the x-direction requires

$$E_{z}(x, y, z) = -\int_{0}^{z} \frac{\partial E_{x}(x, y, \widetilde{z})}{\partial x} d\widetilde{z} .$$
(6)

The numerical calculation approximates the partial derivative by the first-order centered difference:

$$\frac{\partial E_x}{\partial x} \approx \frac{E_x(x+dx, y=0, z) - E_x(x-dx, y=0, z)}{2 \cdot dx},\tag{7}$$

where the beam trajectory defines the (x,z) evaluation points:

$$x(z) = z \tan \theta , \tag{8}$$

and the half-crossing angle of the two laser pulses is  $\theta$ .

The net fields experienced by the electron beam traversing the LEAP cell are given by

$$E_{z}(s) = \left[\sqrt{\frac{2R}{1+R}} + \sqrt{\frac{2}{1+R}}\right] (\cos(\theta) \cdot E_{z}(s) - \sin(\theta) \cdot E_{x}(s))$$

$$E_{x}(s) = \frac{1}{2\gamma^{2}} \left[\sqrt{\frac{2R}{1+R}} + \sqrt{\frac{2}{1+R}}\right] \cdot (\sin(\theta)E_{z}(s) + \cos(\theta)E_{x}(s))$$
(9)

where  $R = P_1/P_2$  is the power ratio of the two laser spots,  $\gamma$  is the relativistic beam energy, and *s* is the coordinate following the electron beam trajectory. The coefficient containing *R* replaces the usual factor of 2 from summing equal power spots, and is plotted in Figure 1 below.

The factor of  $(1/2\gamma^2)$  appearing in the transverse field arises from the progressive cancellation of the transverse forces as  $\gamma \rightarrow \infty$  [3]. The resultant transverse field strength calculated is, in fact, the transverse component of the Lorentz force on the electron beam, expressed as an equivalent, net electric field strength.



**Figure 1.** Effect of power asymmetry on LEAP cell gradient. For every point, the sum of the powers  $(P_1+P_2=2P)$  is the same. This is a plot of the bracketed expression in eq. (9).

The synchronous longitudinal field is obtained by numerically integrating (7) and multiplying by  $(-exp(j\omega t))$ , where  $t=z/\beta c$ . In this example, as in the Sprangle-Esarey-Krall (SEK) treatment [4], the Born approximation is used (i.e. particle trajectories are assumed to be unchanged by interaction with the field.) For convenience, the SEK form of the on-axis electric field from both beams is reproduced here:

$$E_{z}(0,0,z,t) = -\frac{2E_{o}\sin(\theta)}{(1+\tilde{z}^{2}\cos^{2}(\theta))}\exp\left(-\frac{(\tilde{z}/\theta_{d})^{2}\sin^{2}(\theta)}{(1+\tilde{z}^{2}\cos^{2}(\theta))}\right)\cos(\psi_{t}),$$
(10)

where  $E_o = \sqrt{Z_o P / (\pi w_o^2)}, \tilde{z} = z / z_R, z_R = \pi w_o^2 / \lambda, \theta_d = w_o / z_R$ , and the phase advance is  $w_d = k_T \cos(\theta) - \omega t + \frac{\tilde{z}^3 \cos^3 \theta \tan^2 \theta}{1 - 2 \cdot \tan^{-1} (\tilde{z} \cos \theta) + \phi}$ 

$$\psi_t = kz\cos(\theta) - \omega t + \frac{1}{\theta_d^2(1 + \tilde{z}^2\cos^2\theta)} - 2\cdot\tan^2(\tilde{z}\cos\theta) + \phi_d$$

#### **Comparison of Theory and Simulation**

The results of the numerical calculation of  $E_z(z)$  are shown in figure 1, together with direct evaluation of the field from equation (6) Parameters used in this and subsequent calculations are detailed in Table 1.

Parameter	Symbol	Value
Laser wavelength	λ	0.8 μm
Laser Pulse Energy (one beam only)	$E_o$	7.5µJ
Laser pulse length (FWHM)	$\Gamma_{t}$	1 ps
Laser crossing half-angle	θ	11.5 mr
Laser power ratio	R	1.0
LEAP cell length	L	1.5 mm
Waist parameter	Wo	80λ
Electron Beam Energy	γ	71
Sampling Density (for numeric calculations)	δ	$\pi^3/\lambda$

**Table 1.** Parameters used in these calculations



**Figure 2**. Synchronous longitudinal and transverse field components of the fundamental  $(TEM_{00})$  mode in the LEAP cell, using parameters in Table 1. The blue trace is the numerical result, the green is equation (10), the red is the transverse field, identically zero for this case.

It is worth noting that the numerical calculation and SEK theory both show the cell is significantly shorter than the slippage length, as is signaled by the large value of the longitudinal electric field at the ends of the cell. Of course the present calculation assumes zero slit width in each case, and the true cell length would have to compensate for the effects of field leakage.

#### **Impedance Calculations for Higher Order Modes**

Several general conclusions may immediately be drawn based on the form of equation 1. The Laguerre polynomials are finite-valued and non-zero at u=0, as can be seen by explicitly writing the first term of the series definition:

$$L_{pm}(u) = \frac{(p+m)!}{m! \, p!} + \sum_{l=1}^{p} \frac{(p+m)! (-u)^{l}}{(m+l)! (p-l)! l!}$$
(11)

where the first term is nonzero for all *p*,*m*. Consequently, the field near r=0 has the form:

$$E_{\perp} \propto \left(\frac{r}{w_o}\right)^n \exp(-\frac{r^2}{w_o^2}) \tag{12}$$

It is clear from (12) that modes with m>0 will have no field on axis, and fields that remain small out to a distance comparable to  $w_o$ . Consequently, these modes will have small interaction impedances that diminish rapidly with increasing *m*. In addition, the amount of Guoy phase shift of the higher order modes depends on the mode number. Specifically, examining equation (1) shows that:

$$\phi_{pm}(z) = (2p + m + 1) \tan^{-1}(z/z_R)$$
(13)

For the LEAP cell,  $L\sim z_R/10$ , so the Guoy phase shift of the lowest mode is barely 6°. With very large radial mode numbers, the Guoy phase shift can be made large, but for increasing radial mode numbers, the central accelerating region becomes small.

For illustrative purposes, TEM<sub>00</sub>, TEM<sub>11</sub>, TEM<sub>10</sub>, TEM<sub>01</sub>, are plotted at a waist location in Figure 2 below. The thin black line segment at the center of each pattern indicates the beam trajectory, shown projected onto the z=0 plane.



**Figure 3**. Focal waist electric field patterns for the  $TEM_{00}$  (upper left),  $TEM_{01}$  (upper right),  $TEM_{10}$  (lower left), and  $TEM_{11}$  (lower right) modes.

Evaluation of the longitudinal impedances of the first 441 modes is presented in Figure 2. Radial eigenindex p and azimuthal eigenindex m are both varied from 0 to 20. Parameters are again drawn from Table 1, except spot size, which is  $w_o = 132\lambda$  for this case.

The longitudinal impedance is calculated from the voltage gain and input power as:

$$Z_{\parallel} = \frac{V^2}{2P} = \frac{\left(\int_{-L/2}^{+L/2} E_z(x, y = 0, z, t = z / \beta c) dz\right)^2}{2P},$$

and the input power in one laser beam is P.

As expected, the impedance is approximately 2.9  $\Omega$  for the TEM<sub>00</sub> mode and drops quickly with increasing *m*. Interestingly, the impedance shows a broad maximum centered on *p*=8, corresponding to the condition that the electron beam traverses the central lobe of the field distribution from the half-power point on one side to the half-power point on the other.



Figure 4. Longitudinal impedance on linear (left) and logarithmic (right) scales.

Transverse impedances are identically zero if the power in both spots are equal and the phases of the two laser beams are chosen to give maximum acceleration. If acceleration phase is chosen, but the spot powers are not equal, the transverse fields will not cancel. Figure 4 below shows the transverse field for the  $TEM_{00}$  mode with the ratio of the laser power in the two spots set at R=2. The longitudinal field is unaffected, but the transverse fields do not cancel.

## Laser Spot Power Imbalance

Simulation of the LEAP cell performance with asymmetric power flow in the two laser spots is straightforward with this model. As can be appreciated from figure 1, the power asymmetry must be large before a noticeable drop in accelerating gradient is seen. Figure 4 shows the longitudinal and transverse field strengths for 2:1 power asymmetry between the two laser spots.



**Figure 4.** Longitudinal and transverse fields from unbalanced laser power in the TEM<sub>00</sub> mode in the LEAP cell. R=2, with other parameters taken from Table 1.

## Discussion

Higher order mode content of the LEAP laser will in general not cause appreciable change in the accelerating gradient or result in unwanted beam deflection for cases likely to be of practical interest. The dominant effect of power asymmetry and HOM content is to decrease the usable accelerating gradient for a given laser fluence on the cell optics.

The m=0 modes are potentially the most damaging to the gradient, as they have high interaction impedances and are easily produced by scraping the laser on a circular iris during transport. Laser mode damage from non-axisymmetric apertures leads to power being wasted in modes that do not interact with the beam, but which can damage the cell.

Simple asymmetry of the two laser beam powers has little direct effect on the gradient, but serves to fairly rapidly limit the achievable gradient through damage caused by the brighter spot.

## References

[1] P. Goldsmith, *Quasioptical Systems: Gaussian Beam Quasioptical Propagation and Applications*, IEEE Press, New York, p. 27 (1998).

[2] G. Arfken, *Mathematical Methods for Physicists*, Academic Press, Inc., Orlando, FL, 1985, see equation (13.48).

[3] R. Noble, "Gaussian Laser Fields in Paraxial Approximation", unpublished notes, Nov. 6, 2001.

[4] P. Sprangle, E. Esarey, J. Krall, Phys. Rev. E, 52(5) 5443, (1995).