

Resistive hose instability for arbitrary skin depth

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Resistive hose growth for moderate or large skin-depth differs from that derived from a conductivity model. Asymptotic growth is calculated for the Bennett equilibrium, incorporating the inductive component in the plasma return current, with phase-mixing accounted for via the "distributed-mass" model. Analytic scalings are checked with numerical solutions of the linearized model. For a ratio of skin-depth to beam radius less than $1/10$ the conductivity model appears quite adequate. When this ratio exceeds $1/4$ inductive corrections appear.

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Of all instabilities of a relativistic electron beam in plasma, the resistive hose effect is the most extensively studied.^{1,2,3,4,5,6,7,8} Yet to-date such studies have been limited to small plasma skin-depth, with analytic work relying solely on conductivity models. The existing literature offers no scaling appropriate in the limit of large skin-depth, where the inductive component in the plasma return current is important and it is precisely in this regime where numerous modern applications of the beam-plasma are to be found.⁹ In this work, asymptotic growth is computed including the inductive correction, following a treatment of the slab-beam problem.¹⁰

The equilibrium plasma is assumed stationary in the beam coordinate, τ , with spatial profile matched to the beam and constant collision rate ν . To the quasineutral Bennett equilibrium we consider first a *rigid* beam displacement, $Y\hat{y}$, with vanishing initial conditions at $\tau=0$. For Y varying slowly in τ , the perturbed state remains quasineutral and the plasma response may be expressed in terms of the perturbed plasma return current J_{e1} , driven by the perturbed axial vector potential A_1 according to

$$\frac{\partial J_{e1}}{\partial \tau} = -\frac{n_e e^2}{m c} \frac{\partial A_1}{\partial \tau} - \nu J_{e1}.$$

The point of departure from the work of Ref. 4 is the inclusion of the time derivative on the left.

The solution of the reduced Maxwell's equations takes the form, $A_1 = -D\partial A_0/\partial r$, where A_0 is the equilibrium Bennett pinch

potential, and the current centroid D is, after a Laplace transform in τ ,

$$\begin{aligned}\tilde{D} &= \frac{\Gamma p + \nu}{\nu p + \Gamma} \tilde{Y} \\ &= \frac{1}{1 + \varepsilon} \left\{ 1 + \varepsilon \frac{\Gamma}{p + \Gamma} \right\} \tilde{Y}.\end{aligned}\tag{1}$$

The Laplace transform variable is p and the tilde denotes the Laplace transform. The dimensionless parameter

$$\varepsilon = \frac{1}{8} k_e^2 a^2,\tag{2}$$

with k_e the plasma wavenumber on-axis and a the Bennett waist. This parameter relates the diffusion time-scale to the collision time, $\tau_D = \varepsilon/\nu$. The conductivity model is recovered by taking the limit $\varepsilon \rightarrow 0$. The decay rate $\Gamma = \nu/(1 + \varepsilon)$.

From the Vlasov equation one can show that momentum conservation takes the form,

$$\frac{\partial^2 \tilde{Y}}{\partial z^2} = -k_s^2 \left\{ \frac{p}{p + \Gamma} \right\} \tilde{Y},$$

where the slosh wavenumber k_s is

$$k_s^2 = \frac{1}{3} \frac{\varepsilon}{1 + \varepsilon} k_\beta^2.\tag{3}$$

The term k_β is the maximum betatron wavenumber $k_\beta^2 \sim 2v/a^2\gamma$, with γ the Lorentz factor for the beam, and the Budker parameter $v=I/I_0$, with $-I$ the beam current and $I_0 \sim 17\text{kA}$. Evidently the slosh wavenumber $k_{sB} = k_\beta/3^{1/2}$ derived from the conductivity model is relevant only when ε is large (small skin-depth). In particular, for small ε the beam centroid experiences *no restoring force*. This is due to the assumption of quasineutrality, *i.e.*, the assumption $|p| \ll \omega_e$. In the large skin-depth limit, the macroscopic restoring (or deflecting) force becomes small since the plasma return current near the beam becomes small. Previous work does not reveal this dependence of slosh wavenumber on skin-depth, having adopted the limit $\varepsilon \gg 1$.

Taking a unit initial displacement at $z=0$, for $\tau > 0$, the asymptotic form is, in the limit of a short pulse, $k_{sB}z \gg \Gamma\tau$,

$$Y(z, \tau) \approx 0.34 A^{-1/2} e^{-A} \cos\left(k_{sB}z - A + \frac{\pi}{8}\right),$$

where the exponent

$$A = (k_{sB}z\Gamma\tau)^{1/2}. \tag{4a}$$

$$= (k_{sB}z\nu\tau)^{1/2} \frac{\varepsilon^{1/4}}{(1+\varepsilon)^{3/4}}. \tag{4b}$$

The *form* of the exponent Eq. (4a) is just that of Eq. (82) in Ref. 4, however for given $\nu\tau$ and $k_{sB}z$, low growth favors large skin-depth (small ε).

To incorporate phase-mixing in the beam electron motion, we adopt the distributed mass model, representing the beam as an

ensemble of oscillators with displacement Y_α , with a distribution in betatron wavenumbers $\alpha^{1/2}k_\beta$ where $0 < \alpha < 1$. The beam centroid takes the form, $\bar{Y}(z, \tau) = \int d\alpha g(\alpha) Y_\alpha(z, \tau)$, where $g = 6\alpha(1-\alpha)$. The equations for the Y_α take the form,

$$\frac{\partial^2}{\partial z^2} Y_\alpha(z, \tau) = -\alpha k_\beta^2 \{ Y_\alpha(z, \tau) - D(z, \tau) \}, \quad (5)$$

where

$$D(z, \tau) = \frac{\Gamma}{v} Y(z, \tau) + \int_0^\tau d\tau' W(\tau - \tau') Y(z, \tau'), \quad (6)$$

after an inverse Laplace transform of Eq. (1). The kernel is

$$W(\tau) = \frac{\varepsilon}{1 + \varepsilon} \Gamma \exp(-\Gamma\tau). \quad (7)$$

To obtain the dispersion relation, Eq. (5) is Fourier transformed in z with wavenumber $k_z = k k_\beta$ and Laplace transformed in τ to reveal

$$\frac{p}{v} = \frac{\chi}{\varepsilon - \chi}, \quad (8)$$

the susceptibility is just that of Ref. 4

$$\chi = 3k^2 + 6k^4 \left\{ (1 - k^2) \ln \left(\frac{k^2 - 1}{k^2} \right) - 1 \right\}.$$

The conductivity model corresponds to the approximation $\varepsilon \gg |\chi|$ in the denominator of Eq. (8). For large ε , with p scaled by τ_D , there is a single solution to the dispersion relation. In general, however, there is a family of solutions parameterized by ε .

Performing inverse Laplace transforms one can show that

$$Y(z, \tau) \approx -\frac{I}{2\pi i} \int dk_z \exp(ik_z z + p\tau) \left(\frac{p + \Gamma}{\Gamma}\right) \frac{1}{k_z},$$

where the contour parallels the real k axis, displaced downward to avoid poles in the integrand, and $p=p(k)$ is given by Eq. (8).

The saturation amplitude is determined from a saddle-point calculation. Saturation occurs at

$$L_{sat} = -\tau I m \frac{\partial p}{\partial k},$$

and the envelope takes the form

$$Y \approx \left(\frac{2}{\pi}\right)^{1/2} \left(\tau k^2 \frac{\partial^2 p}{\partial k^2}\right)^{-1/2} \frac{p + \Gamma}{\Gamma} \exp(p\tau),$$

or

$$|Y| \approx Y_0 \frac{\exp(p_r \tau)}{(\Gamma \tau)^{1/2}}, \quad (9)$$

where $p_r = \text{Re} p$. The saddle point and the various derivatives are computed numerically and the results are found to be well-fit on the interval $10^{-4} < \varepsilon < 10^4$, by the following expressions in $\rho = 0.1 \ln[\varepsilon/(1+\varepsilon)]$,

$$\frac{p_r}{\Gamma} \approx 0.69 \exp\{-5.23\rho + 7.41\rho^2 + 3.81\rho^3\}, \quad (10)$$

$$\frac{p_i}{\Gamma} \approx 1.02 \exp\{-2.72\rho + 12.46\rho^2 + 6.86\rho^3\} \quad (11)$$

$$\frac{k_s L_{sat}}{\Gamma \tau} \approx 3.7 \exp\{-11.28\rho + 6.67\rho^2\}, \quad (12)$$

$$Y_0 \approx 0.52 \exp\{5.54\rho + 0.62\rho^2\}. \quad (13)$$

The exact results are depicted in Fig. 1 to illustrate the dependence on ε . Resonance occurs at frequencies $\pm p_i$ and wavenumbers $\mp k_r$, well-fit by

$$\frac{k_r}{k_s} \approx \sum_{n=0}^4 \kappa_n \mu^n, \quad (14)$$

where we abbreviate $\mu = \varepsilon \ln[\varepsilon/(1+\varepsilon)]$, and $-1 \leq \mu \leq 0$. The constants are $\kappa_0=1$, $\kappa_1=0.4915$, $\kappa_2=0.6525$, $\kappa_3=0.7947$, and $\kappa_4=0.5359$.

The results for saturation amplitude and length are compared to those from the numerical solution of Eq. (5) in Fig. 2, showing fair agreement. Evolution in z for $\nu\tau \sim 10.7$ is illustrated in Fig. 3.

We have seen that the effect of the induced current in the large skin-depth limit is to increase the saturation amplitude and length. For long range propagation, small skin-depth is favored. At the same time, for short range, growth tends to be less rapid, as reflected in

the exponent of Eq. (4). Roughly, when $\varepsilon > 10$, the conductivity model should be quite adequate.

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FIG. 1. Depicted is the dependence on skin-depth parameter ε , of **(a)** saturation length L_{sat} **(b)** saturation exponent p_r and **(c)** saturation amplitude factor. Overlaid are the approximate fits of Eqs. (10)-(12).

FIG. 2. Analytic results for **(a)** saturation length and **(b)** saturation amplitude from Eq. (9) are compared to the results of numerical solution of Eq. (5) giving fair agreement.

FIG. 3. Comparison of evolution in z for for $\varepsilon=0.5$ and $\varepsilon=2$ with $\nu\tau\sim 11$.

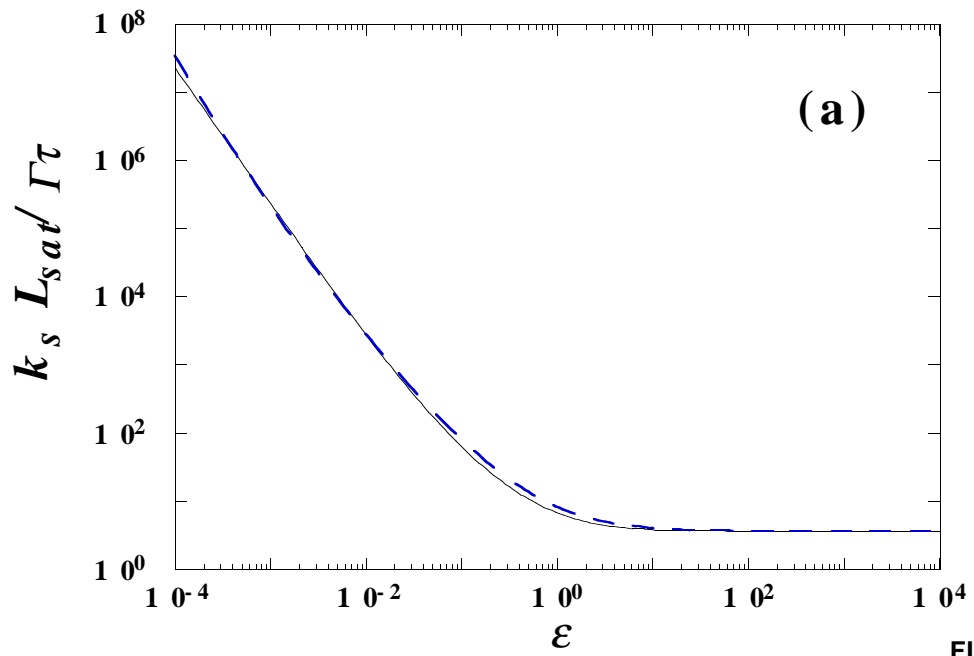


FIG. 1

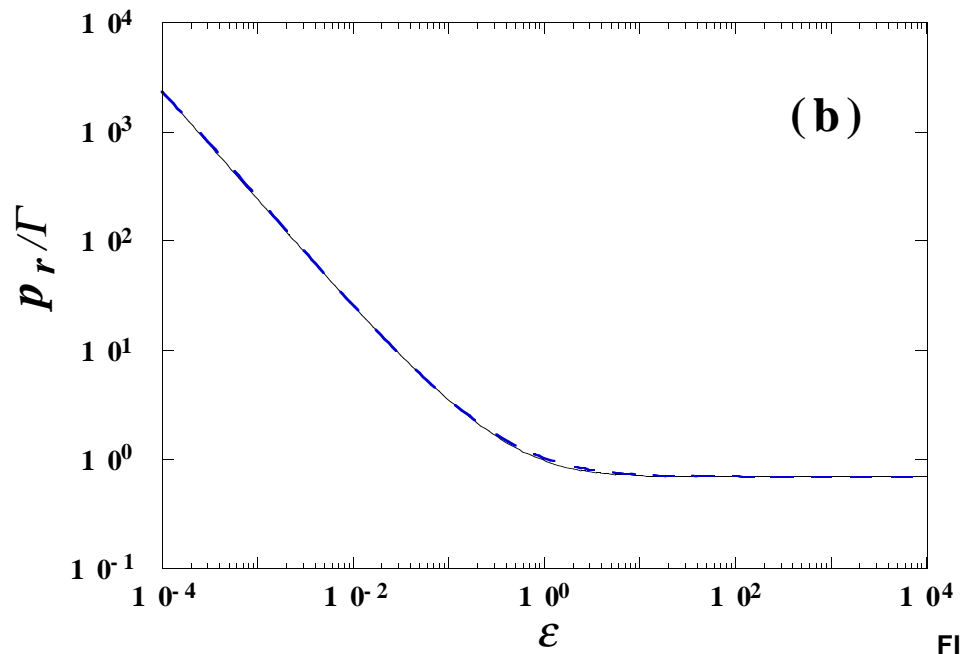


FIG. 1

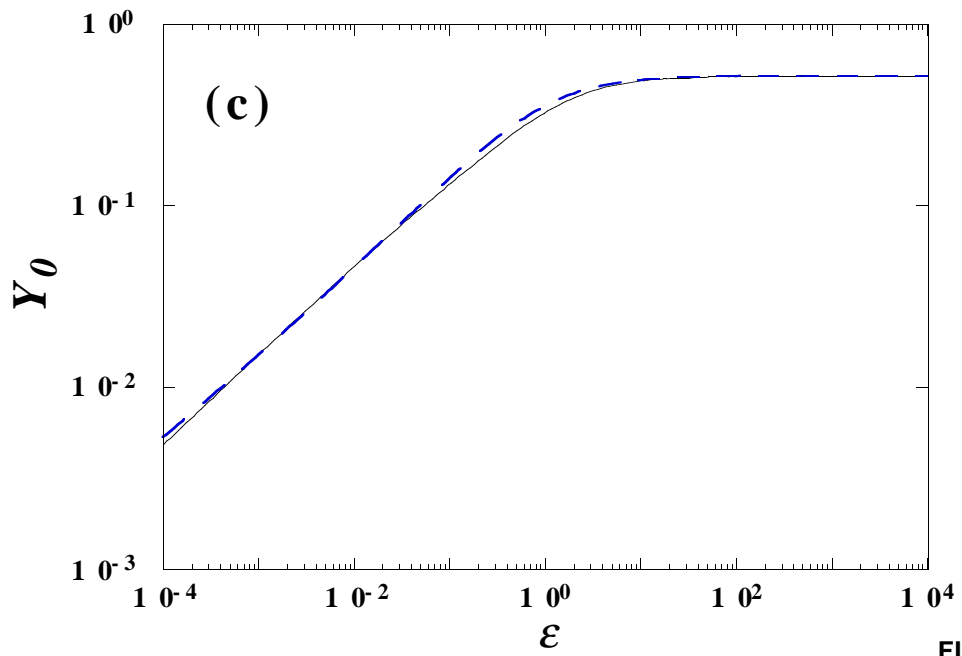


FIG. 1

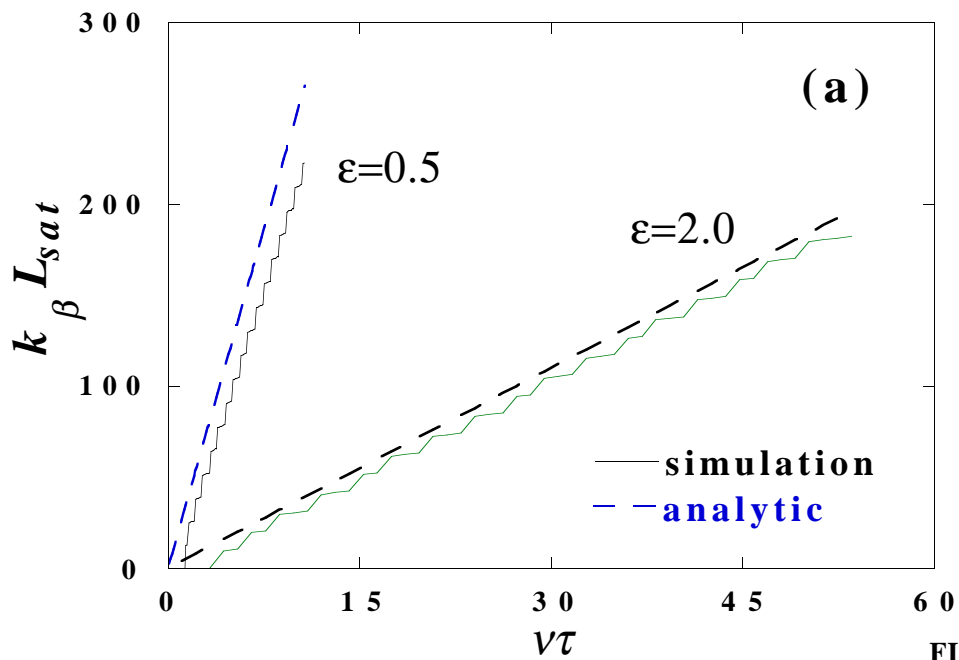


FIG. 2

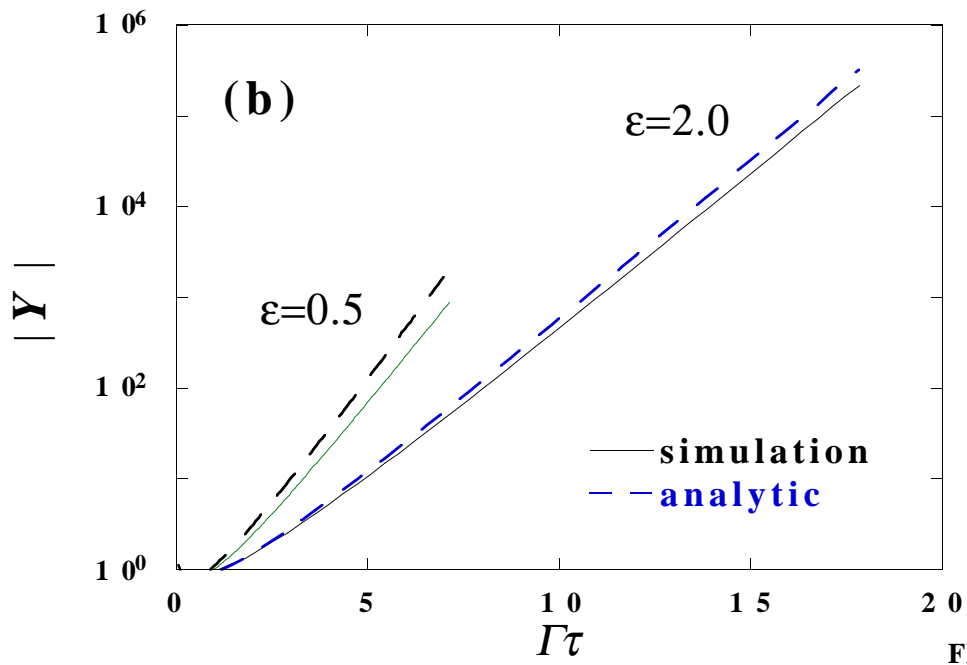


FIG. 2

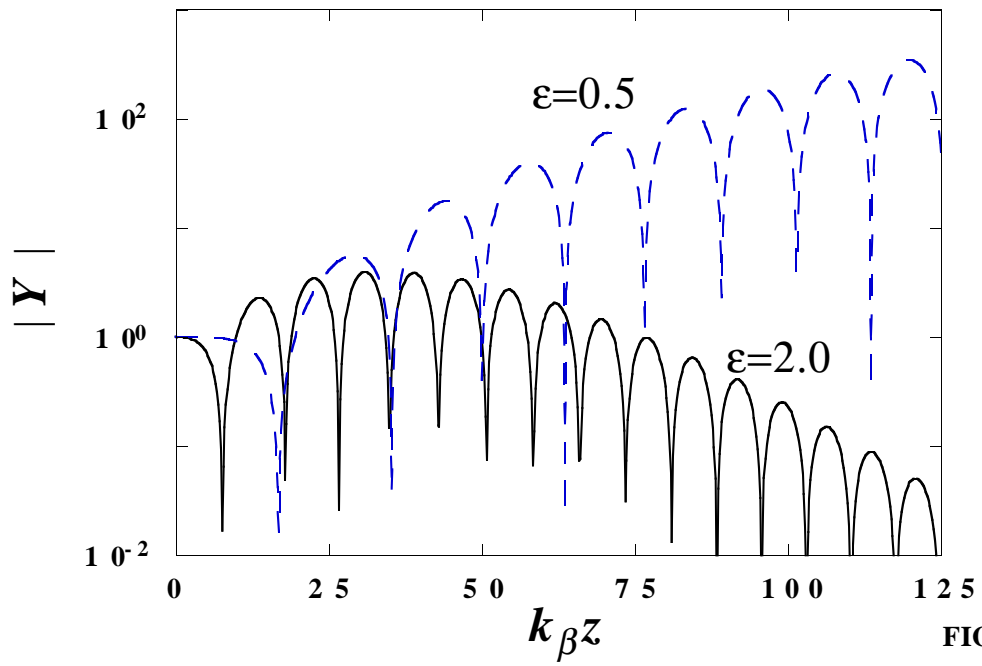


FIG. 3