ARDB Technical Note

RF Circuit and Resonator Scalings for a W-Band Active RF System

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The basic circuit scalings for a standing-wave resonator are noted, for application with an active switch to be employed after resonator filling. Circuit parameters achievable are then investigated for three configurations: (1) TE_{01p} mode in circular guide and (2) TEM_{00} mode of a confocal quasi-optical resonator. These modes are of interest for the first stage of energy storage in the matrix accelerator, due to the high achievable wall Q's. For the second stage of pulse compression and distribution, in the "primary" resonant line, we consider the (3) TE_{10p} mode of a rectangular pillbox. Pulsed heating is quantified.

Introduction

At W-Band structure time-scales are quite short. The natural fill time for a W-Band structure is in the range of 10ns, corresponding to wall Q's on the order of 2700. Pulsed heating suggests that even shorter exposure times for the copper would be preferable. Meanwhile, modulator pulse lengths are on the order of 1μs. To make these time-scales "meet" one must either devise a new rf generation scheme that produces short pulses directly, or one must devise a means of compressing 1μs W-Band signals to the 10ns time scale. The former approach would correspond to the two-beam accelerator schemes, or an Auston-switch[1] based scheme. In this note we consider some basic constraints on the latter problem, rf pulse compression at W-Band, making use of resonant energy storage and active switching. We do not address the mechanism of switching.

Efficiency is a concern; low intrinsic efficiency would render the concept untenable in the face of collider requirements and site-power. As we will see, this circumstance frustrates but does not inhibit the use of multiple compression stages. In previous discusssion of the rf network for the matrix accelerator scheme, the picture was put forward of two resonant lines, one following the ubitron, and the other referred to as the "primary", a resonant line that serves the function of rf distribution. In this scheme, the requirement on the rf input to the primary line is 200MW in a 10ns pulse. The 10ns pulse length implies a 5ft length of resonant line, assuming unloaded guide operated not too close to cut-off. The product of power and pulse length implies an energy in the output pulse from the resonant line of U=2 Joule. This note was begun with this picture in mind. A concept with higher intrinsic efficiency would however include only one stage of resonant energy storage. This would require a primary line with a very high wall Q. In fact, the 1μs time scale implies that high wall Q's of order 2×10^5-9×10^5 will be required. Such high Q's, we will see, are difficult to achieve except in over-moded resonators. Over-moded resonators, one intuits, will impose a more challenging design problem, in view of the need for strong coupling, after switching, to a high-impedance transmission line in which the large accelerating fields are to be developed (the "secondary"). One is concerned that, after switching, the energy storage mode will couple to other modes of the resonator, and fail to discharge.

We may summarize in advance the qualitative implications of the calculations here. Taken together with the energy per meter per pulse requirements of the accelerator, the foregoing considerations imply that the optimal solution for the rf network powering the accelerator will (1) include only one stage of energy storage (2) employ a resonator and switch designed to suppress conversion to adjacent modes (3) be capable of coupling strongly, after switching, to the high-impedance accelerating structure. The feasbility of such a scheme will ultimately hinge on the switch --- not merely the power handling capability, rise-time, lifetime of the carriers, and energetics --- but on an artful geometry that suppresses coupling to adjacent modes, by means of symmetry.

This conflict between high-*Q* and active switching into a high-impedance line can be seen clearly in the case of the confocal quasi-optical resonator, considered below. How one would configure active elements to switch this cavity into 50 parallel transmission lines is unclear. This might be useful however in the case of two-stage pulse compression. Lurking in the background among these concerns is pulsed heating. We will see that high-*Q* resonator geometries may be found, with pulsed heating in the 100°K-200°K range. As we are already aware, such pulsed temperature rises are in themselves a major concern.

Circuit Scalings

Regardless of the implementation in copper, certain funamental relations constrain the circuit behavior of a resonator. For the high pulse compression ratios of interest here (charging time >> discharge time), and for a resonant line terminated in an active opening switch, we may model the resonant line during charge-up (active element = short) as a standing-wave cavity. The cavity voltage phasor V_c follows the forward voltage waveform phasor V_f (from the planar ubitron), when driven on resonance, according to

$$\frac{dV_c}{dt} + \frac{1}{T_f} V_c = \frac{2\beta}{1+\beta} \frac{1}{T_f} V_f ,$$

with

$$\beta = \frac{Q_w}{Q_e} ,$$

the ratio of wall *Q* to external *Q*,

$$T_f = \frac{T_0}{1+\beta} ,$$

the loaded fill time, and

$$T_0 = \frac{2Q_w}{\omega} ,$$

the field decrement time due to wall-losses. The angular frequency of the rf signal is $\omega = 2\pi f$, with *f*=91.39GHz=32 x 2.856GHz. In the following we picture the source as being isolated from the line; this may be accomplished with a hybrid tee arrangement, with the source feeding two lines.

For turn on at *t*=0 of constant input signal, the solution is

$$V_c = \frac{2\beta}{\beta+1} V_f \left(1 - e^{-t/T_f}\right).$$

The reverse voltage on the input line is,

$$V_r = V_c - V_f = \frac{\beta - 1}{\beta + 1} V_f - \frac{2\beta}{\beta + 1} e^{-t/T_f} V_f .$$

To relate voltages to power and energy we can merely use energy conservation. I summarize the results derived in the notes on microwave linacs[2] . Implicit in the use of a "voltage" to describe the cavity excitation, is some normalization constant *[R/Q]* relating energy stored in the cavity to "voltage",

$$U = \frac{V_c^2}{\omega [R/Q]}.$$

In addition, there is a "transformer ratio", *n*,

$$n = \sqrt{\frac{Q_e [R/Q]}{2Z_c}} ,$$

relating net power flow to the cavity, to the forward and reverse voltages,

$$P_{net} = P_f - P_r = \frac{V_f^2 - V_r^2}{2n^2 Z_c} .$$

In these expressions, Z_C is the characteristic impedance of the waveguide mode, in the input line,

$$Z_c = Z_0 \frac{\lambda_g}{\lambda} ,$$

with $Z_0 \approx 376.7\Omega$. Here TE-mode in the connecting guide is assumed, λ is the free-space wavelength and λ_g is the guide wavelength.

Putting all this together one finds that the parameter *[R/Q]* drops out and energy stored in the cavity is

$$U = \frac{2\beta}{1 + \beta} P_f T_f \left(1 - e^{-t/T_f}\right)^2 .$$

Given the short length of the line (and the high-*Q* presumed by all this), the attenuation in two transits of the line is small. Thus essentially all of this energy is available for discharge with the opening of the switch at the end of the line. In this case, the efficiency of this stage of the rf system is

$$\eta_{store} = \frac{U}{P_f\, t} = \frac{2\beta}{1 + \beta} \frac{\left(1 - e^{-\tau}\right)^2}{\tau} ,$$

where $\tau = t / T_f$. Efficiency is a maximum as a function of τ, for τ=1.26, and is given by

$$\eta_{store} \approx 0.815\frac{\beta}{1+\beta}, \qquad (\tau \approx 1.26).$$

Maximum efficiency in such a system is evidently 81.5%. In addition to efficiency though, peak power is also a concern,

$$P_f = \xi\frac{U}{T_0}\frac{(1+\beta)^2}{\beta} \approx \frac{U}{T_0}\frac{(1+\beta)^2}{\beta}, \qquad (\tau \approx 1.26)$$
$$\xi = \frac{1}{2}\frac{1}{\left(1-e^{-\tau}\right)^2} \approx 0.974.$$

Let us consider two specific cases to illustrate the trade-offs one can make, based on β.

For β=5, efficiency is 68.0%, and the peak power requirement is $P_f \approx 7.2U/T_0$. Thus if one requires storage of U=2J, with a $P_f \approx 5MW$ source, then the damping time will be $T_0 \approx 2.9\mu s$. This implies a wall Q for the resonant line mode of $Q_w \approx 8.3\times10^5$. This is an extremely high wall Q for a resonance at W-Band.

For β=1, efficiency is 40.8%, and the peak power requirement is $P_f \approx 4U/T_0$. Thus if one requires storage of U=2J, with a $P_f \approx 5MW$ source, then the damping time will be $T_0 \approx 1.6\mu s$, and $Q_w \approx 4.6\times10^5$. This is still a pretty high wall Q. It is however the minimum as a function of β, for fixed U/P_f.

These considerations suggest we take a look at how to achieve high wall Q's. Before proceeding to specific geometries and achievable Q's, let us note here a complete summary of the energetics of the problem. We express quantities in terms of,

$$\mu = \frac{\beta-1}{\beta+1}, \qquad \alpha = \frac{2\beta}{\beta+1}, \qquad P_f = \beta\frac{V_f^2}{R_s}.$$

Reflected power is

$$P_r = \beta\frac{V_r^2}{R_s} = \left(\mu^2 - 2\alpha\mu e^{-\tau} + \alpha^2 e^{-2\tau}\right)P_f,$$

and dissipated power is,

$$P_w = \frac{\omega U}{Q_w} = \frac{V_c^2}{R_s} = \frac{\alpha^2}{\beta}\left(1-e^{-\tau}\right)^2 P_f.$$

The energy in the forward pulse is

$$U_f = T_f\int_0^\tau d\tau' P_f = P_f T_f \tau = \beta\frac{V_f^2}{R_s}T_f\tau,$$

while the reflected energy is,

$$U_r = T_f \int_0^\tau d\tau' P_r = \left\{ \mu^2 \tau - 2\alpha\mu\left(1 - e^{-\tau}\right) + \frac{1}{2}\alpha^2\left(1 - e^{-2\tau}\right) \right\} \frac{U_f}{\tau},$$

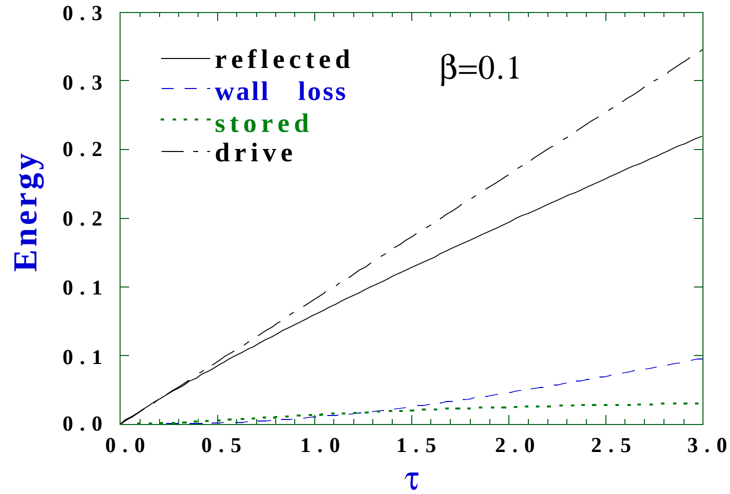

Fig. 1 Evolution of incident, stored, reflected, and dissipated energy for an undercoupled cavity.

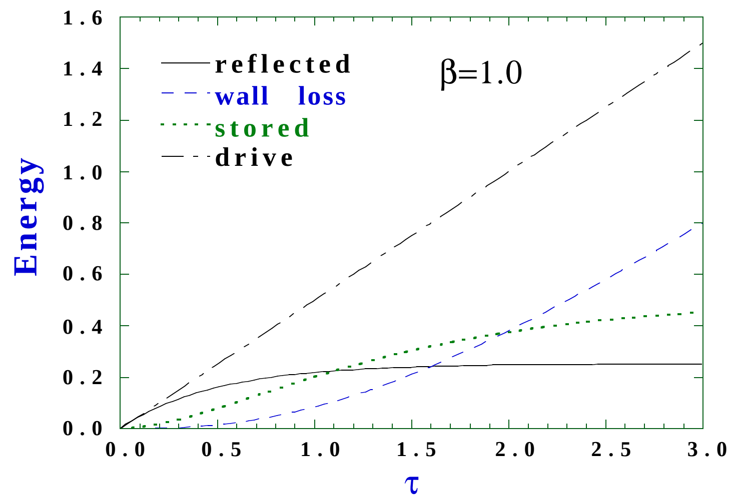

Fig. 2 Evolution of incident, stored, reflected, and dissipated energy for a critically-coupled cavity.

and the energy dissipated in the walls is

$$U_w = T_f \int_0^\tau d\tau' P_w = \left\{ \tau - 2\left(1 - e^{-\tau}\right) + \frac{1}{2}\left(1 - e^{-2\tau}\right) \right\} \frac{\alpha^2}{\beta} \frac{U_f}{\tau}.$$

Finally the stored energy takes the form,

$$U_s = \frac{V_c^2}{\omega[R/Q]} = \frac{1}{2}(1+\beta)T_f \frac{V_c^2}{R_s} = \frac{(1+\beta)}{2\beta}\alpha^2\left(1-e^{-\tau}\right)^2 \frac{U_f}{\tau}.$$

The temporal behavior of these quantities is depicted in Figs. 1-3.

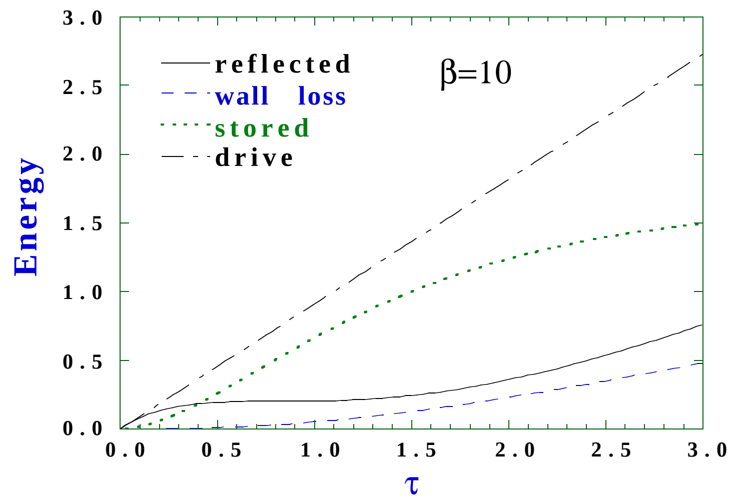

Fig. 3 Evolution of incident, stored, reflected, and dissipated energy for an over-coupled cavity.

TE$_{01p}$ Mode in Circular Guide

The scalings for the TE$_{01p}$ mode of cylindrical pipe have been set down by Montgomery, *et al.*;[3] our application is rather like that of the SLED system employed on the linac (TE$_{015}$) except that our p will be much larger. Recall the "0" subscript refers to the azimuthal mode number, the "1" to the first radial mode (first zero of J_0') and the index "p" is the number of half guide wavelengths along the length of the terminated line. Wall Q is given by losses along the line, and losses at the endcaps,

$$\frac{1}{Q_w} = \frac{1}{\pi}\frac{\lambda^2}{\lambda_g}\alpha + \frac{4}{p\pi}\frac{\lambda^3}{\lambda_g^3}\frac{R_s}{Z_0},$$

Losses on the endcaps will be relatively negligible. The attenuation constant is

$$\alpha = \frac{R_s}{Z_0}\frac{1}{a}\left(\frac{\lambda}{\lambda_c}\right)^2\left\{1-\left(\frac{\lambda}{\lambda_c}\right)^2\right\}^{-1/2},$$

the constant $Z_0 \approx 376.7\Omega$, and

$$R_s \approx 8.3m\Omega\sqrt{f(GHz)} \approx 80m\Omega,$$

is the surface resistance, here evaluated for room temperature copper. The quantity

$$\lambda_c = \frac{2\pi a}{j'_{01}},$$

is the cut-off wavelength of the TE_{01} mode in straight guide, *a* is the pipe radius, and $j'_{01} \approx 3.832$ the first zero of J'_0. The guide wavelength is

$$\lambda_g = \lambda\left\{1-\left(\frac{\lambda}{\lambda_c}\right)^2\right\}^{-1/2},$$

and $\lambda \approx 3.3mm$ is the free-space wavelength. The resonance condition for the mode is

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{p\pi}{L}\right)^2 + \left(\frac{j'_{01}}{a}\right)^2,$$

with *L* the length of the line.

Accepting the additional constraints, $L \approx 5\,ft$, and an input for wall *Q*, it is straightforward to plug in numbers and determine the longitudinal mode index that corresponds to the required *Q*. Note that other modes are clustered nearby in frequency and longitudinal wavenumber, TE_{11} ($j'_{11} \approx 1.841$), TE_{21} ($j'_{21} \approx 3.054$), TM_{01} ($j_{01} \approx 2.405$), and TM_{11} ($j_{11} \approx 3.832$, degenerate) with typical separations of 10-20MHz; however, these don't couple by symmetry. The concern would be the higher radial modes. A simple expedient to avoid conversion to higher radial modes is to operate below cutoff for these modes, and this requires,

$$\beta_0 \le \beta_{c-TE02} = \frac{j'_{02}}{a} \Leftrightarrow a \le 1.12\lambda, \qquad (TE_{02}\text{ cut-off condition})$$

with $j'_{02} = 7.0156$. Equality here corresponds to a TE_{01} cutoff of $\beta_c = 0.674\beta$. For operation at 91.39GHz, this corresponds to a tube radius of $a = 3.69\times10^{-3}m$, or a diameter of 29/100" corresponding to WC29. In this case the maximum wall *Q* is $5.5\text{x}10^4$ in room-temperature copper. Other examples: for a 0.561cm radius tube, operating in *p*=868 mode, attenuation is $1.4\text{x}10^{-2}$ dB/ft, and wall *Q* is $1.9\text{x}10^5$. Cut-off wavelength is 0.92cm. For a 1.05cm tube radius, operating in *p*=912 mode, attenuation is $2\text{x}10^{-3}$ dB/ft, and wall *Q* is $9.5\text{x}10^5$. Cut-off wavelength is 1.7cm.

Next let us consider heating. Average heating is easily computed. For a critically coupled line, and a 2J requirement on the output pulse, energy dissipated in the resonant line, per pulse, will be 2.9J. At 120Hz, this corresponds to 350W. For the 0.56 cm tube radius, surface area is 536cm^2 (neglecting endcaps), and time and area averaged power dissipation is less than 0.65W/cm^2. This is sufficiently low that water cooling is not a challenge.

To quantify the effect of pulsed heating, we should compute the heating at the anti-nodes ("hot-spots"), where maximum thermal stress will occur. Let us consider in detail the TE_{mnp} modes; our work will entail a derivation of some of the results quoted above.

In the time-domain, fields are

$$\vec{E} = \Re\left(\tilde{E}_\perp e^{j\omega t}\right),\ \vec{H} = \Re\left(\left[\tilde{H}_\perp + \tilde{H}_z \hat{z}\right] e^{j\omega t}\right),$$

with complex vector components,

$$Z_0 H_z = \psi \sin(\beta z), \qquad Z_0 \tilde{H}_\perp = \frac{\beta}{\beta_c^2}\cos(\beta z)\nabla_\perp \psi, \qquad \tilde{E}_\perp = j\frac{\beta_0}{\beta_c^2}\sin(\beta z)\,\hat{z}\times\nabla_\perp\psi,$$

and

$$\psi = E_0 J_m(\beta_c r)e^{jm\phi}, \quad \beta_c = \frac{j'_{mn}}{a}, \qquad \beta = \frac{p\pi}{L}, \qquad \beta_0 = \frac{\omega}{c} = \left(\beta_c^2 + \beta^2\right)^{1/2}.$$

Here a is the pipe radius, L is the cavity length, z is the coordinate along the cavity axis, r is the radial coordinate, ϕ is the angle in the transverse plane.

The stored energy is given by an integral over the cavity volume,

$$U = \int d^3\vec{r}\left\{\frac{1}{2}\varepsilon_0 \vec{E}\bullet\vec{E} + \frac{1}{2}\mu_0 \vec{H}\bullet\vec{H}\right\} = \int d^3\vec{r}\,\varepsilon_0\left\langle \vec{E}\bullet\vec{E}\right\rangle$$
$$= \frac{1}{2cZ_0}\int d^3\vec{r}\left|\tilde{E}_\perp^2\right| = \frac{L}{4cZ_0}\frac{\beta_0^2}{\beta_c^4}\int_0^a r\,dr\int_0^{2\pi} d\phi\left|\nabla_\perp\psi\right|^2 = E_0^2\frac{\pi L a^2}{4cZ_0}\frac{\beta_0^2}{\beta_c^2}J_0^2\left(j'_{0n}\right).$$

The brackets in the second equality indicate a time-average, and in the last equality we have evaulated the result for the m=0 modes, and made use of the identities,

$$\int_0^a r\,dr\,J_1^2\left(j_{1n}r\right) = \frac{1}{2}a^2 J_2^2\left(j_{1n}\right), \qquad J_0(\xi) + J_2(\xi) = \frac{2}{\xi}J_1(\xi),$$

so that $J_2\left(j'_{0n}\right) = -J_0\left(j'_{0n}\right)$.

Power dissipation per unit area on the walls takes the form,

$$\frac{dP}{dA} = \frac{1}{2}R_s\left|\tilde{H}\right|^2,$$

and, more explicitly,

$$\frac{dP}{dA} = \frac{R_s}{2Z_0^2}\frac{\beta^2}{\beta_c^4}\left|\nabla_\perp\psi\right|^2, \qquad \text{(endcaps)}$$
$$\frac{dP}{dA} = \frac{R_s}{2Z_0^2}\left\{\cos^2(\beta z)\frac{\beta^2}{\beta_c^4}\left|\nabla_\perp\psi\right|^2 + \sin^2(\beta z)\left|\psi\right|^2\right\}. \qquad \text{(sidewalls)}$$

For the m=0 modes, we have, more explicitly,

$$\left.\frac{dP}{dA}\right|_{endcaps} = E_0^2\frac{R_s}{2Z_0^2}\frac{\beta^2}{\beta_c^2}J_1^2\left(j'_{0n}\frac{r}{a}\right), \qquad \left.\frac{dP}{dA}\right|_{sidewalls} = E_0^2\frac{R_s}{2Z_0^2}J_0^2\left(j'_{0n}\right)\sin^2(\beta z).$$

Maxima are

$$\max \frac{dP}{dA}\bigg|_{endcaps} = \frac{dP}{dA}\left(j'_{0n}\frac{r}{a} = j'_{1n} \right) = E_0^2 \frac{R_s}{2Z_0^2}\frac{\beta^2}{\beta_c^2} J_1^2\left(j'_{1n}\right),$$
$$\max \frac{dP}{dA}\bigg|_{sidewalls} = \frac{dP}{dA}\left(\beta z = \left[N + \frac{1}{2} \right]\pi \right) = E_0^2 \frac{R_s}{2Z_0^2} J_0^2\left(j'_{0n}\right).$$

Considering the lowest radial mode $J_0\left(j'_{01}\right) = -0.40276$, $J_1\left(j'_{10}\right) = 0.58179$ the ratio is,

$$\max \frac{dP}{dA}\bigg|_{sidewalls} / \max \frac{dP}{dA}\bigg|_{endcaps} = \left[\frac{\beta_c}{\beta} \frac{J_0\left(j'_{01}\right)}{J_1\left(j'_{10}\right)} \right]^2 = \left[\frac{\beta_c}{\beta} 0.692 \right]^2.$$

Thus for $\beta \geq 0.692\beta_c$ maximum heating occurs on the endcaps. This condition may be expressed as $\beta_0 \geq 1.22\beta_c$ or $a \geq 0.74\lambda$. Using, $j'_{10} = 1.8226$ and $j'_{01} = 3.8217$, one sees that on the endcaps the maximum heating occurs at

$$r = a\frac{j'_{10}}{j'_{01}} \approx 0.477a, \qquad \text{(location of maximum heating on endcaps)}$$

corresponding to a ring situated about half-way to the sidewall.

It is convenient to reduce the result for maximum heating to a figure of merit for the maxium heat dissipation at an anti-node relative to the dissipation averaged over the cavity walls. For this purpose, we should compute the average dissipation in the walls. This amounts to simply a check of the result employed in a previous section for the wall Q. Net dissipation on the endcaps and sidewalls is,

$$P_{endcaps} = 2E_0^2 \int_0^a r\,dr \int_0^{2\pi} d\phi \frac{R_s}{2Z_0^2}\frac{\beta^2}{\beta_c^2} J_1^2\left(j'_{0n}\frac{r}{a} \right) = E_0^2 \pi a^2 \frac{R_s}{Z_0^2}\frac{\beta^2}{\beta_c^2} J_0^2\left(j'_{0n}\right),$$
$$P_{sidewalls} = E_0^2 \frac{R_s}{2Z_0^2} J_0^2\left(j'_{0n}\right) \int_0^L dz \int_0^{2\pi} a\,d\phi \sin^2(\beta z) = E_0^2 \frac{1}{2}\pi a L \frac{R_s}{Z_0^2} J_0^2\left(j'_{0n}\right),$$
$$P_{diss} = P_{endcaps} + P_{sidewalls} = E_0^2 \frac{1}{2}\frac{R_s}{Z_0^2} J_0^2\left(j'_{0n}\right)\pi a L \left\{ 2\frac{\beta^2}{\beta_c^2}\frac{a}{L} + 1 \right\}.$$

Thus

$$\frac{1}{Q_w} = \frac{P_{diss}}{\omega U} = \frac{P_{endcaps} + P_{sidewalls}}{\omega U} = \frac{\frac{1}{2}\frac{R_s}{Z_0^2} J_0^2\left(j'_{0n}\right)\pi a L \left\{ 2\frac{\beta^2}{\beta_c^2}\frac{a}{L} + 1 \right\}}{\omega \frac{\pi L a^2}{4cZ_0}\frac{\beta_0^2}{\beta_c^2} J_0^2\left(j'_{0n}\right)}$$
$$= 2\frac{R_s}{Z_0}\frac{1}{\beta_0^3}\left\{ \frac{2}{L}\beta^2 + \frac{1}{a}\beta_c^2 \right\} = \frac{2}{j'_{0n}}\frac{R_s}{Z_0}\frac{1}{\beta_0^3}\left\{ \frac{2j'_{0n}}{p\pi}\beta^3 + \beta_c^3 \right\}$$

or,

$$\frac{1}{Q_w} = \frac{2}{j'_{0n}} \frac{R_s}{Z_0} \left\{ \left(\frac{2 j'_{0n}}{p\pi} \right) \left(1 - \frac{\beta_c^2}{\beta_0^2} \right)^{3/2} + \frac{\beta_c^3}{\beta_0^3} \right\},$$

or, for our 91.39GHz parameters,

$$Q_w = 9.02 \times 10^3 \left\{ \frac{2.44}{p} \left(1 - \frac{\beta_c^2}{\beta_0^2} \right)^{3/2} + \frac{\beta_c^3}{\beta_0^3} \right\}^{-1}.$$

In passing, we note that if the cutoff condition for the TE_{02} mode is employed,

$$\beta_0 = \beta_{c-TE02} = \frac{j'_{02}}{j'_{01}} \beta_{c-TE01} = 1.831 \beta_{c-TE01},$$

then only one free parameter remains, the longitudinal mode index *p*,

$$Q_w = 5.54 \times 10^4 \left\{ 1 + \frac{8.78}{p} \right\}^{-1},$$

and in this case the maximum wall *Q* is about 55,000.

In terms of wall *Q*, we may express the time and area averaged power dissipation per unit area as

$$\bar{S} = \overline{\frac{dP}{dA}} = \frac{1}{2\pi a^2 + 2\pi a L} \frac{\omega U}{Q_w},$$

and the maximum heating may be quantified in the form,

$$\eta_{endcaps} = \frac{1}{\bar{S}} \max \left. \frac{dP}{dA} \right|_{endcaps} = 2\pi a^2 \left(1 + \frac{L}{a} \right) Q_w \frac{\frac{R_s}{2Z_0^2} \frac{\beta^2}{\beta_c^2} J_1^2(j'_{1n})}{\omega \frac{\pi L a^2}{4cZ_0} \frac{\beta_0^2}{\beta_c^2} J_0^2(j'_{0n})}$$

$$= 2 \frac{J_1^2(j'_{1n})}{J_0^2(j'_{0n})} \left(1 - \frac{\beta_c^2}{\beta_0^2} \right) \frac{\left\{ \frac{j'_{0n}}{p\pi} \left(1 - \frac{\beta_c^2}{\beta_0^2} \right)^{1/2} + \frac{\beta_c}{\beta_0} \right\}}{\left\{ \frac{2 j'_{0n}}{p\pi} \left(1 - \frac{\beta_c^2}{\beta_0^2} \right)^{3/2} + \frac{\beta_c^3}{\beta_0^3} \right\}}$$

and for a cavity operated at cutoff for the TE_{02} mode ($\beta_0 = 1.831 \beta_{c1},\ a = 1.12\lambda,\ L = 0.6 p\lambda$) we have,

$$\eta_{endcaps} \approx 9.82 \left(\frac{p + 1.9}{p + 8.8} \right).$$

Notice that while the net dissipation on the endcaps is relatively small, the dissipation per unit area may be almost an order of magnitude larger than the average, in the case of a long cavity $L/a>>1$. As for the sidewalls,

$$\begin{aligned}\eta_{sidewalls} &= \frac{1}{\bar{S}}\max\left.\frac{dP}{dA}\right|_{sidewalls} = \eta_{endcaps}\left[\frac{\beta_c}{\beta}\frac{J_0\left(j'_{01}\right)}{J_1\left(j'_{10}\right)}\right]^2 \\ &= \eta_{endcaps}\frac{j'^2_{01}}{j'^2_{02}-j'^2_{01}}\left[\frac{J_0\left(j'_{01}\right)}{J_1\left(j'_{10}\right)}\right]^2 = 0.204\eta_{endcaps}\end{aligned},$$

and in the last line we assume operation at cutoff for the TE_{02} mode. Maximum temperature rise at a point on the surface subjected to dissipated power per unit area S, for time t, takes the form,

$$\Delta T(t) = \frac{2S}{\sqrt{\pi\kappa C}}t^{1/2},$$

where for room temperature copper the thermal conductivity and volume-specific heat capacity are,

$$\kappa = 401\frac{W}{°K-m}, \qquad C = 3.45\times 10^6\frac{J}{°K-m^3}.$$

Acceptable temperature rise is somewhere between the melting point of copper, 1356.5°K, and the cyclic stress limit for annealed copper, 40°C. To put this pulsed temperature rise expression in practical units, we express

$$S = \eta\bar{S} = \eta\frac{\omega U}{Q_w}\frac{1}{A},$$

with the cavity surface area $A = 2\pi a(a+L)$, and

$$t = \tau T_f = \frac{2Q_w}{\omega}\frac{\tau}{1+\beta},$$

where, here, β refers to the cavity coupling parameter and not the axial wavenumber. Putting these results together,

$$\Delta T = \frac{4}{\pi^{1/2}}\frac{1}{\left(\kappa C T_f\right)^{1/2}}\eta\frac{U}{A}\frac{\tau^{1/2}}{(1+\beta)},$$

with U in Joules, and a, L in meters. Operated at cutoff for the TE_{02} mode ($\beta_0 = 1.831\beta_{c1}$, $a \approx 1.12\lambda \approx 3.7\times 10^{-3}m$), and evaluted at the maximum (on the endcaps, at $r \approx 0.477a \approx 1.8\times 10^{-3}m$), and taking a long cavity ($\eta_{endcaps} \approx 9.82$), operated for maximum efficiency ($\tau \approx 1.26$), with $A \approx 4.6\times 10^{-5}m^2\left(p+1.9\right)$, we have

$$\Delta T = \frac{8.3\times 10^3\,^\circ K}{(p+1.9)}\frac{U(J)}{T_f\,(\mu s)^{1/2}}.$$

To illustrate these considerations. For the 5ft resonator, p=780 corresponds to cutoff for the TE_{02} mode, a wall Q of $5.6\text{x}10^4$, and a pulsed temperature rise $1.5\text{x}10^2$ °K for 5J energy stored, and 123ns pulse length with incident power 102.5MW (τ=1.26,β=1). At the other extreme, p=913 corresponds to a=1.08cm, a wall Q of $1.0\text{x}10^6$, and a pulsed temperature rise of $1.0\text{x}10^2$ °K for 5J energy stored, and 2.2μs pulse length with incident power to the resonator of 5.6MW (τ=1.26,β=1). TE_{0n} modes above cut-off in this case are those with

$$j'_{0n} < \frac{2\pi a}{\lambda} = 20.6,$$

and these are n=3, 4, 5, 6, and 7, corresponding to roots 10.17, 13.32, 16.47, and 19.6.[4]

TEM_{00} Mode in a Quasi-Optical Resonator

Given the high Q required by the circuit scalings, it is natural to consider also a quasi-optical resonator. Scalings may be found, for example, in Yariv's text.[5] The vacuum TEM_{00} mode electric field takes the form,

$$\vec{E}(\vec{r}) = E_0\frac{w_0}{w(z)}\exp\left\{-r_\perp^2\left(\frac{1}{w(z)^2}+\frac{ik}{2R(z)}\right)-ikz+i\eta(z)\right\},$$

with z is the propagation direction, w the waist, R the radius of curvature of the wavefront, and $k=2\pi/\lambda$. These quantities are related through the confocal parameter,

$$z_0 = \frac{\pi w_0^2}{\lambda},$$

according to

$$w(z) = w_0\left\{1+\left(\frac{z}{z_0}\right)^2\right\}^{1/2}, \qquad R(z) = z+\frac{z_0^2}{z}, \qquad \eta(z) = \tan^{-1}\left(\frac{z}{z_0}\right).$$

Here we have designated the point z=0 as the minimum waist position.

A resonator designed to operate in the TEM_{00} mode can be fashioned by placing spherical mirrors at two points in z, enclosing the origin and matching the wavefront radius of curvature at those points. A symmetrical resonator is symmetric about the waist position, *i.e.*, with one mirro at $z=-l/2$ and one at $z=l/2$, with l the mirror separation. For such a resonator one has the relations,

$$w_0 = \left(\frac{\lambda}{\pi}\right)^{1/2}\left(\frac{l}{2}\right)^{1/4}\left(R-\frac{l}{2}\right)^{1/4}, \qquad z_0^2 = \frac{1}{4}(2R-l)l,$$

$$w_1 = \left(\frac{\lambda l}{2\pi}\right)^{1/2} \left(\frac{2R^2}{l\left(R-\frac{1}{2}l\right)}\right)^{1/4}, \qquad \frac{w_1}{w_0} = \left(\frac{R}{R-\frac{1}{2}l}\right)^{1/2},$$

with w_1 the mode waist at the mirror. The resonance condition for the resonator takes the form,

$$kl - 2\tan^{-1}\left(\frac{l}{2z_0}\right) = N\pi.$$

A confocal resonator is one for which the mirrors share a common focal point, which is then at the center of the cavity,

$$R = l = 2z_0, \quad w_0 = \left(\frac{\lambda l}{2\pi}\right)^{1/2}, \frac{w_1}{w_0} = 2^{1/2}, \quad kl = \left(N + \frac{1}{2}\right)\pi,$$

or $N \approx 2l/\lambda$.

For illustration, with $l \equiv 5ns = 1.5m$, and $\lambda = 3.3\times 10^{-3} m$, we have,

$$w_0 = 2.8\times 10^{-2} m, \quad w_1 = 4.0\times 10^{-2} m, \quad R = 1.5m, \quad N \approx 909$$

Nearest modes are the TEM_{10} and TEM_{01}, separated in frequency by

$$\frac{c\Delta k}{2\pi} = \frac{c}{4l} \approx 50\, MHz.$$

To compute the Q of the cavity, let us neglect first diffraction losses due to finite mirror size, and consider the loss upon reflection from a conducting plane. Incident power per unit area, in a plane-wave, and absent the wall is

$$\frac{dP_{inc}}{dA} = \frac{1}{2} Z_0 \left|\tilde{H}\right|^2.$$

In the presence of the wall, and in steady-state, a reflected wave is setup, increasing the magnetic field amplitude by x2 at the wall, so that steady-state power dissipated in the wall is

$$\frac{dP_{diss}}{dA} = 2R_s \left|\tilde{H}\right|^2.$$

This implies that reflected power is reduced, on one bounce by,

$$P_{ref} = \left(1 - 4\frac{R_s}{Z_0}\right) P_{inc}.$$

Thus energy stored in the cavity must decay on a time-scale T_0,

$$\frac{dU}{dt} = -\frac{U}{T_0},$$

with

$$\frac{1}{T_0} = 4\frac{R_s}{Z_0}\frac{c}{l}.$$

The cavity wall Q is then given by

$$Q = \frac{1}{2}\omega T_0 = \frac{\pi}{8}\left(N + \frac{1}{2}\right)\frac{Z_0}{R_s} \approx 1.9\times 10^3 N.$$

For our example, $Q \approx 1.7\times 10^6$, with an energy decrement of 8.5×10^{-4} per bounce. We set the mirror size by asking that the diffraction loss per bounce be much smaller than this, let us say, 10% of this loss. Then

$$0.1\times 8.5\times 10^{-4} = \frac{\int_r^\infty r'dr' \exp\left(-\frac{2r'^2}{w_1^2}\right)}{\int_0^\infty r'dr' \exp\left(-\frac{2r'^2}{w_1^2}\right)} = \exp\left(-\frac{2r^2}{w_1^2}\right),$$

or a mirror radius of $r \approx 2.16 w_1 \approx 8.6\times 10^{-2} m$.

$\mathbf{TE_{10p}}$ Mode in a Planar Resonator

For the primary cavity of the matrix accelerator, energy storage in a rectangular pillbox is favored by the presumed simplicity of fabrication. The geometry of the problem, and the desire to maintain a single depth dimension suggest the TE_{10p} mode as a suitable candidate.

The TE_{10p} mode field components in a rectangular pillbox may be expressed as,

$$\tilde{E}_y = \sin(\beta_x x)\sin(\beta_z z), \qquad Z_0\tilde{H}_x = -j\frac{\beta}{\beta_0}\sin(\beta_x x)\cos(\beta z), \qquad Z_0\tilde{H}_z = j\frac{\beta_x}{\beta_0}\cos(\beta_x x)\sin(\beta z).$$

The wavenumbers and resonant frequency are

$$\beta_x = \frac{\pi}{a}, \qquad \beta = \frac{p\pi}{L}, \qquad \beta_0 = \frac{\omega}{c} = \sqrt{\beta_x^2 + \beta^2}.$$

The long dimension (z-dimension) is L, and the width (x-dimension) is a. We will denote the height (y-dimension) b. The wall Q may be computed from

$$\frac{1}{Q_w} = \frac{\delta}{2} \frac{\int_{wall} dS \left|\tilde{H}\right|^2}{\int_{volume} dV \left|\tilde{H}\right|^2},$$

with δ the skin-depth, and the result is

$$\frac{1}{Q_w} = \frac{R_s}{Z_0} \frac{2}{\beta_0} \left\{ \frac{1}{b} + \frac{2}{L} \frac{\beta^2}{\beta_0^2} + \frac{2}{a} \frac{\beta_x^2}{\beta_0^2} \right\} = \frac{4}{\pi} \frac{R_s}{Z_0} \left\{ \frac{\pi}{2\beta_0 b} + \frac{\left(1 + \frac{p^2 a^3}{L^3}\right)}{\left(1 + \frac{p^2 a^2}{L^2}\right)^{3/2}} \right\}.$$

The time-average Poynting flux into the walls, at the location of maximum dissipated power density on the walls, is

$$S_{\max} = \frac{\omega U_1}{V} \eta \delta \ ,$$

where $V = abL$ is the cavity volume, and

$$\eta = 2 \times \max \begin{cases} 1 - \frac{\beta_x^2}{\beta_0^2} \\ \frac{\beta_x^2}{\beta_0^2} \end{cases}.$$

Note that $\eta \geq 1$, with equality if $pa = L$.

For the matrix geometry, with $pa = L$, the depth $b = \lambda 2^{-1/2}$ and width $a = \lambda 2^{-1/2}$ are favored, if constrained to a single depth dimension, based on scalings for the secondary line, as described in a separate technical note. In this case,

$$Q_w \approx 3530.$$

Scalings for Two-Stage Matrix Accelerator Charge-Up

With these general features in mind, let us analyze the system depicted in Fig. 4. We suppose that our power source is a tube providing a square-pulse, of peak power P_t with a pulse length T_t and we take these quantities to be fixed. Values we have in mind are 50MW and 1μs. The energy U_s stored in one resonant energy storage line is determined from

$$\frac{1}{2} P_t = \xi \frac{U_s}{T_{0s}} \frac{\left(1 + \beta_s\right)^2}{\beta_s},$$

with

$$T_t = \frac{T_{0s}}{(1+\beta_s)}\tau,$$

with τ˜1.26. This cavity is discharged and provides then to each "primary" storage cavity a peak power,

$$\frac{1}{4}P_p = \frac{1}{4}\frac{U_s}{T_p} = \xi \frac{U_p}{T_{0p}}\frac{(1+\beta_p)^2}{\beta_p},$$

with pulse length (discharge time),

$$T_p = \frac{T_{0p}}{(1+\beta_p)}\tau,$$

just twice the length of the storage line, divided by the speed of light..

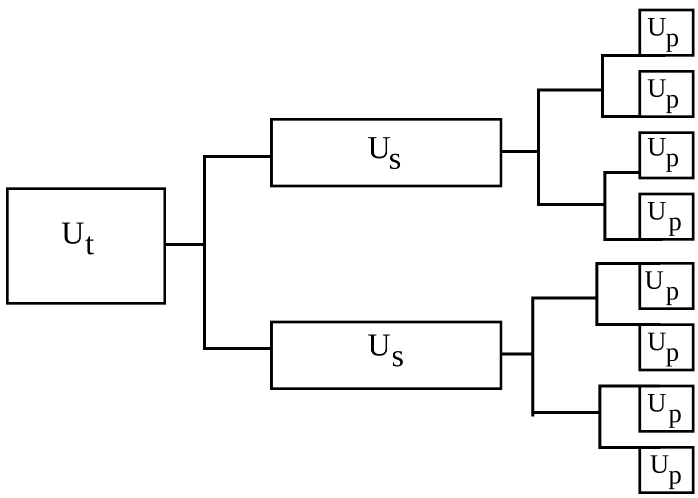

Fig. 4. RF distribution and pulse compression scheme for one tube-station on a matrixed linac. The tube produces energy U_t and energy U_S is stored in each of two resonant storage lines. These lines are discharged, and charge up eight "primary" storage cavities to energy U_p per cavitiy. These cavities are subsequently discharged into a linac structure.

In the meantime, there is a requirement on the energy stored per primary cell, U_1, based on the accelerator scalings. In the worst case this is 0.168J and this number could be as low as 0.128J, with improvements in the secondary line design. (The primary cell length for these illustrative numbers was 1.22mm). Thus the figure U_p determines the number of primary cells that may be powered by this tube,

$$N_p = 8\frac{U_p}{U_1},$$

and thus the number of tubes required per unit length. For our working example, we must power 820 primary cells.

Let us now proceed to rewrite these relations in a form more amenable to optimization. The total stored energy in the two resonant lines, prior to discharge, is

$$2U_s = U_t \frac{\beta_s}{(1+\beta_s)} \frac{1}{\xi\tau},$$

where $U_t = P_t T_t$ is the total energy in the initial rf pulse. The wall Q of the first storage line is constrained to

$$Q_s = \frac{1}{2}\omega T_{0s} = \frac{(1+\beta_s)}{2\tau}\omega T_t.$$

After discharge of the first storage line, and charge-up of the primaries, the total energy stored in the primaries is

$$8U_p = \underbrace{U_s}_{P_p T_p} \frac{2}{\xi\tau} \frac{(1+\beta_p)}{\beta_p} = U_t \frac{\beta_s}{(1+\beta_s)} \frac{\beta_p}{(1+\beta_p)} \frac{1}{\xi^2\tau^2} \approx 0.664 U_t \frac{\beta_s}{(1+\beta_s)} \frac{\beta_p}{(1+\beta_p)}.$$

The wall Q for the primaries is constrained to

$$Q_p = \frac{1}{2}\omega T_{0p} = \frac{(1+\beta_p)}{2\tau}\omega T_p.$$

Note that the net efficiency is

$$\eta_{t\to s\to p} = \frac{8U_p}{U_t} \approx 66.4\% \frac{\beta_s}{(1+\beta_s)} \frac{\beta_p}{(1+\beta_p)},$$

and is constrained by the optimal efficiency of a single stage (81.5%) squared. The number of primary cells powered by this tube is

$$N_p \approx 0.664 \frac{U_t}{U_1} \frac{\beta_s}{(1+\beta_s)} \frac{\beta_p}{(1+\beta_p)}.$$

Thus in the limit even of infinite wall Q's, the maximum number of primary cells one could power with 50J rf pulse is 198 (for 0.168J per cell) to 390 (at 0.128J per cell), corresponding to 4 tubes and 2 tubes respectively, for a 1 GeV linac.

Let us next set down a couple of examples, based on our investigation of wall Q's. Starting with a primary wall Q, $Q_p \approx 3530$. We have

$$T_p = \frac{2Q_p}{\omega} \frac{\tau}{(1+\beta_p)} \approx \frac{15.5ns}{(1+\beta_p)},$$

and this corresponds to a physical length

$$L_s = \frac{1}{2} c T_p \approx \frac{2.33 \times 10^2 cm}{\left(1+\beta_p\right)} \approx \frac{7.75 ft}{\left(1+\beta_p\right)}.$$

As for the *Q* of this storage cavity, we require

$$Q_s = \frac{\left(1+\beta_s\right)}{2\tau} \omega T_t \approx 2.28 \times 10^5 \left(1+\beta_s\right).$$

Since $Q_s > 5.5 \times 10^4$, we see that an overmoded cavity will be required. Let us suppose we employ a confocal resonator, so that

$$Q_s \approx 3.7 \times 10^3 \frac{L_s}{\lambda} \approx \frac{2.6 \times 10^6}{\left(1+\beta_p\right)}.$$

Combining these results, we have $\left(1+\beta_p\right)\left(1+\beta_s\right) \approx 11.5$. Subject to this constraint, it is not hard to show that efficiency is optmizied when $\beta_p \approx \beta_s \approx 2.4$, in which case $\eta_{t \to s \to p} \approx 33\%$, about 50% of the theoretical maximum. The wall *Q* is $Q_s \approx 7.7 \times 10^5$, and the cavity length is 69cm, or 2.3ft. In this case, one could power 99 primary cells with one tube, at 0.168J per cell, or 195 primary cells at 0.128J per cell. These numbers correspond to 8 tubes and 4 tubes respectively, for a 1 GeV linac. Each matrix accelerator structure would include either 12 or 24 cells, respectively.

Conclusions

To reach *Q*'s in excess of 55,000 in a TE_{01p} circular-tube resonator made of room-temperature copper, one must operate above cut-off for higher radial modes. High *Q*'s can be achieved in a confocal resonator geometry. If high-*Q*'s prove difficult, energy recovery on the hybrid-tee load arm may be worthwhile.

In the meantime, a working parameter set to aim for in design is:

For the storage line:

$Q_s \approx 7.7 \times 10^5$, $L_s = 69cm$, $\beta_s \approx 2.4$

For the primary

$Q_p \approx 3530$, 24 cells, $\beta_p \approx 2.4$

For the secondary

0.128J stored per primary cell (a $0.174c$ initial group velocity).

With these parameters we can contemplate a 4 tube 1 GeV linac. Moreover, with one 12.5MW tube, and one storage cavity, we could power a single accelerator up to the design no-load gradient, 1.13GeV/m.

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