Cherenkov Radiation in a Dielectric Tube

Levi Schächter and David Schieber have calculated the energy loss due to Cherenkov radiation for a particle of charge q propagating in a hole of radius R is an infinite dielectric tube.[1] They find that E_z field is retarding, and at $s = 0$ which is the location of q

$$E_z(s=0) = \frac{q}{4\pi\varepsilon_0 R^2} F_N \tag{1.1}$$

where F_N is a "normalized force"

$$F_N = 2 \text{ for } \gamma \gg \frac{\varepsilon}{\sqrt{\varepsilon - 1}}. \tag{1.2}$$

In that limit, the effective, retarding gradient is

$$G = \frac{qF_N}{4\pi\varepsilon_0 R^2} = \frac{qcZ_H}{\lambda^2}. \tag{1.3}$$

where

$$Z_H = \frac{F_N}{4\pi\varepsilon_0 c\left(R/\lambda\right)^2} = Z_0 \frac{1}{2\pi\left(R/\lambda\right)^2}. \tag{1.4}$$

As a numerical example, for a structure with $R/\lambda = 0.678$,[2] $Z_H = 0.346 Z_0 = 131\Omega$.

Georges Dôme has performed a more general calculation.[3] (Other references for wakefields in a dielectric tube are King-Yuen Ng[4] and M. Rosing and W. Gai.[5]) Dôme's result (eq. 9 in his paper) can be expressed in terms of impedance as

$$Z_H(s) = 2Z_H \upsilon(s) \qquad \left(s > 0\right) \tag{1.5}$$

where the distance from the source charge to a trailing test particle is s. Note the factor of two between the value at $s = 0$ and $s = 0^+$. This is the usual factor coming from the fundamental theorem of beam loading. $\upsilon(s)$ is given by

$$\upsilon(s) = \frac{1}{\pi} \operatorname{Im} \int_0^\infty dz \frac{\exp\left(iz \dfrac{s}{R\sqrt{\varepsilon_r - 1}}\right)}{z - 2\varepsilon_r \dfrac{H_1^{(2)}\left(z\right)}{H_0^{(2)}\left(z\right)}} \tag{1.6}$$

$H_0^{(2)}$ and $H_1^{(2)}(z)$ are Hankel functions and the relation $H_0^{(2)'}\left(z\right) = -H_1^{(2)}(z)$ has been used. This integral must be evaluated numerically. Break it up into two pieces to perform the integration – one from 0 to U and the other from U to ∞.

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[1] Levi Schächter and David Schieber, NIM **A388**, 8 (1997).
[2] Xintian Eddie Lin, Phys. Rev. ST - Accel & Beams **4**, 051301 (2001).
[3] G. Dôme, Proceedings of the Second European Accelerator Conference 628 (1990).
[4] King-Yuen Ng, Phys. Rev D **42**, 1819 (1990).
[5] M. Rosing and W. Gai, Phys. Rev D **42**, 1829 (1990).

$$\upsilon(s)=\frac{1}{\pi}\operatorname{Im}\int_0^U dz\frac{\exp\left(iz\frac{s}{R\sqrt{\varepsilon_r-1}}\right)}{z-2\varepsilon_r\frac{H_1^{(2)}(z)}{H_0^{(2)}(z)}}+\frac{1}{\pi}\operatorname{Im}\int_U^\infty dz\frac{\exp\left(iz\frac{s}{R\sqrt{\varepsilon_r-1}}\right)}{z-2\varepsilon_r\frac{H_1^{(2)}(z)}{H_0^{(2)}(z)}} \tag{1.7}$$
$$=\upsilon_1(s,U)+\upsilon_2(s,U)$$

The first integral can be performed using MATLAB functions

$$H_0^{(2)}(z)=besselh(0,2,z) \tag{1.8}$$

and

$$H_1^{(2)}(z)=besselh(1,2,z) \tag{1.9}$$

So,

$$\upsilon_1(s,U)=\frac{1}{\pi}\operatorname{Im}\int_0^U dz\frac{\exp\left(iz\frac{s}{R\sqrt{\varepsilon_r-1}}\right)}{z-2\varepsilon_r\frac{besselh(1,2,z)}{besselh(0,2,z)}} \tag{1.10}$$

The argument of this integral has been coded as *dome13.m*.

U should be chosen large enough that the asymptotic expansions of the Hankel functions can be used (Abramowitz & Stegun (9.2.8))

$$\lim_{|z|\to\infty}\frac{H_1^{(2)}(z)}{H_0^{(2)}(z)}=i \tag{1.11}$$

Then*

$$\upsilon_2(s,U)=\frac{1}{\pi}\operatorname{Im}\int_U^\infty dz\frac{\exp\left(iz\frac{s}{R\sqrt{\varepsilon_r-1}}\right)}{z-2i\varepsilon_r}$$
$$=\frac{1}{\pi}\operatorname{Im}\left(\exp\left(-\frac{2\varepsilon_r s}{R\sqrt{\varepsilon_r-1}}\right)E_1\left(\frac{-is\left(U-2i\varepsilon_r\right)}{R\sqrt{\varepsilon_r-1}}\right)\right) \tag{1.12}$$

E_1 is the exponential integral

$$E_1(t)=expint(t) \tag{1.13}$$

The evaluation of $\upsilon(s)$ is done in the program *cherenkov_wake.m*.
The figure below gives the wake for Eddie Lin's structure.

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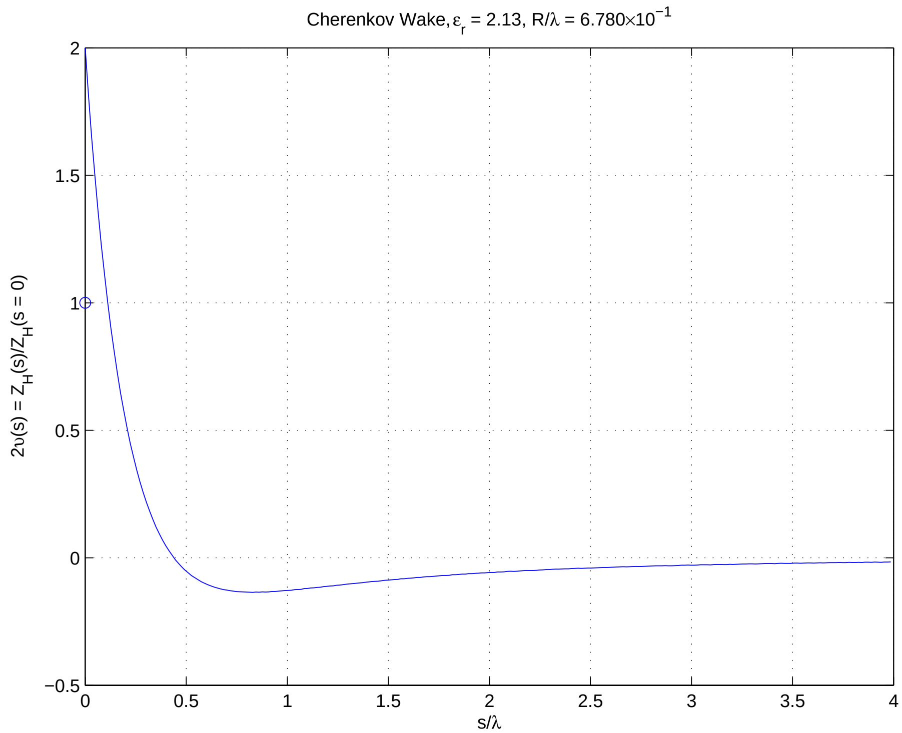
* 7/14/2004 – The previous version of this equation had a sign mistake in the argument of E_1. This did not affect the plots because the code always chose U such that $sU=2n\pi$, i.e. on the real axis, where $ImE_1 = 0$. There will be a future ARDB note on Cherenkov radiation where a similar expression is derived. ("Cherenkov Radiation in a Dielectric Tube with Frequency Dependent Dielectric Constant")

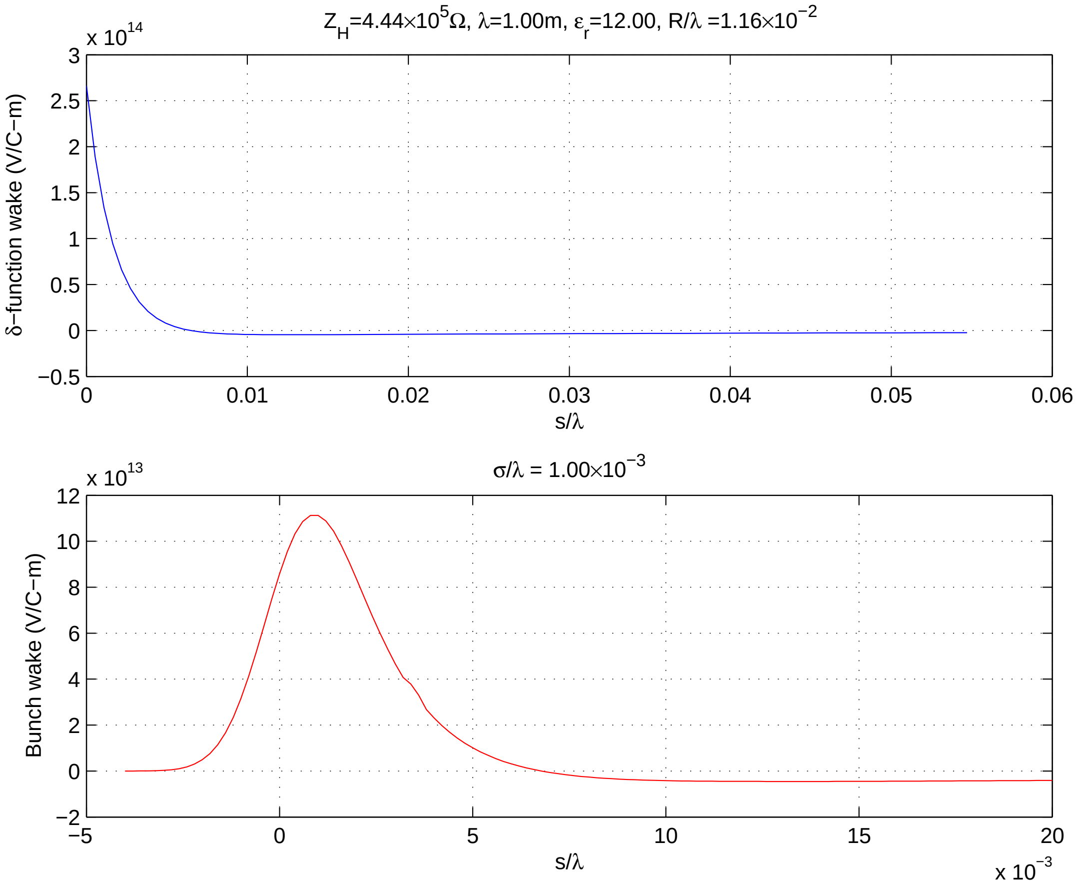


Cherenkov Wake for Eddie Line's Structure with $R/\lambda = 0.678$ and $\varepsilon_r = 2.13$.

This same calculation can be used to calculate the δ-function wakefield by using $\lambda = 1$ m in the calculation because λ only sets the scale for the impedance. Let $V_\delta(s)$ denote the δ-function wakefield, then the wakefield for a bunch with normalized charge density $\rho(s)$ is

$$V(s) = \int_{-\infty}^{\infty} \rho(s') V_\delta\left(s - s'\right) ds' \tag{1.14}$$

The result below has been calculated for $R = 1.163$ cm and ρ a Gaussian with $\sigma = 1$ mm.



***δ*-function wake and bunch wake for *R = 1.163 cm, ε_r = 12 and σ = 1* mm**