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Analytic Consistency of Physical Optics Propagators

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Abstract

We demonstrate that despite the superficial divergences in the matrix-valued physical optics propagators, the integrals remain finite, so the divergences do not make the algorithm inconsistent.

1 Introduction

The physical optics vector diffraction technique is a practical method for computing electric and magnetic fields in a variety of quasioptical systems relevant for accelerator applications. This technique, described for instance in [1], involves defining equivalent currents on physical or fictitious surfaces in the system and using matrix-valued Green functions to compute the fields due to these currents.

A superficial examination of the propagators given in [1] seem to indicate singular behavior as the field point becomes very close to a source surface, and claims have been made that the algorithm is inconsistent due to this singularity. Here we show that in fact no such singularity occurs, and therefore that modifications to the propagators in order to avoid this apparent singularity are not necessary.

Since we are interested in exploring the divergence, we need only evaluate the integral in a very small area around the singularity. We take the dimensions of our integration area to be much smaller than characteristic lengths of both the variation of the current and the curvature of the surface, as well as much smaller than a wavelength. We can therefore take the surface to be a plane with constant surface current. We let the surface lie in the xy-plane and have electric and magnetic currents currents $\mathbf{J} = J_y\hat{\mathbf{y}}$ and $\mathbf{M} = M_x\hat{\mathbf{x}}$ respectively at all points, and we integrate over a small disc of radius ε around the origin.

We cannot compute the field directly at the surface, since there is a discontinuity in the fields there (recall that this is the motivation for introducing the currents in the first place). Instead, we compute the field at a point $(0, 0, z)$ very close to the surface, and show that the integral does not diverge as $z \to 0$. Since we are taking this limit we can use the approximation $z \ll \varepsilon$. Thus our approximations can be summarized as

$$z \ll \varepsilon \ll \lambda.$$

The superficial divergences appear, for instance, in the expressions for the electric field due to the electric and magnetic currents; the expressions for the magnetic field are similar. Since the electric and magnetic currents can be defined independently, the fields due to each current must be finite. We examine the expressions for the electric field due to each current separately.

2 Convergence of $\mathbf{E}_J$

The expression for the electric field due to the electric current, from equation (2.34) in [1], is

$$\mathbf{E}_J(\mathbf{r}) = -ikZ_0 \int_S \frac{e^{-ikR}}{4\pi R} \left[\left(1 - \frac{i}{kR} - \frac{1}{k^2R^2}\right)\mathbf{J} + \left(-1 + \frac{3i}{kR} + \frac{3}{k^2R^2}\right)(\hat{\mathbf{R}}\cdot\mathbf{J})\hat{\mathbf{R}}\right] d^2\mathbf{r}'.$$

For a field point $(0,0,z)$ and source point $\mathbf{r}' = (x', y', 0)$, we have

$$\hat{\mathbf{R}} \cdot \mathbf{J} = -J_y \frac{y'}{R}, \qquad \hat{\mathbf{R}} = -\hat{\mathbf{x}}\frac{x'}{R} - \hat{\mathbf{y}}\frac{y'}{R} + \hat{\mathbf{z}}\frac{z}{R},$$

and it is then apparent that by symmetry $\mathbf{E}_J$ must point in the $\hat{\mathbf{y}}$-direction, since terms odd in y' will vanish in the integral. This is because we have taken the field point to lie on the z-axis, but this choice was made without any loss of generality. We then have

$$E_y = -\frac{ikZ_0 J_y}{4\pi} \int_S \frac{e^{-ikR}}{R} \left[\left(1 - \frac{i}{kR} - \frac{1}{k^2R^2}\right) + \left(-1 + \frac{3i}{kR} + \frac{3}{k^2R^2}\right)\frac{y'^2}{R^2} \right] d^2\mathbf{r}'.$$

For $|\mathbf{r}'| = r'$, we have $y' = r'\sin\theta$ in polar coordinates, so we can compute the integral:

$$\begin{aligned}
E_y &= -\frac{ikZ_0 J_y}{4\pi} \int_0^\varepsilon \int_0^{2\pi} \frac{e^{-ikR}}{R} \left[\left(1 - \frac{i}{kR} - \frac{1}{k^2R^2}\right) + \left(-1 + \frac{3i}{kR} + \frac{3}{k^2R^2}\right)\frac{r'^2\sin^2\theta}{R^2} \right] r'\, d\theta\, dr' \\
&= -\frac{ikZ_0 J_y}{4\pi} \int_0^\varepsilon \frac{r'e^{-ikR}}{R} \left[2\pi\left(1 - \frac{i}{kR} - \frac{1}{k^2R^2}\right) + \pi\left(-1 + \frac{3i}{kR} + \frac{3}{k^2R^2}\right)\frac{r'^2}{R^2} \right] dr' \\
&= -\frac{ikZ_0 J_y}{4} \int_0^\varepsilon \frac{r'e^{-ikR}}{R} \left[\left(2 - \frac{r'^2}{R^2}\right) + \left(\frac{i}{kR} + \frac{1}{k^2R^2}\right)\left(\frac{3r'^2}{R^2} - 2\right) \right] dr'.
\end{aligned}$$

Since $R = \sqrt{r'^2 + z^2}$, the quantity r'/R does not diverge as $z \to 0$, for any r'. Thus the integral of the first term in the brackets is is certainly finite; we wish to examine only the potentially divergent terms. Since $R \ll \lambda$, we can approximate $e^{-ikR} \approx 1 - ikR$; note that terms of higher order in R will not have even superficial divergences. We then have

$$\begin{aligned}
E_y &= -\frac{ikZ_0 J_y}{4} \int_0^\varepsilon \frac{r'}{R}(1 - ikR)\left(\frac{i}{kR} + \frac{1}{k^2R^2}\right)\left(\frac{3r'^2}{R^2} - 2\right) dr' \\
&= -\frac{ikZ_0 J_y}{4} \int_0^\varepsilon \frac{r'}{R}\left(1 + \frac{1}{k^2R^2}\right)\left(\frac{3r'^2}{R^2} - 2\right) dr'.
\end{aligned}$$

Again ignoring finite terms, we obtain

$$\begin{aligned}
E_y &= -\frac{iZ_0 J_y}{4k} \int_0^\varepsilon \frac{r'}{R^3}\left(\frac{3r'^2}{R^2} - 2\right) dr' = \frac{iZ_0 J_y}{4k} \int_0^\varepsilon \frac{r'(2R^2 - 3r'^2)}{R^5} dr' \\
&= \frac{iZ_0 J_y}{4k} \int_0^\varepsilon \frac{r'(2z^2 - r'^2)}{(r'^2 + z^2)^{5/2}} dr' = \frac{iZ_0 J_y}{4k} \left[\frac{r'^2}{(r'^2 + z^2)^{3/2}} \right]_{r=0}^{\varepsilon} \\
&= \frac{iZ_0 J_y}{4k} \cdot \frac{\varepsilon^2}{(\varepsilon^2 + z^2)^{3/2}} \\
&\to \frac{iZ_0 J_y}{4k\varepsilon}
\end{aligned}$$

as $z \to 0$. This quantity is finite for any $\varepsilon > 0$, so there is no singularity in the fields. The appearance of ε in the denominator may appear to be a divergence, but remember that we do not take $\varepsilon \to 0$; in fact ε need only be small enough for our approximations to be valid. Also, note that the total current in the disc scales as $J_{\text{tot}} \sim J_y\varepsilon^2$, so the field scales as $E_y \sim J_{\text{tot}}/\varepsilon^3$, which is the behavior we expect in the reactive near field.

3 Convergence of $\mathbf{E}_M$

The expression for the electric field due to the magnetic current, from equation (2.37) in [1], is

$$\mathbf{E}_M(\mathbf{r}) = \int_S \frac{e^{-ikR}}{4\pi R}\left(\frac{1}{R}+ik\right)\hat{\mathbf{R}}\times\mathbf{M}\,d^2\mathbf{r}'.$$

For a source point $\mathbf{r}' = (x', y', 0)$, we have

$$\hat{\mathbf{R}}\times\mathbf{M} = \frac{M_x}{R}(y'\hat{\mathbf{z}} + z\hat{\mathbf{y}}).$$

The first term is odd in y' so it will vanish in the integral, since again we take the field point to be on the z-axis without loss of generality. Then $\mathbf{E}$ points in the $\hat{\mathbf{y}}$-direction, and we have

$$E_y = \int_0^\varepsilon \frac{e^{-ikR}}{4\pi R}\left(\frac{1}{R}+ik\right)\frac{M_x z}{R}2\pi r'\,dr'.$$

As in the previous section we take $e^{-ikR}\approx 1 - ikR$, and obtain

$$\begin{aligned} E_y &= \frac{M_x z}{2}\int_0^\varepsilon \frac{r'}{R^2}(1-ikR)\left(\frac{1}{R}+ik\right)\,dr' \\ &= \frac{M_x z}{2}\int_0^\varepsilon \frac{r'}{R^3}(1+k^2R^2)\,dr' \end{aligned}$$

Keeping only the divergent term,

$$\begin{aligned} E_y &= \frac{M_x z}{2}\int_0^\varepsilon \frac{r'}{(r'^2+z^2)^{3/2}}\,dr' \\ &= -\frac{M_x z}{2}\left[\frac{1}{\sqrt{r'^2+z^2}}\right]_{r'=0}^{\varepsilon} \\ &= \frac{M_x}{2}\left(1-\frac{z}{\sqrt{\varepsilon^2+z^2}}\right) \\ &\to \frac{M_x}{2} \end{aligned}$$

as $z\to 0$. Thus there are no divergences in the fields.

Acknowledgements

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References

[1] L. Diaz and T. Milligan, *Antenna Engineering using Physical Optics: Practical CAD Techniques and Software*, Artech House, 1996