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## Beam Envelope Dependence on Plasma Density

The beam envelope on either the Downstream OTR or Integrated Cherenkov is routinely being used to measure plasma density. The plasma channel that produces the focusing a) has time dependence within the bunch, b) has chromatic aberrations, and c) probably has multipoles in addition to being a quadrupole. Nonetheless treating the plasma as a thick quadrupole has proven to be powerful. The following is a derivation of the dependence of the beam envelope on plasma density.

Let  $\alpha_0$ ,  $\beta_0$  and  $\gamma_0$  ( $\beta_0\gamma_0 - \alpha_0^2 = 1$ ) be the Twiss parameters at the entrance to the oven. Let  $L_{oven}$  be the length of the oven and L be the distance from the downstream end of the oven to the downstream detector (either the Downstream OTR or the Integrated Cherenkov).  $\beta$  at the detector is given by

$$\beta = C^2 \beta_0 - 2CS\alpha_0 + S^2 \gamma_0$$

where C and S are the cosine-like and sine-like trajectories starting at the plasma entrance. They are given by

$$C = \cos \varphi - \sqrt{k}L\sin \varphi, \quad S = \frac{1}{\sqrt{k}}\sin \varphi + L\cos \varphi.$$

k is the plasma focusing strength, and  $\varphi$  is the phase advance through the plasma.  $\varphi$  is given by

$$\varphi = \sqrt{k L_{oven}},$$

and k in terms of the ion density,  $n_I$ , beam energy,  $\gamma$ , and classical electron radius,  $r_e$ , is

$$k = \frac{2\pi r_e n_I}{\gamma}$$

*C* and *S* are periodic in  $\varphi$  with period  $2\pi$ , and  $\varphi$  can be thought of loosely as the betatron phase. Minima and maxima of the beam envelope can be determined by solving the equation

$$\frac{d\beta}{dk} = 2\left[C\beta_0 \frac{dC}{dk} - \alpha_0 \left(C\frac{dS}{dk} + S\frac{dC}{dk}\right) + S\gamma_0 \frac{dS}{dk}\right] = 0.$$
 (1)

The derivatives are

$$\frac{dC}{dk} = -\frac{1}{2\sqrt{k}} \left( (L + L_{oven}) \sin \varphi + LL_{oven} \sqrt{k} \cos \varphi \right),$$
$$\frac{dS}{dk} = -\frac{1}{2\sqrt{k}} \left( (LL_{oven} + \frac{1}{k}) \sin \varphi - \frac{L_{oven}}{\sqrt{k}} \cos \varphi \right).$$

This equation for  $d\beta/dk$  could be solved to give the minima in the beam envelope.

E-157 note (ARDB-217) is an example of the application of equation (1). In that note the values of the laser energy readout at several minima and maxima are used to generate a set of simultaneous equations that are solved for the incoming Twiss parameters and the proportionality constant between the laser energy meter and the phase shift  $\varphi$  in the plasma.

Alternatively, the beam envelope can be plotted. The figure below is for reasonable E-157 parameters,  $\beta_0 = 0.8$  m,  $\alpha_0 = = -0.25$ , 0, 0.25. These correspond to the focal point from the FFTB being at different locations with respect to the front of the plasma. There are a number of comments



Beam envelope at the Downstream OTR for  $\beta_0 = 0.80$  m, and different  $\alpha_0$ 's. The figure below is for reasonable E-157 parameters,  $\beta_0 = 0.8$  m,  $\alpha_0 = -0.25$ , 0, 0.25. The latter correspond to the focal point of the FFTB being 0.19 m before the plasma, at the plasma entrance, and 0.19 m downstream of the plasma, respectively.

- 1) The details of the envelope depend on the focal point of the FFTB, but for reasonable focal positions the first minimum in the envelope occurs at  $\varphi \sim \pi/2$  and the second minimum at  $\varphi \sim 1.1\pi$ .
- 2) The first minimum at  $\varphi \sim \pi/2$  corresponds to the plasma acting as a relatively thin lens and just focusing the beam at the Downstream OTR.
- 3) The second minimum at  $\varphi \sim l.1\pi$  corresponds to particles undergoing  $\sim \frac{1}{2}$  betatron oscillation, i.e. crossing the axis once. This is one example of the general result that the  $\beta$ -function advances with phase at twice the rate of individual particle motion.
- 4) Other examples in the figure are the minima after the first at  $\varphi \sim \pi$ ,  $2\pi$ ,  $3\pi$ , .... The initial conditions are less important as the number of minima increase.
- 5) The desired condition for run 5,  $\varphi = 3\pi$ , corresponds to the fourth envelope minimum.