Applied Physics 453C Microwave Linear Accelerators

Homework No. 1 Solutions

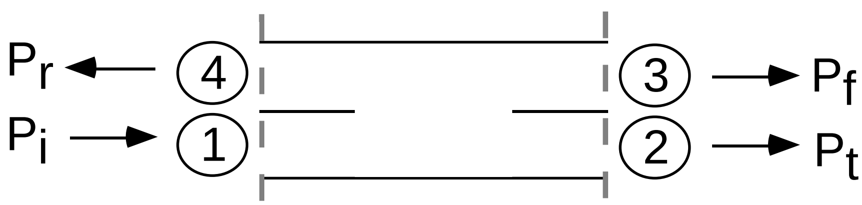
Problem 1.1 (Lorentz Reciprocity) Consider a volume V enclosed in a perfectly conducting boundary, and Maxwell's Equations in the frequency domain. Suppose that with a source term $\tilde{J}_1$ the solution to Maxwell's Equations corresponds to electric field $\tilde{E}_1$. Similarly, suppose that with source term $\tilde{J}_2$, a field $\tilde{E}_2$ results. Show that

$$\int_V \tilde{E}_1 \bullet \tilde{J}_2 dV = \int_V \tilde{E}_2 \bullet \tilde{J}_1 dV.$$

Hint: Consider $\nabla \bullet \left(\tilde{E}_1 \times \tilde{H}_2 - \tilde{E}_2 \times \tilde{H}_1\right)$.

It is claimed that an electromagnetic machine has been devised, that employs currents to produce microwave fields that then accelerate a beam of particles. The beam it is claimed, has only a small effect on the source currents. Explain qualitatively why this might be consistent with Lorentz Reciprocity. If further description of the system revealed inconsistencies with Lorentz Reciprocity, could you conclude that the system is unphysical?

Problem 1.2 (Helmholtz Theorem) Given a sufficiently well-behaved vector field $\vec{A}$, show that it may be represented as a sum of a gradient and a curl. You may want to make use of the relations, $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$, and $\nabla^2\vec{A} = \vec{\nabla}(\vec{\nabla}\bullet\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$. Is this decomposition unique? Can a vector field whose divergence vanishes within a finite volume V be represented as merely the curl of a vector, regardless of boundary conditions on ∂V?



Problem 1.3 (Directional Coupler) A directional coupler is a four port device described by coupling C and directivity D , defined with respect to the quantities indicated in the sketch: power incident on port 1 P_i, transmitted power to port 2, P_t, forward power on port 3 P_f and reverse power P_r on port 4. The coupling and directivity are

$$C = 10\log_{10}\left(\frac{P_i}{P_f}\right), \qquad \text{(coupling)} \qquad D = 10\log_{10}\left(\frac{P_f}{P_r}\right), \qquad \text{(directivity)}$$

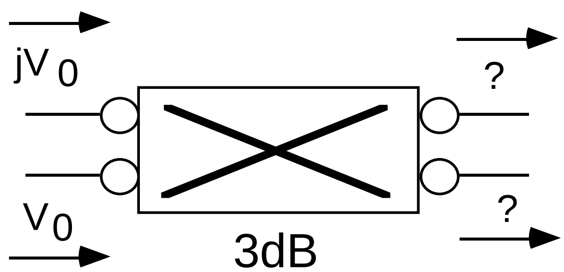
The isolation is defined according to

$$I = 10\log_{10}\left(\frac{P_i}{P_r}\right). \qquad \text{(isolation)}$$

An *ideal* directional coupler is described by the S-matrix,

$$\mathbf{S} = \begin{pmatrix} 0 & \alpha & j\sqrt{1-\alpha^2} & 0 \\ \alpha & 0 & 0 & j\sqrt{1-\alpha^2} \\ j\sqrt{1-\alpha^2} & 0 & 0 & \alpha \\ 0 & j\sqrt{1-\alpha^2} & \alpha & 0 \end{pmatrix}.$$

(a) What is the directivity of an ideal directional coupler? Relate the coupling C to the S-matrix parameter α. **(b)**The SLAC modified Bethe hole coupler has a coupling of 52dB and 80dB isolation. What is the directivity? **(c)** An ideal "3dB" coupler has C=10log$_{10}$2˜3dB and infinite directivity. What is the S-matrix? **(d)**Compute the outgoing signals for the situation depicted in the sketch, below, and explain how a 3dB coupler and a phase-shifter can be used to make a switch.



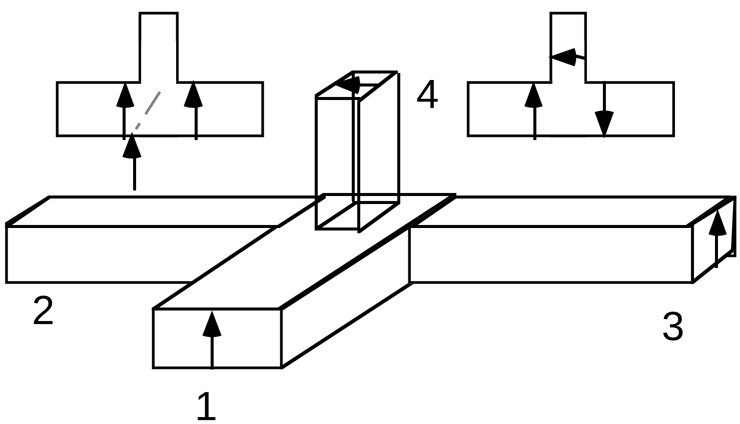


Illustration of a hybrid tee.

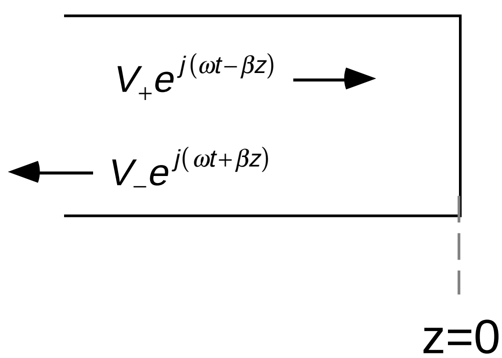
Problem 1.4 (**Magic Tee**) An ideal magic-tee is a four port device described by the S-matrix,

$$\mathbf{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

It is formally equivalent to a 3dB coupler, however, the natural choice of reference planes makes it easy to distinguish. The appearance is sketched in the Fig. 1. **(a)** Make a sketch of field lines to justify the form the S-Matrix takes. **(b)** Show that with voltages incident on arms 2 and 3, arm 4 provides a difference signal, and arm 3 a sum signal. **(c)** Explain the function of the setup depicted in Fig. 2 for powering two cavities with one source.

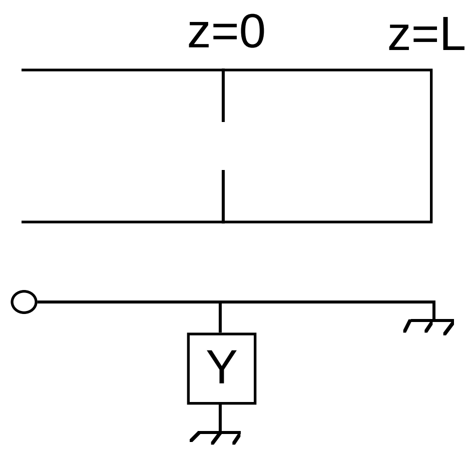
Problem 1.5 (**impedance transformation through a line**) Consider a transmission line with characteristic impedance Z_c, and a signal with wavenumber β on the line, as indicated in the sketch. Show that the impedance looking into terminal #1, $Z_1=V_1/I_1$, may be related to the impedance looking into terminal #2, $Z_2=V_2/I_2$, according to

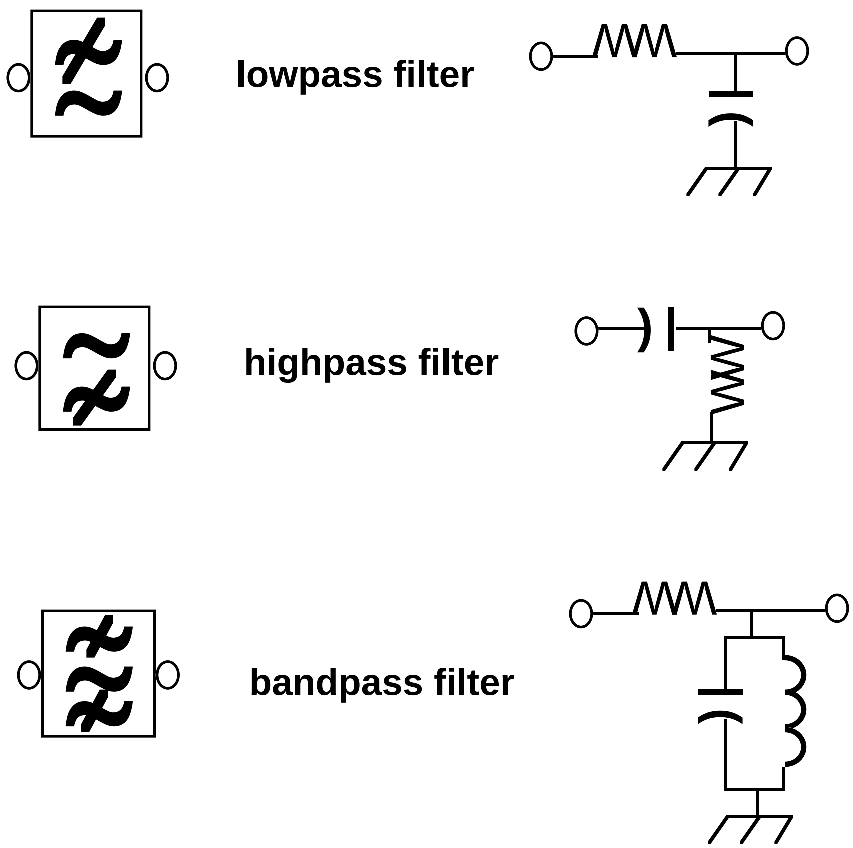
$$\frac{Z_2}{Z_c} = \frac{\frac{Z_1}{Z_c} - j\tan(\beta L)}{1 - j\frac{Z_1}{Z_c}\tan(\beta L)}.$$



Problem 1.6 Consider a waveguide with a conducting plate placed on one end, and operated below cutoff for all but the fundamental mode ("fundamental mode guide"). Determine the relation between forward and reflected wave amplitudes V_+ and V_- that satisifies conducting boundary conditions at z=0. What is the VSWR in this line? Suppose that a lossless obstacle were placed in the line, before the plate. In this case what would the VSWR be?

Problem 1.7 Consider fundamental mode lossless guide with an iris and a short as illustrated in the sketch. In terms of angular frequency ω, wavenumber β, and shunt admittance Y of the iris, determine the reflection coefficient as a function of z, the distance to the left of the iris. If the admittance is a pure susceptance, what is the VSWR on the line?

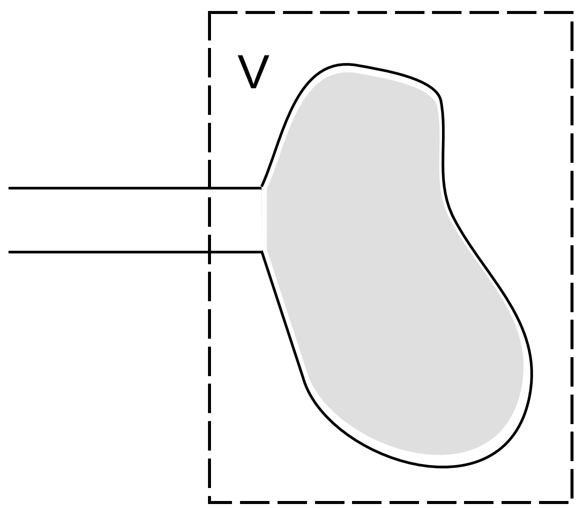




Problem 1.8 Solve for the transfer function of the three filters depicted below in terms of circuit parameters R,L,C and angular frequency ω. Solve also for the response of the circuits in the *time-doman*, by considering the response to a delta-function applied voltage. Why is the low-pass circuit also called an *integrator*, and under what conditions is the output voltage faithful to the integral of the input voltage? Under what conditions on the input voltage is the high-pass filter a *differentiator*?

Problem 1.9 **(a)** Express $\tilde{E} = \hat{y}\sin(\beta_c x)\exp(j\omega t - j\beta z)$ as a superposition of plane-waves. Compute the corresponding $\tilde{H}$. **(b)** Determine conditions on β_c such that these fields are a solution of Maxwell's Equations in an infinite medium of permeability μ and permittivity ε. **(c)** Show that at the planes $x=0$, $x=\pi/\beta_c$, $y=0$, and $y=b$ conducting boundary conditions are satisfied. Assuming $\beta=3^{1/2}\beta_c$, sketch the constant phase fronts in the x-z plane for each plane-wave in the superposition. What angle do the plane-wave components make with the z-axis? **(d)** If these signals were being generated at the $z=0$ plane in the waveguide, and the signal generator were turned off, how long would it take for the signals to begin to die off at a plane $z=L$ in the waveguide?

Problem 1.10 Using Green's theorem, show that if a waveguide mode has zero cutoff, then the axial electric field vanishes.



Problem 1.11 Consider a waveguide operated in fundamental mode, attached to a cavity, all encased in a perfect conductor. Considering a volume V as indicated in the adjacent sketch, show that the "impedance looking into the cavity" $Z = V_1 / I_1$ (with V_1 and I_1 the fundamental mode voltage and current coefficients) takes the form

$$Z = \frac{1}{|I_1|^2}\left\{\int_V \tilde{E} \bullet \tilde{J}^* + 4j\omega\int_V (w_m - w_e)\,dV\right\}.$$

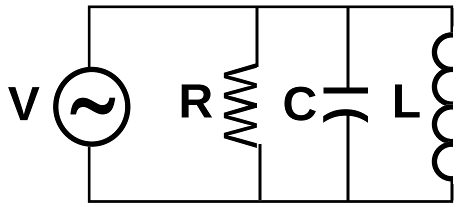
Expressing this as $Z=R+jX$, identify the equivalent resistance and reactance.

Problem 1.12 **(Foster's Reactance Theorem)** Consider the problem of Exercise 2.12, for the case of a lossless passive termination ($\tilde{J} = 0$). From Maxwell's Equations in the frequency domain, show that

$$\int_{\partial V}\left\{\tilde{E} \times \frac{\partial \tilde{H}^*}{\partial \omega} - \tilde{H} \times \frac{\partial \tilde{E}^*}{\partial \omega}\right\} \bullet d\vec{S} = -4j(w_m + w_e) = \tilde{V}\frac{\partial \tilde{I}^*}{\partial \omega} + \frac{\partial \tilde{V}^*}{\partial \omega}\tilde{I},$$

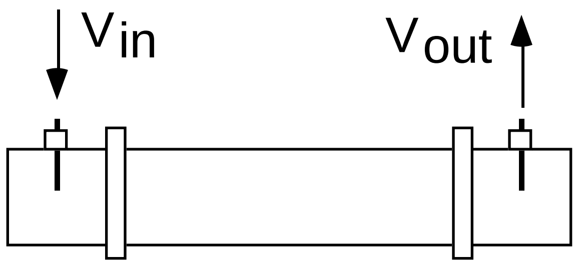
Using $\tilde{V} = jX\tilde{I}$, conclude that

$$\frac{\partial X}{\partial \omega} = 4\frac{\left(w_m + w_e\right)}{\tilde{I}\tilde{I}^*} > 0.$$



Problem 1.13 Confirm Foster's Theorem explicitly for the case of a parallel resonant circuit, as illustrated in the adjacent sketch.

Problem 1.14 A 50kW signal at 2856MHz is coupled into 2 feet of WR187. Neglecting reflections at the couplers, and attenuation, what is the power flowing out? Travelling with this signal is a 10μW signal at 4140MHz. Assuming that the guide width is twice the height ($a=2b$), is this system operating exclusively in the fundamental mode? Estimate the power flowing out for the 4140MHz signal.



Problem 1.15 Compute the attenuation constant for the TE_{10} mode of rectangular guide. Suggest a reasonable operating range for WR90.

Problem 1.16 Consider a beam consisting of a series of point bunches of charge Q_b space at intervals T,

$$I_b(t) = \sum_{n=-\infty}^{\infty} Q_b \delta(t - nT).$$

Show that this may be represented by the Fourier series,

$$I_b(t) = \frac{Q_b}{T}\sum_{m=-\infty}^{\infty} e^{jm\omega t},$$

with $\omega=2\pi/T$. In this way demonstrate that the first harmonic current component of a well-bunched beam $\tilde{I}_b = 2\bar{I}_b$, with $\bar{I}_b$ the average current within the pulse.

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$$\int_V \tilde{E}_1 \bullet \tilde{J}_2 dV = \int_V \tilde{E}_2 \bullet \tilde{J}_1 dV .$$

Hint: Consider $\nabla \bullet \left(\tilde{E}_1 \times \tilde{H}_2 - \tilde{E}_2 \times \tilde{H}_1 \right)$.

It is claimed that an electromagnetic machine has been devised, that employs currents to produce microwave fields that then accelerate a beam of particles. The beam it is claimed, has only a small effect on the source currents. Explain qualitatively why this might be consistent with Lorentz Reciprocity. If further description of the system revealed inconsistencies with Lorentz Reciprocity, could you conclude that the system is unphysical?

Solution: We make use of a vector identity from Appendix A, and Maxwell's Equations,

$$\begin{aligned}\nabla \bullet \left(\tilde{E}_1 \times \tilde{H}_2 \right) &= \tilde{H}_2 \bullet \nabla \times \tilde{E}_1 - \tilde{E}_1 \bullet \nabla \times \tilde{H}_2 \\ &= \tilde{H}_2 \bullet \left(-j\omega\mu \tilde{H}_1 \right) - \tilde{E}_1 \bullet \left(j\omega\varepsilon \tilde{E}_2 + \tilde{J}_2 \right),\end{aligned}$$

$$\begin{aligned}\nabla \bullet \left(\tilde{E}_2 \times \tilde{H}_1 \right) &= \tilde{H}_1 \bullet \nabla \times \tilde{E}_2 - \tilde{E}_2 \bullet \nabla \times \tilde{H}_1 \\ &= \tilde{H}_1 \bullet \left(-j\omega\mu \tilde{H}_2 \right) - \tilde{E}_2 \bullet \left(j\omega\varepsilon \tilde{E}_1 + \tilde{J}_1 \right).\end{aligned}$$

Subtracting these two, we find

$$\nabla \bullet \left(\tilde{E}_1 \times \tilde{H}_2 - \tilde{E}_2 \times \tilde{H}_1 \right) = -\tilde{E}_1 \bullet \tilde{J}_2 + \tilde{E}_2 \bullet \tilde{J}_1,$$

and integrating over the volume, making use of the perfectly conducting boundary conditions, we have,

$$\begin{aligned}\int dV \left(-\tilde{E}_1 \bullet \tilde{J}_2 + \tilde{E}_2 \bullet \tilde{J}_1 \right) &= \int dV \nabla \bullet \left(\tilde{E}_1 \times \tilde{H}_2 - \tilde{E}_2 \times \tilde{H}_1 \right) \\ &= \int d\vec{S} \bullet \left(\tilde{E}_1 \times \tilde{H}_2 - \tilde{E}_2 \times \tilde{H}_1 \right) \\ &= \int dA \left\{ \underbrace{\left(\hat{n} \times \tilde{E}_1 \right)}_{0} \bullet \tilde{H}_2 - \underbrace{\left(\hat{n} \times \tilde{E}_2 \right)}_{0} \bullet \tilde{H}_1 \right\}. \\ &= 0\end{aligned}$$

Accelerators function in some sense as transformers, drawing their power from large currents passing through low voltages, and depositing it in small currents passing through high voltages. As we have derived it, the results applies only to a lossless system. In general, one can say that if the system in question is closed (no energy leaving it), and if the permeability and permittivity are symmetric tensors (*"reciprocal medium"*) then the result still follows. Notable examples exist of non-reciprocal media, and devices employing them --- *e.g.*, isolators and circulators.

Problem 1.2 (Helmholtz Theorem) Given a sufficiently well-behaved vector field $\vec{A}$, show that it may be represented as a sum of a gradient and a curl. You may want to make use of the relations, $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$, and $\nabla^2\vec{A} = \vec{\nabla}(\vec{\nabla}\bullet\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$. Is this decomposition unique? Can a vector field whose divergence vanishes within a finite volume V be represented as merely the curl of a vector, regardless of boundary conditions on ∂V?

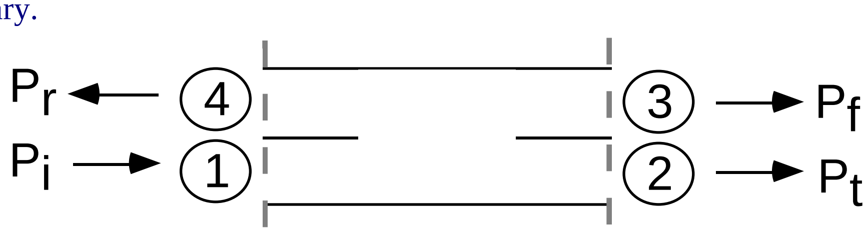
Solution: We consider a neighborhood of a point $\vec{r}$, and express the vector field $\vec{A}$ as an integral over a volume V enclosing $\vec{r}$,

$$\begin{aligned}\vec{A}(\vec{r}) &= \int dV\, \vec{A}(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,') \\ &= \int dV\, \vec{A}(\vec{r}\,')\left(-\frac{1}{4\pi}\right)\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \\ &= -\frac{1}{4\pi}\nabla^2\int dV\, \frac{\vec{A}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\ &= -\frac{1}{4\pi}\left(\vec{\nabla}\left[\vec{\nabla}\bullet\int dV\, \frac{\vec{A}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right] - \vec{\nabla}\times\left[\vec{\nabla}\times\int dV\, \frac{\vec{A}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right]\right)\end{aligned}$$

In this way one can see that the vector field may be represented locally as a sum of a gradient and a curl. Such a decomposition is not unique in general. For example consider

$$\vec{A}(\vec{r}) = \hat{x}y + \hat{y}x = \vec{\nabla}\times\left[\tfrac{1}{2}\left(y^2 - x^2\right)\hat{z}\right] = \vec{\nabla}(xy).$$

The short answer to the last question is that a field with vanishing divergence over some volume V is *solenoidal* (may be represented as a curl) provided the normal component vanishes at the boundary. A field with vanishing curl is *lamellar* (may be represented as a gradient) provided the tangential component vanishes at the boundary.



Problem 1.3 (Directional Coupler) A directional coupler is a four port device described by coupling C and directivity D , defined with respect to the quantities indicated in the sketch: power incident on port 1 P_i, transmitted power to port 2, P_t,

forward power on port 3 P_f and reverse power P_r on port 4. The coupling and directivity are

$$C = 10\log_{10}\left(\frac{P_i}{P_f}\right), \qquad \text{(coupling)}$$

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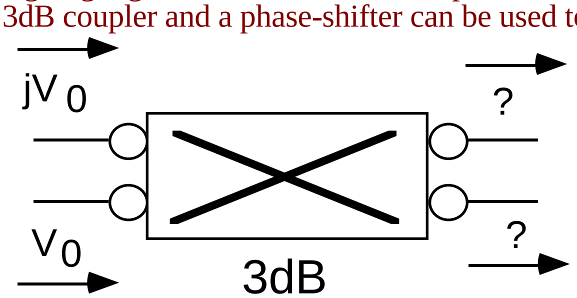
The isolation is defined according to

$$I = 10\log_{10}\left(\frac{P_i}{P_r}\right). \qquad \text{(isolation)}$$

An *ideal* directional coupler is described by the S-matrix,

$$\mathbf{S} = \begin{pmatrix} 0 & \alpha & j\sqrt{1-\alpha^2} & 0 \\ \alpha & 0 & 0 & j\sqrt{1-\alpha^2} \\ j\sqrt{1-\alpha^2} & 0 & 0 & \alpha \\ 0 & j\sqrt{1-\alpha^2} & \alpha & 0 \end{pmatrix}.$$

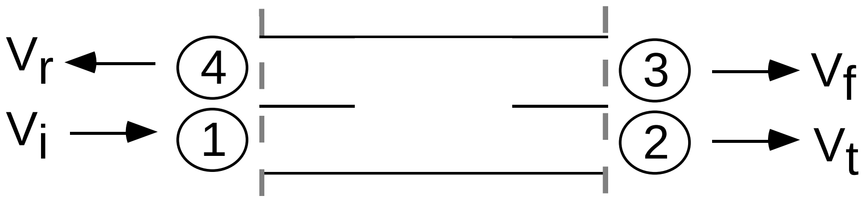
(a) What is the directivity of an ideal directional coupler? Relate the coupling C to the S-matrix parameter α. **(b)**The SLAC modified Bethe hole coupler has a coupling of 52dB and 80dB isolation. What is the directivity? **(c)** An ideal "3dB" coupler has $C=10\log_{10}2\tilde{}3$dB and infinite directivity. What is the S-matrix? **(d)**Compute the outgoing signals for the situation depicted in the sketch, below, and explain how a 3dB coupler and a phase-shifter can be used to make a switch.



Solution: (a) An ideal directional coupler satisfies,

$$\begin{pmatrix} 0 \\ V_t \\ V_f \\ V_r \end{pmatrix} = \begin{pmatrix} 0 & \alpha & j\sqrt{1-\alpha^2} & 0 \\ \alpha & 0 & 0 & j\sqrt{1-\alpha^2} \\ j\sqrt{1-\alpha^2} & 0 & 0 & \alpha \\ 0 & j\sqrt{1-\alpha^2} & \alpha & 0 \end{pmatrix} \begin{pmatrix} V_i \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where we define the voltages according to the following sketch



Since power is proportional to the squared modulus of these voltage phasors, we have

$$P_t = \alpha^2 P_i, \qquad P_f = \left(1-\alpha^2\right)P_i, \qquad P_r = 0.$$

The coupler characteristics are, in terms of the S-matrix parameter, α,

$$C = -10\log_{10}\left(1-\alpha^2\right),$$

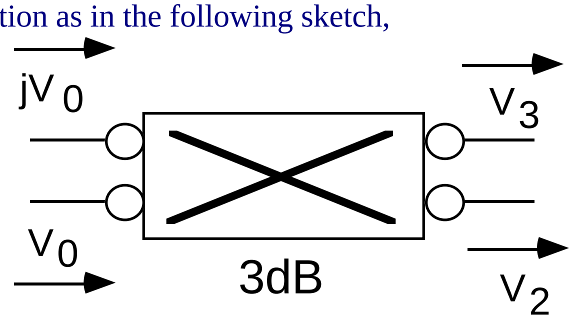
$$D = 10\log_{10}\left(\frac{1-\alpha^2}{0}\right) \to \infty, \quad \text{(directivity of an ideal directional coupler)}$$

$$I = 10\log_{10}\left(\frac{1}{0}\right) \to \infty. \qquad \text{(isolation of an ideal directional coupler)}$$

(b) From the definitions we see that $C+D=I$, so that D=80-52=28dB. **(c)** For a 3dB coupler we have

$$\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & j & 0 \\ 1 & 0 & 0 & j \\ j & 0 & 0 & 1 \\ 0 & j & 1 & 0 \end{pmatrix}.$$

(d) Adopting notation as in the following sketch,



,

the S-matrix tells us that

$$\begin{pmatrix} 0 \\ V_2 \\ V_3 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & j & 0 \\ 1 & 0 & 0 & j \\ j & 0 & 0 & 1 \\ 0 & j & 1 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \\ 0 \\ jV_0 \end{pmatrix},$$

or

$$V_2 = \frac{1}{\sqrt{2}}\left(V_0 + jjV_0\right) = 0,$$

$$V_3 = \frac{1}{\sqrt{2}}\left(jV_0 + jV_0\right) = \sqrt{2}\,jV_0.$$

Evidently a 180° phase-flip for the input signal on one of the arms, can redirect the output from the upper arm to the lower arm:

$$\begin{pmatrix} 0 \\ V_2 \\ V_3 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & j & 0 \\ 1 & 0 & 0 & j \\ j & 0 & 0 & 1 \\ 0 & j & 1 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \\ 0 \\ -jV_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}V_0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 1.4 (**Magic Tee**) An ideal magic-tee is a four port device described by the S-matrix,

$$\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

It is formally equivalent to a 3dB coupler, however, the natural choice of reference planes makes it easy to distinguish. The appearance is sketched in the Fig. 1. **(a)** Make a sketch of field lines to justify the form the S-Matrix takes. **(b)** Show that with voltages incident on arms 2 and 3, arm 4 provides a difference signal, and arm 3 a sum signal. **(c)** Explain the function of the setup depicted in Fig. 2 for powering two cavities with one source.

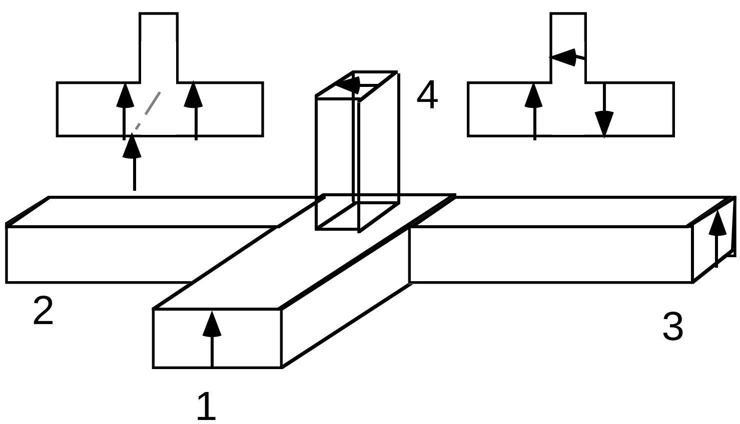
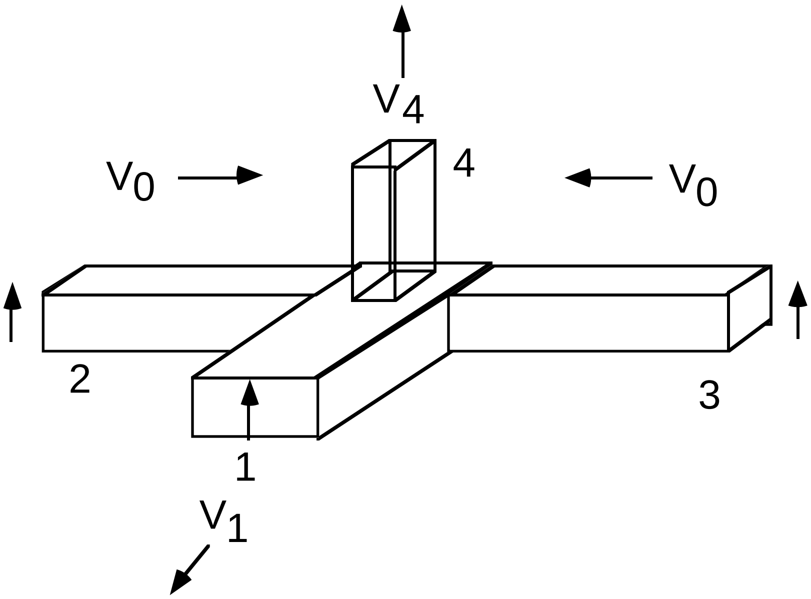


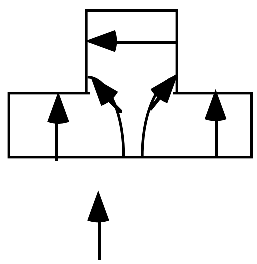
Illustration of a hybrid tee.

Solution: (a) To appreciate the S-matrix for this device consider two cases. The first consists of in-phase signals incident on lines 3 and 4:



In this case the S-matrix tells us that

$$\begin{pmatrix} V_1 \\ 0 \\ 0 \\ V_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ V_0 \\ V_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}V_0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$



i.e., the two signals interfere constructively on line 1, and destructively on line 4. One can check that this is reasonable by considering the field lines in the vicinity of the waveguide junction. From the symmetry of the geometry it is clear that the two signals should add in line 1, and subtract in line 2. Symmetry does not guarantee that the match into this device is perfect (*i.e.*, it does not guarantee zero reflected signal on lines 2 and 3), and in practice this requires the addition of a tuning post, and a step. These matching elements make the difference between a "four-port tee" and a "magic tee".

The second case consists of two signals 180° out of phase on lines 3 and 4, where one can show, by similar arguments, that the signals add constructively on line 4, and destructively on line 1. In general, line 1 provides a sum signal, and line 4, a difference signal. **(b)** To see this we simply apply the S-matrix.

$$\begin{pmatrix} V_1 \\ 0 \\ 0 \\ V_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ V_2 \\ V_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{V_2 + V_3}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{V_2 - V_3}{\sqrt{2}} \end{pmatrix}.$$

(c) This problem calls on our experience with the filling of a cavity. We know that, even for a critically coupled cavity, a large reflected signal results during the transient filling of the cavity. Were we to employ one klystron connected by

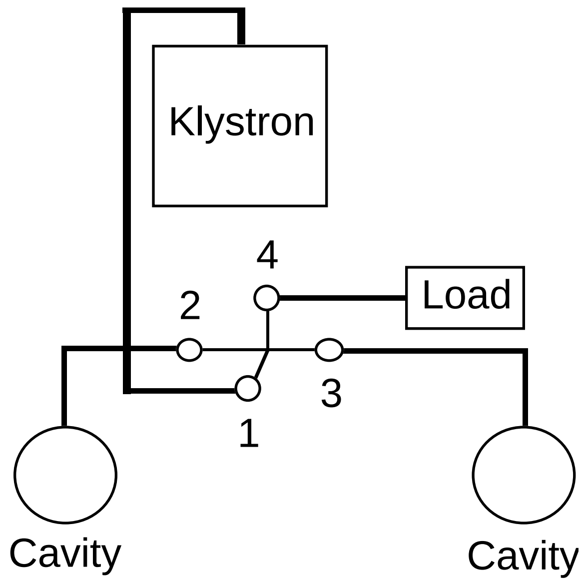
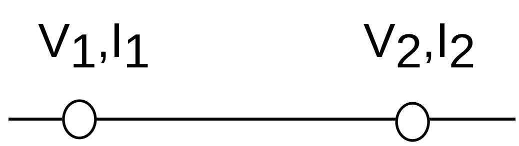


Fig. 2 Scheme for powering two cavities from one klystron.

waveguide (or any reciprocal network) to a single cavity, this reflected signal would return to the klystron, possibly damaging it. With the help of a magic-tee, however, we can avoid this problem, by powering two cavities, as illustrated in the adjacent sketch. Consider the case where the length of guide from port 2 to its cavity, differs by 1/4 of a guide wavelength from the length of guide from port 3 to its cavity. Assuming the cavities are identical, the reflected signals then appear at the magic tee 180° out of phase, and emerge through port 4 where they can be dissipated harmlessly in a load.



Problem 1.5 (**impedance transformation through a line**) Consider a transmission line with characteristic impedance Z_c, and a signal with wavenumber β on the line, as indicated in the sketch. Show that the impedance looking into terminal #1, $Z_1=V_1/I_1$, may be related to the impedance looking into terminal #2, $Z_2=V_2/I_2$, according to

$$\frac{Z_2}{Z_c}=\frac{\dfrac{Z_1}{Z_c}-j\tan(\beta L)}{1-j\dfrac{Z_1}{Z_c}\tan(\beta L)}.$$

Solution: In our study of transmission lines we noted that a fundamental-mode excitation on uniform guide could be described by the voltage and current waveforms consisting of a forward wave and a reflected wave,

$$\tilde{V}(z)=\tilde{V}_+e^{-j\beta z}+\tilde{V}_-e^{j\beta z},$$
$$Z_c\tilde{I}(z)=\tilde{V}_+e^{-j\beta z}-\tilde{V}_-e^{j\beta z}.$$

The impedance at a point z then takes the form,

$$\frac{Z(z)}{Z_c}=\frac{\tilde{V}(z)}{Z_c\tilde{I}(z)}=\frac{\tilde{V}_+e^{-j\beta z}+\tilde{V}_-e^{j\beta z}}{\tilde{V}_+e^{-j\beta z}-\tilde{V}_-e^{j\beta z}}=\frac{1+re^{2j\beta z}}{1-re^{2j\beta z}},$$

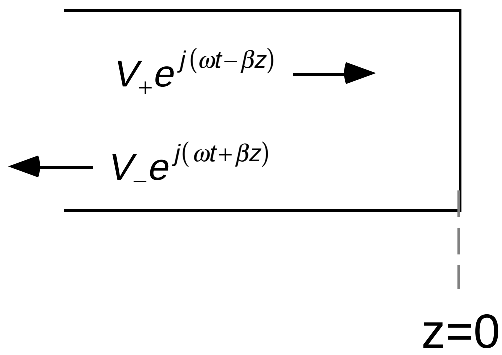
where we abbreviate the reflection coefficient $r=\tilde{V}_-/\tilde{V}_+$. Notice that one can solve for this coefficient in terms of the impedance at z=0---let us call it Z_1,

$$r=\frac{\frac{Z_1}{Z_c}-1}{\frac{Z_1}{Z_c}+1}.$$

The impedance at $z=L$ --- call it Z_2, may then be expressed as,

$$\begin{aligned}\frac{Z_2}{Z_c}&=\frac{1+\left(\dfrac{\frac{Z_1}{Z_c}-1}{\frac{Z_1}{Z_c}+1}\right)e^{2j\beta L}}{1-\left(\dfrac{\frac{Z_1}{Z_c}-1}{\frac{Z_1}{Z_c}+1}\right)e^{2j\beta L}}=\frac{\frac{Z_1}{Z_c}+1+\left(\frac{Z_1}{Z_c}-1\right)e^{2j\beta L}}{\frac{Z_1}{Z_c}+1-\left(\frac{Z_1}{Z_c}-1\right)e^{2j\beta L}}\\&=\frac{\frac{Z_1}{Z_c}\left(1+e^{2j\beta L}\right)+\left(1-e^{2j\beta L}\right)}{\frac{Z_1}{Z_c}\left(1-e^{2j\beta L}\right)+\left(1+e^{2j\beta L}\right)}=\frac{\frac{Z_1}{Z_c}\cos(\beta L)-j\sin(\beta L)}{-\frac{Z_1}{Z_c}j\sin(\beta L)+\cos(\beta L)}\\&=\frac{\frac{Z_1}{Z_c}-j\tan(\beta L)}{1-\frac{Z_1}{Z_c}j\tan(\beta L)}\end{aligned}$$

as claimed.



Problem 1.6 Consider a waveguide with a conducting plate placed on one end, and operated below cutoff for all but the fundamental mode ("fundamental mode guide"). Determine the relation between forward and reflected wave amplitudes V_+ and V_- that satisifies conducting boundary conditions at z=0. What is the VSWR in this line? Suppose that a lossless obstacle were placed in the line, before the plate. In this case what would the VSWR be?

Solution: The tangential electric field should vanish at the conducting plane. Thus the voltage,

$$\tilde{V}(z) = \tilde{V}_+ e^{-j\beta z} + \tilde{V}_- e^{j\beta z},$$

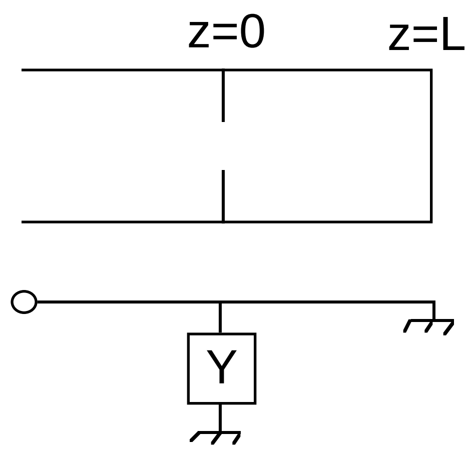
(by definition, the coefficient of the tangential electric field) should vanish at the plane,

$$\tilde{V}(z=0) = \tilde{V}_+ + \tilde{V}_- = 0,$$

and $\tilde{V}_+ = -\tilde{V}_-$. Without resorting to formulae and starting only with the definition of VSWR, one can see that the maximum voltage amplitude is $2|\tilde{V}_+|$, and the minimum is zero. Thus the ratio of these (the VSWR) is infinite. One can also arrive at this by appealing to previous results. Referring to the previous problem, $\tilde{V}_+ = -\tilde{V}_-$ corresponds to a reflection coefficent r=-1. In terms of the reflection coefficient, we found previously that the voltage standing wave ratio is

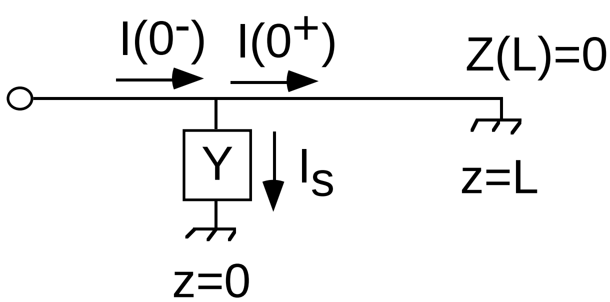
$$VSWR = \frac{1+|r|}{1-|r|},$$

and this is infinite. If an obstacle is placed in the line, and it is a lossless obstacle, then there is nowhere for the incident power to go, except back. Power is totally reflected. Thus moving well to the left of the obstacle, one must find equal amplitude forward and reflected waves, and the VSWR must again be infinite.



Problem 1.7 Consider fundamental mode lossless guide with an iris and a short as illustrated in the sketch. In terms of angular frequency ω, wavenumber β, and shunt admittance Y of the iris, determine the reflection coefficient as a function of z, the distance to the left of the iris. If the admittance is a pure susceptance, what is the VSWR on the line?

Solution: Answering the last question first, if the admittance is a pure susceptance, that means it is a lossless obstacle, and all incident power is totally reflected. Moving well to the left of the obstacle, one must then find equal amplitude forward and reflected voltage phasors, and the VSWR is then infinite, according to the reasoning of the previous problem. To compute the reflection coefficient, we make use of the impedance transformation on a line. Referring to the adjacent sketch, we see that *Z(L)*=0 since the voltage is zero there. The impedance *Z(0+)* is then related by impedance transformation through a line of length -*L*,



$$\frac{Z(0^+)}{Z_c} = \frac{\dfrac{Z(L)}{Z_c} + j\tan(\beta L)}{1 + j\dfrac{Z(L)}{Z_c}\tan(\beta L)} = j\tan(\beta L).$$

To transform the impedance across the iris, we recall that the shunt current is given by $I_s = YV(0)$, and thus

$$\frac{1}{Z(0^-)} = \frac{I(0^-)}{V(0)} = \frac{I(0^+) + I_s}{V(0)} = \frac{I(0^+)}{V(0)} + Y = \frac{1}{Z(0^+)} + Y.$$

Solving this we have,

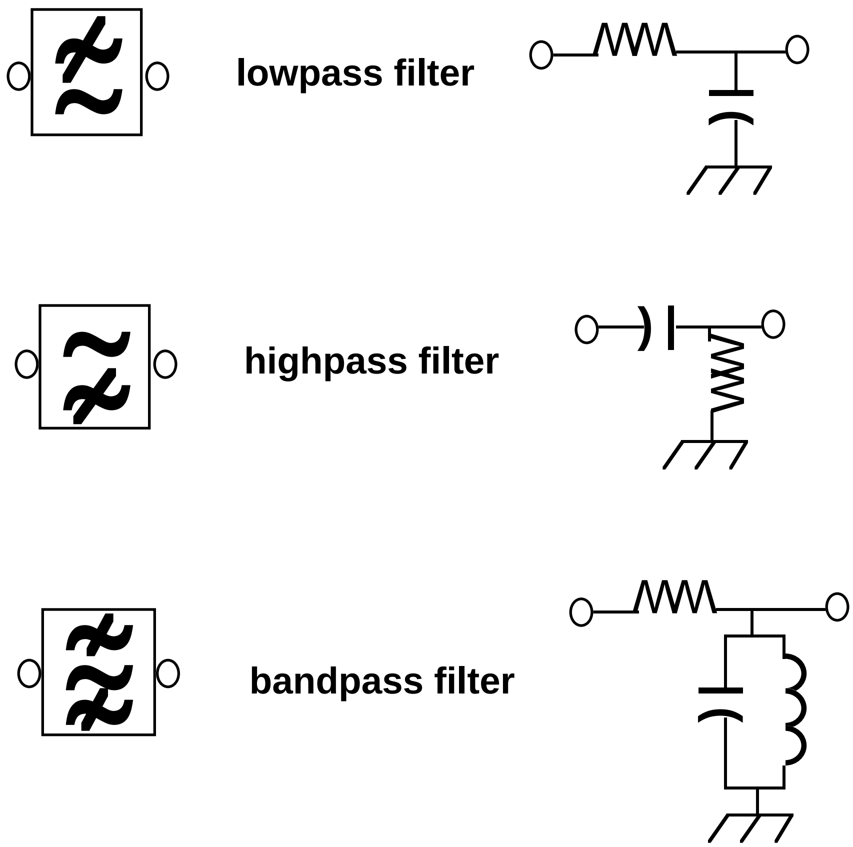
$$\frac{Z(0^-)}{Z_c} = \frac{j\tan(\beta L)}{1 + \hat{Y}j\tan(\beta L)},$$

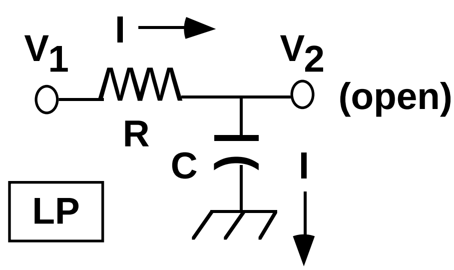
where $\hat{Y} = YZ_c$ is the normalized shunt admittance of the iris. The reflection coefficient at z=0- is then,

$$r = \frac{\frac{Z_1}{Z_c} - 1}{\frac{Z_1}{Z_c} + 1} = \frac{\left(1 - \hat{Y}\right)j\tan(\beta L) - 1}{\left(1 + \hat{Y}\right)j\tan(\beta L) + 1},$$

and the reflection coefficient at any point z to the left of the iris is simply $re^{2j\beta z}$.

Problem 1.8 Solve for the transfer function of the three filters depicted below in terms of circuit parameters *R,L,C* and angular frequency *ω*. Solve also for the response of the circuits in the *time-doman*, by considering the response to a delta-function applied voltage. Why is the low-pass circuit also called an *integrator*, and under what conditions is the output voltage faithful to the integral of the input voltage? Under what conditions on the input voltage is the high-pass filter a *differentiator*?





Solution: We consider the low-pass filter first, with the notation of the adjacent sketch. In terms of phasors, the charge on the capacitor satisfies $\tilde{Q} = C\tilde{V}_2$, so that the current flowing through it satisfies, $\tilde{I} = j\omega C\tilde{V}_2$. The voltage drop through the resistor is then $\tilde{V}_1 - \tilde{V}_2 = \tilde{I}R = j\omega RC\tilde{V}_2$. Evidently the open-circuit voltage is then

$$\tilde{V}_2 = \frac{\tilde{V}_1}{1 + j\omega\tau},$$

where we define a time-constant for the circuit, $\tau = RC$. One can see that high frequencies are suppressed, so that the circuit acts as a low-pass filter. The 3dB point is $\omega = 1/\tau$. Notice also that if the time scale τ is long compared to the time-scales characterizing the waveform V_1, then $\omega\tau >> 1$, and

$$\tilde{V}_2 \approx \frac{\tilde{V}_1}{j\omega\tau} \Leftrightarrow V_2(t) = \frac{1}{\tau}\int^t dt' V_1(t') \ .$$

In this limit, the circuit integrates the applied voltage. To solve for the response to a delta-function excitation, we rexpress the exact result as

$$(1 + j\omega\tau)\tilde{V}_2 = \tilde{V}_1 \Leftrightarrow V_2 + \tau\frac{dV_2}{dt} = V_1,$$

and integrate starting from t=0, with initial conditions $V_2(0) = 0$, $dV_2/dt(0^-) = 0$, subject to $V_1(t) = \delta(t)$. Thus

$$\begin{aligned}\int_{0^-}^{0^+} V_1 dt &= \int_{0^-}^{0^+} \delta(t) dt = 1 \\ &= \int_{0^-}^{0^+}\left(V_2 + \tau\frac{dV_2}{dt}\right)dt = \tau\left[V_2(0^+) - V_2(0^-)\right] = \tau V_2(0^+)\end{aligned}.$$

For t>0, V_2 satisfies,

$$V_2 + \tau\frac{dV_2}{dt} = 0,$$

and so the solution is $V_2(t) = e^{-t/\tau}$. Using superposition, one can show that the general solution must then be

$$V_2(t) = \frac{1}{\tau}\int_{-\infty}^{t} dt'\ V_1(t')\exp\left(\frac{t'-t}{\tau}\right),$$

One can check that this is the solution, by differentiating.

A similar analysis for the high-pass filter yields,

$$\tilde{V}_2 = \frac{j\omega\tau}{1 + j\omega\tau}\tilde{V}_1,$$

so that low-frequencies are suppressed. The 3-dB point is again $\omega=1/\tau$. If the time-scale τ is short compared to the time-scales characterizing the incident waveform, V_2, then

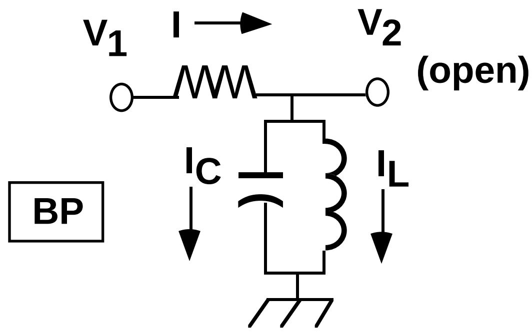
$$\tilde{V}_2 \approx j\omega\tau\tilde{V}_1 \Leftrightarrow V_2(t) = \tau\frac{dV_1}{dt},$$

and the circuit acts as a differentiator. To produce the output voltage V_2 from a delta-function impulse, we return to the exact result, writing it as

$$(1 + j\omega\tau)\tilde{V}_2 = j\omega\tau\tilde{V}_1 \Leftrightarrow V_2 + \tau\frac{dV_2}{dt} = \tau\frac{dV_1}{dt}.$$

From our analysis for the LP filter, we know that the solution takes the form,

$$V_2(t) = \int_{-\infty}^{t} dt'\frac{dV_1}{dt'}\exp\left(\frac{t'-t}{\tau}\right).$$



Analysis of the bandpass filter is a bit more involved. Adopting the notation of the adjacent sketch, and denoting the inductance L, resistance R, and capacitance C, one has the relations, from Kirchoff's Laws,

$$\tilde{V}_2 = \tilde{V}_1 - \tilde{I}R = j\omega L\tilde{I}_L = \frac{1}{j\omega C}\tilde{I}_C,$$
$$\tilde{I} = \tilde{I}_L + \tilde{I}_C.$$

Abbreviating

$$\omega_0 = \frac{1}{\sqrt{LC}},$$

the first relation implies,

$$\tilde{I}_L = -\frac{\omega_0^2}{\omega^2}\tilde{I}_C,$$

and the second implies

$$\tilde{I} = \left(1 - \frac{\omega_0^2}{\omega^2}\right)\tilde{I}_C = j\omega C\left(1 - \frac{\omega_0^2}{\omega^2}\right)\tilde{V}_2.$$

We then have,

$$\tilde{V}_1 = \tilde{V}_2 + \tilde{I}R = \tilde{V}_2 + \frac{R}{j\omega L}\left(1 - \frac{\omega^2}{\omega_0^2}\right)\tilde{V}_2,$$

and we may solve for the open-circuit voltage,

$$\tilde{V}_2 = \frac{\tilde{V}_1}{1 + \frac{R}{j\omega L}\left(1 - \frac{\omega^2}{\omega_0^2}\right)} = \frac{\tilde{V}_1}{1 - jQ\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)},$$

where we have abbreviated

$$Q = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}.$$

We can also express this in terms of a tuning angle, ψ,

$$\tan\psi = Q\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right),$$

as

$$\tilde{V}_2 = \frac{\tilde{V}_1}{1 - j\tan\psi} = \tilde{V}_1 \cos\psi e^{j\psi}.$$

We can put our results back in the time-domain, first rearranging terms,

$$\left\{1 - jQ\left(\tfrac{\omega_0}{\omega} - \tfrac{\omega}{\omega_0}\right)\right\}\tilde{V}_2 = \tilde{V}_1 \Rightarrow \left\{-\omega^2 + j\frac{\omega\omega_0}{Q} + \omega_0^2\right\}\tilde{V}_2 = j\frac{\omega\omega_0}{Q}\tilde{V}_1,$$

so that

$$\left\{\frac{d^2}{dt^2} + \frac{\omega_0}{Q}\frac{d}{dt} + \omega_0^2\right\}V_2 = \frac{\omega_0}{Q}\frac{dV_1}{dt}.$$

One can check by differentiating that the solution takes the form,

$$V_2(t) = \int_{-\infty}^{t} dt' \frac{\omega_0}{Q}\frac{dV_1}{dt'}\frac{\sin\Omega(t-t')}{\Omega}\exp\left[-\tfrac{\omega_0}{2Q}(t-t')\right],$$

with

$$\Omega = \omega_0\sqrt{1 - \frac{1}{4Q^2}}.$$

Problem 1.9 **(a)** Express $\tilde{E} = \hat{y}\sin(\beta_c x)\exp(j\omega t - j\beta z)$ as a superposition of plane-waves. Compute the corresponding $\tilde{H}$. **(b)** Determine conditions on β_c such that these fields are a solution of Maxwell's Equations in an infinite medium of permeability μ and permittivity ε. **(c)** Show that at the planes $x=0$, $x=\pi/\beta_c$, $y=0$, and $y=b$ conducting boundary conditions are satisfied. Assuming $\beta=3^{1/2}\beta_c$, sketch the constant phase fronts in the *x*-*z* plane for each plane-wave in the superposition. What angle do the plane-wave components make with the *z*-axis? **(d)** If these signals were being generated at the $z=0$ plane in the waveguide, and the signal generator were turned off, how long would it take for the signals to begin to die off at a plane $z=L$ in the waveguide?

Solution: In terms of plane-waves, this is

$$\begin{aligned}\tilde{E} &= \hat{y}\sin(\beta_c x)\exp(j\omega t - j\beta z)\\ &= \hat{y}\frac{1}{2j}\left(\exp(j\omega t - j\beta z + j\beta_c x) - \exp(j\omega t - j\beta z - j\beta_c x)\right).\end{aligned}$$

The magnetic field can be computed from Faraday's Law,

$$-j\omega\mu\tilde{H} = \vec{\nabla}\times\tilde{E} = \hat{x}\left(-\frac{\partial\tilde{E}_y}{\partial z}\right) + \hat{z}\left(\frac{\partial\tilde{E}_y}{\partial x}\right),$$

or

$$Z_c\tilde{H}_x = \tilde{E}_y,$$
$$Z_c\tilde{H}_z = j\frac{\beta_c}{\beta}\cos(\beta_c x)\exp(j\omega t - j\beta z),$$

where we abbreviate,

$$Z_c = \sqrt{\frac{\mu}{\varepsilon}}\frac{\omega}{\beta c}.$$

(b) The wave equation for the electric field takes the form,

$$0 = \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{-\beta_c^2} + \underbrace{\frac{\partial^2}{\partial y^2}}_{0} + \underbrace{\frac{\partial^2}{\partial z^2}}_{-\beta_z^2} \underbrace{-\mu\varepsilon\frac{\partial^2}{\partial t^2}}_{\mu\varepsilon\omega^2}\right)\tilde{E},$$

so that $\mu\varepsilon\omega^2 = \beta_c^2 + \beta_z^2$.

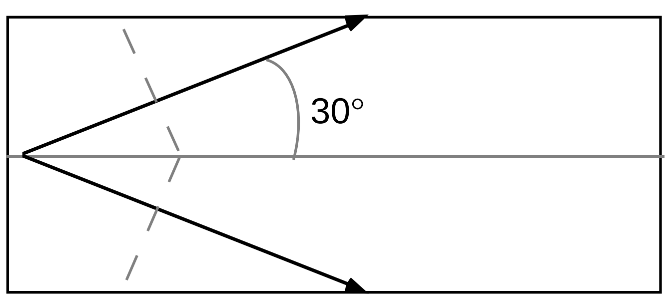
(c) Conducting boundary conditions require that normal H and tangential E vanish. This amounts to $\tilde{H}_x = 0$ at $x=0$ and $x=a$, and the condition $\tilde{E}_y = 0$ on the same planes. Since $\tilde{H}_x \propto \tilde{E}_y$, this is assured for $\beta_c = \pi / a$. Constant phase-fronts correspond to

$\omega t - \beta z \mp \beta_c x = \text{constant} = \omega t - \vec{k}_\pm \bullet \vec{r}$,

where $\vec{k}_\pm = \hat{z}\beta \pm \hat{x}\beta_c$. For $\beta = 3^{1/2}\beta_c$, we have

$$\frac{\vec{k}_\pm}{\left|\vec{k}_\pm\right|} = \hat{z}\frac{\sqrt{3}}{2} \pm \hat{x}\frac{1}{2} = \hat{z}\cos 30^\circ \pm \hat{x}\sin 30^\circ,$$

and these propagation vectors, representing the normals to the surfaces of constant phase, make an angle of 30° with respect to the z-axis, as illustrated in the adjacent sketch.



(d) To travel a length *2L* in the *z*-direction, one wave must travel a path length *2l*, where cos30°=*L/l*. The time required to travel the length *2l* is $2l\sqrt{\mu\varepsilon}$, where $1/\sqrt{\mu\varepsilon}$ is the speed of light in the medium. Thus the speed relative to the z-axis is

$$v_g = \frac{1}{\sqrt{\mu\varepsilon}}\cos 30^\circ .$$

(Observe that this is the same as $d\omega / d\beta$). The time for the turn-off of the signal to be observed at *z=L* is L/v_g.

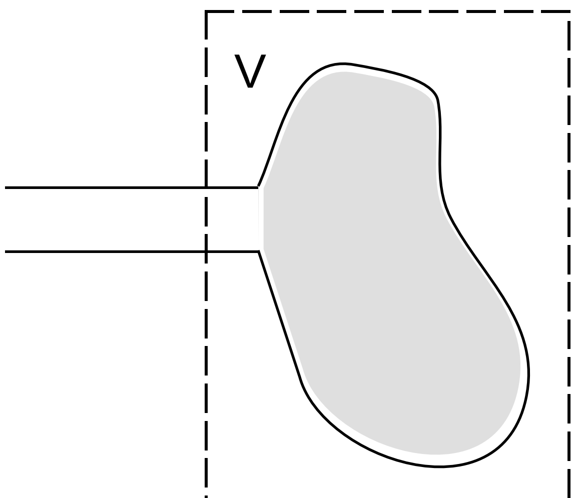
Problem 1.10 Using Green's theorem, show that if a waveguide mode has zero cutoff, then the axial electric field vanishes.

Solution: the Helmholtz equation in the waveguide is $\left(\nabla_\perp^2 + \beta_c^2\right)E_z = 0$, and for zero cutoff this is just. $\nabla_\perp^2 E_z = 0$. In this case, we have, from Green's theorem,

$$0 = \oint dl\, E_z \hat{n} \bullet \vec{\nabla}_\perp E_z = \int dA \left\{ E_z \nabla_\perp^2 E_z + \vec{\nabla}_\perp E_z \bullet \vec{\nabla}_\perp E_z \right\}$$

$$= \int dA \left| \vec{\nabla}_\perp E_z \right|^2$$

In the first line, the conducting boundary condition implies that the axial electric field vanishes on the waveguide periphery. The last line implies $\left|\vec{\nabla}_\perp E_z\right|=0$ everywhere in the waveguide cross-section, so that E_z is constant. Since E_z is zero on the boundary, E_z is zero everywhere in the waveguide.



Problem 1.11 Consider a waveguide operated in fundamental mode, attached to a cavity, all encased in a perfect conductor. Considering a volume *V* as indicated in the adjacent sketch, show that the "impedance looking into the cavity" $Z = V_1 / I_1$ (with V_1 and I_1 the fundamental mode voltage and current coefficients) takes the form

$$Z = \frac{1}{|I_1|^2}\left\{\int_V \tilde{E}\bullet\tilde{J}^* + 4j\omega\int_V (w_m - w_e)\,dV\right\}.$$

Expressing this as $Z=R+jX$, identify the equivalent resistance and reactance.

Solution: We start with Maxwell's Equations,

$$\tilde{E}\bullet\tilde{J}^* = \tilde{E}\bullet\left(\vec{\nabla}\times\tilde{H}^* + j\omega\varepsilon^*\tilde{E}^*\right),$$

and we employ a vector identity,

$$\vec{\nabla}\bullet\left(\tilde{E}\times\tilde{H}^*\right) = \tilde{H}^*\bullet\left(\vec{\nabla}\times\tilde{E}\right) - \tilde{E}\bullet\left(\vec{\nabla}\times\tilde{H}^*\right).$$

Combining these and employing Faraday's Law, $\vec{\nabla}\times\tilde{E} = -j\omega\mu\tilde{H}$, we obtain

$$\tilde{E}\bullet\tilde{J}^* = -j\omega\tilde{H}^*\bullet\mu\tilde{H} + j\omega\tilde{E}\bullet\varepsilon^*\tilde{E}^* - \vec{\nabla}\bullet\left(\tilde{E}\times\tilde{H}^*\right).$$

This we integrate over the volume, to obtain,

$$\begin{aligned} -\int_{\partial V} d\vec{S}\bullet\tilde{E}\times\tilde{H}^* &= -\int_V dV\,\vec{\nabla}\bullet\left(\tilde{E}\times\tilde{H}^*\right) \\ &= \int_V dV\,\tilde{E}\bullet\tilde{J}^* + j\omega\int_V dV\left(\tilde{E}\bullet\varepsilon^*\tilde{E}^* + \tilde{H}^*\bullet\mu\tilde{H}\right), \\ &= \int_V dV\,\tilde{E}\bullet\tilde{J}^* + 4j\omega\left(w_m - w_e\right) \end{aligned}$$

and in the last line we restrict consideration to a lossless medium. We can also express the Poynting Flux through the outward normal, down the guide, in terms of the voltage and current waveforms describing the waveguide fundamental mode excitation, taking account of a sign for the orientation,

$$VI^* = \int_V dV\,\tilde{E}\bullet\tilde{J}^* + 4j\omega\left(w_m - w_e\right).$$

This result we may express in terms of an impedance $Z=V/I$, as

$$Z = \frac{1}{II^*}\left\{\int_V dV\,\tilde{E}\bullet\tilde{J}^* + 4j\omega\left(w_m - w_e\right)\right\}.$$

Problem 1.12 (Foster's Reactance Theorem) Consider the problem of Exercise 2.12, for the case of a lossless passive termination ($\tilde{J}=0$). From Maxwell's Equations in the frequency domain, show that

$$\int_{\partial V}\left\{\tilde{E}\times\frac{\partial\tilde{H}^*}{\partial\omega} - \tilde{H}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right\}\bullet d\vec{S} = -4j\left(w_m + w_e\right) = \tilde{V}\frac{\partial\tilde{I}^*}{\partial\omega} + \frac{\partial\tilde{V}^*}{\partial\omega}\tilde{I},$$

Using $\tilde{V} = jX\tilde{I}$, conclude that

$$\frac{\partial X}{\partial\omega} = 4\frac{\left(w_m + w_e\right)}{\tilde{I}\tilde{I}^*} > 0.$$

Solution: Starting from Maxwell's Equations, we have

$$\vec{\nabla}\times\tilde{H}=j\omega\varepsilon\tilde{E}, \qquad \vec{\nabla}\times\tilde{E}=-j\omega\mu\tilde{H}.$$

Differentiating with respect to frequency we have,

$$\vec{\nabla}\times\frac{\partial\tilde{E}}{\partial\omega}=-j\tfrac{\partial(\omega\mu)}{\partial\omega}\tilde{H}-j\omega\mu\frac{\partial\tilde{H}}{\partial\omega}, \qquad \vec{\nabla}\times\frac{\partial\tilde{H}}{\partial\omega}=j\tfrac{\partial(\omega\varepsilon)}{\partial\omega}\tilde{E}+j\omega\varepsilon\frac{\partial\tilde{E}}{\partial\omega}.$$

Next we make use of a vector identity, combined with Maxwell's Equations,

$$\begin{aligned}\vec{\nabla}\bullet\left(\tilde{E}\times\frac{\partial\tilde{H}^*}{\partial\omega}\right)&=\frac{\partial\tilde{H}^*}{\partial\omega}\bullet\left(\vec{\nabla}\times\tilde{E}\right)-\tilde{E}\bullet\left(\vec{\nabla}\times\frac{\partial\tilde{H}^*}{\partial\omega}\right)\\&=\frac{\partial\tilde{H}^*}{\partial\omega}\bullet\left(-j\omega\mu\tilde{H}\right)-\tilde{E}\bullet\left(\vec{\nabla}\times\frac{\partial\tilde{H}^*}{\partial\omega}\right)\end{aligned},$$

$$\begin{aligned}\vec{\nabla}\bullet\left(\tilde{H}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right)&=\frac{\partial\tilde{E}^*}{\partial\omega}\bullet\left(\vec{\nabla}\times\tilde{H}\right)-\tilde{H}\bullet\left(\vec{\nabla}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right)\\&=\frac{\partial\tilde{E}^*}{\partial\omega}\bullet\left(j\omega\varepsilon\tilde{E}\right)-\tilde{H}\bullet\left(\vec{\nabla}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right)\end{aligned}.$$

Subtracting these we have then

$$\begin{aligned}\vec{\nabla}\bullet\left(\tilde{E}\times\frac{\partial\tilde{H}^*}{\partial\omega}-\tilde{H}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right)&=\frac{\partial\tilde{H}^*}{\partial\omega}\bullet\left(-j\omega\mu\tilde{H}\right)-\tilde{E}\bullet\left\{-j\tfrac{\partial(\omega\varepsilon^*)}{\partial\omega}\tilde{E}^*-j\omega\varepsilon^*\frac{\partial\tilde{E}^*}{\partial\omega}\right\}\\&\quad-\frac{\partial\tilde{E}^*}{\partial\omega}\bullet\left(j\omega\varepsilon\tilde{E}\right)+\tilde{H}\bullet\left\{j\tfrac{\partial(\omega\mu^*)}{\partial\omega}\tilde{H}^*+j\omega\mu^*\frac{\partial\tilde{H}^*}{\partial\omega}\right\},\\&=j\tilde{E}\bullet\tfrac{\partial(\omega\varepsilon^*)}{\partial\omega}\tilde{E}^*+j\tilde{H}\bullet\tfrac{\partial(\omega\mu^*)}{\partial\omega}\tilde{H}\end{aligned}$$

and we have assumed the medium is reciprocal and lossless. Integrating over the volume we have,

$$\int_{\partial V}d\vec{S}\bullet\left(\tilde{E}\times\frac{\partial\tilde{H}^*}{\partial\omega}-\tilde{H}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right)=4j\left(w_e+w_m\right).$$

This surface integral may be related to the voltage and current waveforms in the connecting guide according to

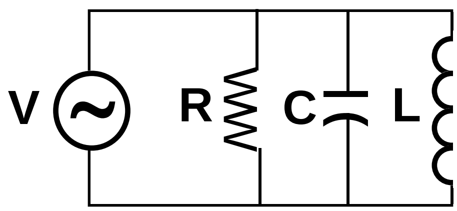
$$\int_{\partial V}d\vec{S}\bullet\left(\tilde{E}\times\frac{\partial\tilde{H}^*}{\partial\omega}-\tilde{H}\times\frac{\partial\tilde{E}^*}{\partial\omega}\right)=\tilde{V}\frac{\partial I^*}{\partial\omega}-I\frac{\partial V^*}{\partial\omega},$$

and using $\tilde{V}=jX\tilde{I}$, we have then

$$4j\left(w_e+w_m\right)=\tilde{V}\frac{\partial I^*}{\partial\omega}-I\frac{\partial}{\partial\omega}\left(-jX^*\tilde{I}^*\right)=jI\tilde{I}^*\frac{\partial X}{\partial\omega},$$

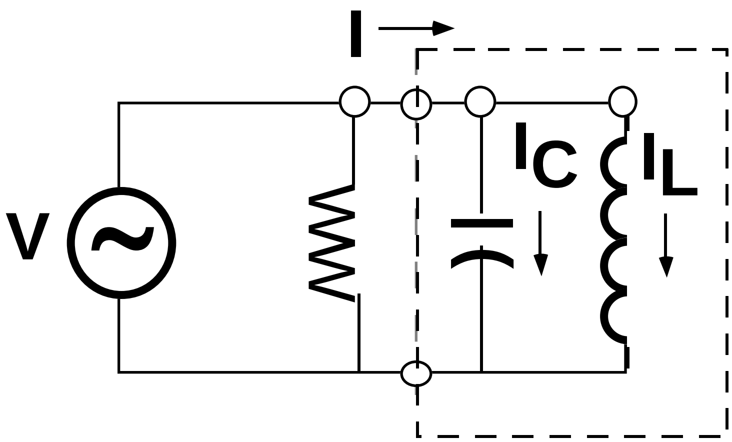
using the fact that χ is real. Thus we obtain,

$$\frac{\partial X}{\partial \omega} = \frac{4}{II^*}\left(w_e + w_m\right) > 0.$$



Problem 1.13 Confirm Foster's Theorem explicitly for the case of a parallel resonant circuit, as illustrated in the adjacent sketch.

Solution: Foster's theorem applies to a lossless system, so it is natural to consider that portion of the ciruit consisting of the parallel inductance and capacitance, as depicted in the adjacent sketch.



The equivalent impedance of the inductance and capacitance in parallel is

$$\frac{1}{Z} = \frac{1}{j\omega L} + j\omega C,$$

or $Z=jX$, with

$$\chi = \left(\frac{1}{\omega L} - \omega C\right)^{-1} = \frac{1}{C}\frac{\omega}{\omega_0^2 - \omega^2}.$$

Thus

$$\frac{\partial \chi}{\partial \omega} = \frac{1}{C}\frac{\omega_0^2 + \omega^2}{\left(\omega_0^2 - \omega^2\right)^2}.$$

To relate this to the conclusion of Foster's Theorem, we should compute the energy stored in the circuit as a function of the current I flowing into it. Kirchoff's Laws tell us

$$\tilde{V} = \frac{\tilde{I}_c}{j\omega C} = j\omega L \tilde{I}_L,$$

$$\tilde{I} = \tilde{I}_c + \tilde{I}_L.$$

With these one may show

$$\tilde{I}_C = \frac{\omega^2}{\omega^2 - \omega_0^2}\tilde{I},$$

$$\tilde{I}_L = \frac{\omega_0^2}{\omega_0^2 - \omega^2}\tilde{I}.$$

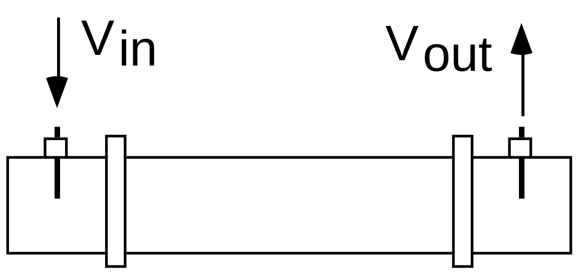
These permit us to express the stored energy in terms of current,

$$\begin{aligned} w_e + w_m &= \frac{1}{4C\omega^2}\left|\tilde{I}_C\right|^2 + \frac{L}{4}\left|\tilde{I}_L\right|^2 \\ &= \frac{1}{4C\omega^2}\left(\frac{\omega^2}{\omega^2 - \omega_0^2}\right)^2\left|\tilde{I}\right|^2 + \frac{L}{4}\left(\frac{\omega_0^2}{\omega^2 - \omega_0^2}\right)^2 |I|^2, \\ &= \frac{1}{4C}\frac{\omega^2 + \omega_0^2}{\left(\omega^2 - \omega_0^2\right)^2}\left|\tilde{I}\right|^2 \end{aligned}$$

and thus to confirm,

$$\frac{4}{\left|\tilde{I}\right|^2}\left(w_e + w_m\right) = \frac{1}{C}\frac{\omega^2 + \omega_0^2}{\left(\omega^2 - \omega_0^2\right)^2} = \frac{\partial \chi}{\partial \omega}.$$

Problem 1.14 A 50kW signal at 2856MHz is coupled into 2 feet of WR187. Neglecting reflections at the couplers, and attenuation, what is the power flowing out? Travelling with this signal is a 10μW signal at 4140MHz. Assuming that the guide width is twice the height (a=2b), is this system operating exclusively in the fundamental mode? Estimate the power flowing out for the 4140MHz signal.



Solution: "WR187" corresponds to a wide dimension a=187/100"=4.75cm. Cutoff occurs when the free-space wavelength $\lambda = 2a$=9.5cm. For 2856MHz the free space wavelength is λ=c/2856MHz=10.5cm, so the S-Band signal is cutoff. To estimate the signal coupled through the guide, consider the adjacent sketch. The output voltage amplitude is

$$\begin{aligned}|V_{out}| &= |V_{in}|\exp\{-|\beta|L\} = |V_{in}|\exp\left\{-L\sqrt{\left(\frac{\pi}{a}\right)^2 - \left(\frac{\omega}{c}\right)^2}\right\} \\ &= |V_{in}|\exp\left\{-2.54\times 12\times 2\sqrt{\left(\frac{\pi}{4.75}\right)^2 - \left(\frac{2\pi}{10.5}\right)^2}\right\}, \\ &= |V_{in}|\exp\{-17.2\}\end{aligned}$$

or

$$A = 10\log_{10}\frac{P_{out}}{P_{in}} = 20\log_{10}\frac{|V_{out}|}{|V_{in}|} = -149dB.$$

Since

$$10\log_{10}\frac{50kW}{1mW} = 77dBm,$$

the output signal is 77dBm-149dB=-72dBm. Meanwhile, 4140MHz corresponds to a 7.2cm free-space wavelength, and this is not cutoff. This signal is

$$10\log_{10}\frac{10\mu W}{1mW} = -20dBm,$$

and so is about 5 orders of magnitude larger than the S-Band signal at the output.

The cutoff for the TE_{20} mode occurs at a free-space wavelength of 9.5cm/2=4.75cm, and this is shorter than any of the wavelengths we are considering --- so the guide is operated in fundamental mode.

Problem 1.15 Compute the attenuation constant for the TE_{10} mode of rectangular guide. Suggest a reasonable operating range for WR90.

Solution: The attenuation constant is given by

$$\alpha = \frac{1}{2}R_s\frac{\oint dl\,|\tilde{H}|^2}{\Re\int d\vec{S}\bullet\tilde{E}\times\tilde{H}^*}.$$

The mode fields we computed in a previous exercise,

$$\tilde{E}_y = \sin(\beta_c x), \qquad Z_c\tilde{H}_x = \sin(\beta_c x), \qquad Z_c\tilde{H}_z = j\frac{\beta_c}{\beta}\cos(\beta_c x).$$

The integrals are

$$\oint dl|Z_c\tilde{H}_x|^2 = a+2b, \quad \oint dl|Z_c\tilde{H}_z|^2 = (a+2b)\frac{c\beta_c}{\omega}, \qquad \int d\vec{S}\bullet\tilde{E}\times Z_c\tilde{H}^* = \frac{1}{2}ab.$$

So that (after some algebra),

$$\alpha = \frac{R_s}{Z_0}\left[\frac{\omega^2}{c^2}\frac{1}{b} + 2\frac{\beta_c^2}{a}\right]\frac{c}{\omega}\frac{1}{\sqrt{\frac{\omega^2}{c^2} - \beta_c^2}}.$$

Keeping in mind the frequency dependence of surface resistivity, one finds that attenuation as a function of frequency has a minimum, diverging to large values for frequencies near cutoff (as the signal resides a longer time per unit length of propagation), and diverging, for high-frequency, due the increase in surface resistivity with frequency. The minimum occurs for $\omega / c \approx 2.4\beta_c$ for guide dimensions $a=2b$. The conventional operating range for WR90, 8-12GHz, takes into account the greater attenuation (and the rapidly increasing impedance) as the frequency nears the 6.6GHz cutoff. The upper figure of 12GHz is just shy of the cutoff for TE_{20}.

Problem 1.16 Consider a beam consisting of a series of point bunches of charge Q_b space at intervals T,

$$I_b(t) = \sum_{n=-\infty}^{\infty} Q_b \delta(t - nT).$$

Show that this may be represented by the Fourier series,

$$I_b(t) = \frac{Q_b}{T}\sum_{m=-\infty}^{\infty} e^{jm\omega t},$$

with $\omega=2\pi/T$. In this way demonstrate that the first harmonic current component of a well-bunched beam $\tilde{I}_b = 2\bar{I}_b$, with $\bar{I}_b$ the average current within the pulse.

Solution: Such a bunch train is periodic with period T, and thus can be expressed as a Fourier series,

$$I_b(t) = \sum_{n=-\infty}^{+\infty} I_n e^{jn\omega t},$$

where $\omega=2\pi/T$ and

$$I_n = \frac{1}{T}\int_0^T dt\; I_b(t) e^{-jn\omega t} = \frac{Q_b}{T}.$$

Thus

$$I_b(t) = \sum_{n=-\infty}^{+\infty} \frac{Q_b}{T} e^{jn\omega t} = \frac{Q_b}{T}\left\{1 + 2\sum_{n=1}^{\infty}\cos(n\omega t)\right\}.$$