LEAP Accelerator Cell Efficiency

This note follows the ideas and notation in ARDB-280 "Photonic Band Gap Fiber Accelerator Efficiency". The efficiency is given by

$$\eta = \frac{\Delta U_{beam}}{\Delta U_{laser}} = \frac{q\beta_g c}{P\left(1-\beta_g\right)}\left(\frac{\sqrt{PZ_c}}{\lambda} - \frac{qcZ_H}{\lambda^2}\right). \tag{1.1}$$

The quantities in this expression are defined in ARDB-280. Calculating the efficiency requires determining the characteristic impedance, Z_C, the wakefield impedance, Z_H, and the group velocity β_g.

Group Velocity

The factor $\beta_g / \left(1-\beta_g\right)$ in eq. (1.1) comes from having the entire length of the structure filled with energy during the entire beam transit time. For the LEAP structure the effective group velocity as seen by the electron beam is

$$\beta_g = \cos\theta \approx 1 - \frac{\theta^2}{2} \tag{2.1}$$

where θ is the crossing angle. Substituting this into eq. (1.1) gives

$$\eta = \frac{\Delta U_{beam}}{\Delta U_{laser}} = \frac{2qc}{P\theta^2}\left(\frac{\sqrt{PZ_c}}{\lambda} - \frac{qcZ_H}{\lambda^2}\right). \tag{2.2}$$

As an example, for θ =20 mrad and a structure length L = 500 μm the necessary pulse length is τ = 3 fsec to overlap with the electron beam for the length of the cell. This is not practical, and the pulse will be longer to accelerate multiple bunches. The efficiency equation becomes

$$\eta = \frac{\Delta U_{beam}}{\Delta U_{laser}} = \frac{qL}{P\tau}\left(\frac{\sqrt{PZ_c}}{\lambda} - \frac{qcZ_H}{\lambda^2}\right). \tag{2.3}$$

Two comments.

- The expression for q_{max} (= charge at maximum efficiency) is unchanged.
- This is not a true efficiency because the pulse length is artificially longer than it needs to be. The resultant inefficiency can be recovered with multiple bunches

Characteristic Impedance

Use the expression for the accelerating gradient from the original paper[1] even though it ignores the effects of slits and leakage radiation. The on-axis accelerating field is

$$G_0 = -4\sqrt{\frac{Z_0 P}{\pi}}\,\frac{\sin\theta}{w_0\left(1+z^2\cos^2\theta/z_r^2\right)}\exp\left(-\frac{z^2\sin^2\theta/w_0^2}{\left(1+z^2\cos^2\theta/z_r^2\right)}\right)\cos\left(\phi_p+\phi_g+\phi_r\right) \tag{2.4}$$

where
θ = the crossing angle
w_0 = the size of the waist in the middle of the cell
$z_r = \pi w_0^2/\lambda$ = Rayleigh range

$\phi_p, \phi_g, \& \phi_r$ = the plane wave phase, the Guoy phase and the radial phase respectively. These phases are given by

$$\phi_p = \frac{\pi z}{\lambda}\left(\theta^2 - \frac{1}{\gamma^2}\right), \phi_g = 2\tan^{-1}\left(\frac{z}{z_r}\right), \phi_r = \frac{\pi z^3\theta^2}{z_r^2\lambda\left(1+z^2/z_r^2\right)}. \tag{2.5}$$

The maximum value of the characteristic impedance is

$$Z_c(\max) = \frac{16 Z_0 \lambda^2\theta^2}{\pi w_0^2} \tag{2.6}$$

For typical LEAP values, $\lambda = 1\mu m, \theta = 20\ mrad$, $w_0 = 50\mu m$, $Z_c(\max) = 3.1\times 10^{-4}\Omega$. This low value of the impedance is a consequence of the predominantly transverse fields and the large cross sectional area of the free-space, Gaussian beams used. The gradient in terms of $Z_c(\max)$ is

$$G_0 \approx -\frac{\sqrt{Z_c(\max)P}}{\lambda}\frac{1}{\left(1+z^2/z_r^2\right)}\exp\left(-\frac{\pi z^2\theta^2/z_r\lambda}{\left(1+z^2/z_r^2\right)}\right)\cos\left(\phi_p+\phi_g+\phi_r\right) \tag{2.7}$$

Wakefield Impedance

Use the method of Sprangle *et al*[2] to calculate the energy incident on different apertures. Calculate the radial electric field of a relativistic, unit, point charge located at $z_0 = ct$. It is

$$E_r(r,z,z_0) = \frac{1}{4\pi\varepsilon_0}\frac{\gamma r}{\left(r^2+\gamma^2\left(z-z_0\right)^2\right)^{3/2}}. \tag{3.1}$$

Assuming a square pulse of length ℓ and charge q; then

$$\begin{aligned} E_r &= \int_{-\ell/2}^{\ell/2}\frac{q dz_0}{\ell}E_r(r,z,z_0) \\ &= \frac{q\gamma}{4\pi\varepsilon_0 \ell r}\left(\frac{\ell/2 - z}{\sqrt{r^2+\gamma^2\left(\ell/2 - z\right)^2}} + \frac{\ell/2 + z}{\sqrt{r^2+\gamma^2\left(\ell/2 + z\right)^2}}\right) \end{aligned} \tag{3.2}$$

$z = 0$ is the middle of the bunch. The electromagnetic energy density is

$$U = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \varepsilon_0 E_r^2. \tag{3.3}$$

The total energy intercepting an aperture of radius r_0 is

$$W = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz \int_{r_0}^{\infty} U r dr \tag{3.4}$$

Approximate the integrals in the following way. The fields will be Lorentz contracted into a pancake of length ℓ, and approximately uniform within the pancake. So, use the value of the electric field at the center of the bunch ($\ell_+ = \ell_- = \ell/2$) and extend the z integral from $-\ell/2$ to $\ell/2$. Then

$$W=\frac{q^2\gamma^2\ell}{2\pi\varepsilon_0}\int_{r_0}^{\infty}\left(\frac{rdr}{r^2\left(4r^2+\gamma^2\ell^2\right)}\right)=\frac{q^2\gamma^2\ell}{4\pi\varepsilon_0}\int_{r_0^2}^{\infty}\left(\frac{dx}{x\left(4x+\gamma^2\ell^2\right)}\right)=\frac{q^2}{4\pi\varepsilon_0\ell}\ln\left(\frac{\gamma^2\ell^2+4r_0^2}{4r_0^2}\right) \quad (3.5)$$

When $\gamma\ell \gg r_0$

$$W=\frac{q^2}{2\pi\varepsilon_0\ell}\ln\left(\frac{\gamma\ell}{2r_0}\right) \quad (3.6)$$

The total energy intercepting an infinite slit of separation 2X is

$$W=2\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dy\int_{X}^{\infty}Udx=\frac{q^2\gamma^2\ell}{4\pi^2\varepsilon_0}\int_{X}^{\infty}\int_{0}^{\infty}\left(\frac{dxdy}{r^2\left(r^2+\gamma^2\ell^2/4\right)}\right) \quad (3.7)$$

The y integral can be done using Gradshteyn & Ryzhik (2.161.1)

$$\int_{0}^{\infty}\left(\frac{dy}{r^2\left(r^2+\gamma^2\ell^2/4\right)}\right)=\frac{2\pi}{\gamma^2\ell^2}\left(\frac{1}{x}-\frac{1}{\sqrt{x^2+\lambda^2\ell^2/4}}\right) \quad (3.8)$$

So

$$W=\frac{q^2}{2\pi\varepsilon_0\ell}\ln\left(\frac{X+\sqrt{X^2+\ell^2\gamma^2/4}}{2X}\right) \quad (3.9)$$

When $\gamma\ell \gg X$

$$W=\frac{q^2}{2\pi\varepsilon_0\ell}\ln\left(\frac{\gamma\ell}{4X}\right) \quad (3.10)$$

This is the energy loss per slit. With the slits a distance L apart, the effective impedance associated with this energy loss is given by

$$W=\frac{q^2cZ_HL}{\lambda^2} \quad (3.11)$$

This gives

$$Z_H=Z_0\frac{\lambda^2}{2\pi\ell L}\ln\left(\frac{\gamma\ell}{4X}\right) \quad (3.12)$$

Efficiency

The efficiency is given by

$$\eta=\frac{\Delta U_{beam}}{\Delta U_{laser}}=\frac{qL}{P\tau}\left(\frac{\sqrt{PZ_c}}{\lambda}-\frac{qcZ_H}{\lambda^2}\right). \quad (4.1)$$

Use the following values in the numerical example below

λ = 1 µm, θ=20 mrad, w_0 = 50 µm, $\ell L=\lambda^2$, $\ell=\lambda/100$, X = 2.5 µm, γ = 10^5, τ = 0.4 psec

These give

$$Z_c(\max)=3.1\times10^{-4}\,\Omega \text{ and } Z_H=0.75Z_0=280\,\Omega.$$

Using a fluence limit of 2J/cm^2 and estimating that the spot size on optics is ~ w_0, the maximum power is P = 4×10^8 W

LEAP Cell at Maximum Efficiency

Quantity	General Expression	For ($Z_c(max)$ = 3.1×10^{-4} Ω, Z_H = 280Ω, P = 6×10^{16} W)
Charge for max. efficiency	$q_{\max} = \dfrac{\sqrt{PZ_C(\max)}\lambda}{2cZ_H}$	$q_{\max}(pC) = 1\times10^{-7}\sqrt{P(W)}$ $= 2$ fC at maximum power $=1.25\times10^4$ electrons
Maximum efficiency	$\eta_{\max} = \dfrac{LZ_C}{4c\tau Z_H}$	$\eta_{\max} \approx 2.3\times10^{-7}$ $\to 2.8\times10^{-3}$ if pulse length is minimum needed
Loaded gradient at max. efficiency	$G = \dfrac{\sqrt{PZ_c(\max)}}{2\lambda}$	$G(Gev/m) = 9\times10^{-6}\sqrt{P(W)}$ $= 0.18$ GeV/m
Beam energy gain at max. efficiency	$\dfrac{dU_{beam}}{dL} = \dfrac{PZ_c(\max)}{4cZ_H}$	$\dfrac{dU_{beam}(J/m)}{dL} = 1\times10^{-15}P(W)$ $= 4\times10^{-7}$ J/m

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[1] Y. C. Huang, D. Zheng, W. M. Tulloch & R. L. Byer, *Applied Physics Letters* **68**, 753 (1996).

[2] P. Sprangle, E. Esarey, B. Hafizi, R. Hubbard, J. Krall & A. Ting in *Advanced Accelerator Concepts*, AIP Proceedings **398**, 96 (1997).