ARDB Technical Note

Electromagnetic Stress in a Rectangular Pillbox

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In this note we compute the maximum time-averaged electromagnetic stress on the walls of a rectangular pillbox accelerator cavity, as a function of stored energy. GLIDCOP has tensile strengths in the range of 380-700MPa. We are interested here to know at what gradient this limit is approached due to the electromagnetic stress. To emphasize, this is not the stress associated with thermal expansion.

TE_{101} Scalings

The TE_{101} mode field components in a rectangular pillbox may be expressed as,

$$\tilde{E}_y = \tilde{E}_0 \sin(\beta_x x)\sin(\beta_z z),$$
$$Z_0 \tilde{H}_x = -j\tilde{E}_0 \frac{\beta}{\beta_0}\sin(\beta_x x)\cos(\beta z), \qquad Z_0 \tilde{H}_z = j\tilde{E}_0 \frac{\beta_x}{\beta_0}\cos(\beta_x x)\sin(\beta z).$$

The wavenumbers and resonant frequency are

$$\beta_x = \frac{\pi}{a}, \qquad \beta = \frac{\pi}{L}, \qquad \beta_0 = \frac{\omega}{c} = \sqrt{\beta_x^2 + \beta^2}\,.$$

The long dimension (z-dimension) is L, and the width (x-dimension) is a. We will denote the height (y-dimension) b, and this is also the "gap" length of the accelerator cavity.

In a separate note we have computed the stored energy

$$U = \frac{V}{8cZ_0}\left|\tilde{E}_0\right|^2,$$

where $V = abL$ is the cavity volume, and the *[R/Q]*

$$\left[\frac{R}{Q}\right] = \frac{\left|\tilde{V}_c\right|^2}{\omega U} = Z_0 \frac{16}{\beta_0^2 aL}\frac{\sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)}$$

where θ is the transit angle,

$$\theta = \frac{\omega b}{c},$$

and T is the transit time factor

$$T = \frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)}.$$

Optimum occurs for θ=133.56°, and corresponds to $b=0.37099\lambda$, and

$$\left[\frac{R}{Q}\right]=\frac{442.5\Omega}{\left[\frac{L}{a}+\frac{a}{L}\right]}.$$

Optimum shunt impedance occurs for $L=a$, and corresponds to *[R/Q]*=221.3Ω. (In practice, addition of beam ports will reduce this figure and we will premise the working examples discussed below on the value *[R/Q]*=176Ω.) This choice of transit angle corresponds to $\beta_0=\sqrt{2}\pi/a$, or $a=L=\lambda 2^{-1/2}$, so that *a*=*L*=0.23195cm, *b*=0.121696cm.

Electromagnetic Stress Tensor

The EM stress tensor is

$$T^{ab}=D^aE^b+H^aB^b-\frac{1}{2}\left(\vec{D}\bullet\vec{E}+\vec{H}\bullet\vec{B}\right)\delta_{ab}=D^aE^b+H^aB^b-u\delta_{ab},$$

and corresponds to an outward pressure on a perfectly conducting boundary,

$$p=-T^{ab}n_bn_a=\frac{1}{2}\left(\mu_0\vec{H}^2-\varepsilon_0\vec{E}^2\right),$$

and time-averaged over an rf period, this is

$$\overline{p}=\frac{1}{4}\left(\mu_0\left|\tilde{H}\right|^2-\varepsilon_0\left|\tilde{E}\right|^2\right),$$

or,

$$\overline{p}=\frac{\left|\tilde{E}_0\right|^2}{4Z_0\,c}\left\{\left(\frac{\beta}{\beta_0}\right)^2\sin^2(\beta_x x)\cos^2(\beta z)+\left(\frac{\beta_x}{\beta_0}\right)^2\cos^2(\beta_x x)\sin^2(\beta z)-\sin^2(\beta_x x)\sin^2(\beta_z z)\right\}.$$

Thus on the *x*=0 side-wall

$$\overline{p}(x=0,z)=\frac{\left|\tilde{E}_0\right|^2}{4Z_0\,c}\left(\frac{\beta_x}{\beta_0}\right)^2\sin^2(\beta z),\qquad \overline{p}_{\max-x-\text{side}}=\frac{\left|\tilde{E}_0\right|^2}{4Z_0\,c}\left(\frac{\beta_x}{\beta_0}\right)^2=2\frac{U}{V}\left(\frac{\beta_x}{\beta_0}\right)^2,$$

on the *z*=0 side-wall

$$\overline{p}(x,z=0)=\frac{\left|\tilde{E}_0\right|^2}{4Z_0\,c}\left(\frac{\beta}{\beta_0}\right)^2\sin^2(\beta_x x),\qquad \overline{p}_{\max-x-\text{side}}=\frac{\left|\tilde{E}_0\right|^2}{4Z_0\,c}\left(\frac{\beta_x}{\beta_0}\right)^2=2\frac{U}{V}\left(\frac{\beta_x}{\beta_0}\right)^2.$$

These walls evidently tend to bulge due to magnetic pressure. For the $\beta=\beta_x=\beta_0 2^{-1/2}$ geometry the maximum pressure is simply $\overline{p}_{\max-z-\text{side}}=\overline{p}_{\max-x-\text{side}}=U/V$, while, on the *y*-wall, where the electric field terminates,

$$\bar{p}_{\text{max-y-wall}} = -2\frac{U}{V}.$$

For the canonical numbers: *[R/Q]*=176Ω, a minimum loaded gradient of 1.01GV/m for the *N*=50 (last) bunch, at 60pC per bunch, an unloaded gradient of 1.13GV/m, no-load cell voltage of 1.4MV (corresponding to a 1.13GV/m accelerating field), stored energy *U*=19mJ. Volume is $6.55\times10^{-3}cm^3$, and energy density is $2.9MJ/m^3$. This corresponds to a maximum cavitation pressure of 2.9MPa on the *x-z* sidewalls, and a cavitation on the y-sidewall with maximum 5.8MPa.