Physics of Free Electron Lasers - Midterm - Helical Wiggler FEL

In this problem you will analyze an FEL based on an ideal helical wiggler field,

$$\vec{B} = -\frac{2B_w}{k_w}\vec{\nabla}\left\{I_1(k_w r)\sin(k_w z - \phi)\right\},$$

where *r* is the radial coordinate, and

$$\vec{\nabla} = \hat{z}\frac{\partial}{\partial z} + \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi},$$

with polar coordinates in the transverse plane, $x=r\cos\phi$, $y=r\sin\phi$, or

$\hat{r} = (\cos\phi, \sin\phi)$, $\hat{\phi} = (-\sin\phi, \cos\phi)$.

(0) Confirm that this magnetic field is consistent with Maxwell's Equations.

(1) Evaluate the wiggler field on-axis. You will want to use $I_1(\xi) \approx \xi/2$.

(2) Considering an almost monochromatic electromagnetic signal,

$$\vec{A} = \frac{mc^2}{e}\mathrm{Im}\left\{\vec{\varepsilon} a \exp(ik_z z - i\omega t)\right\},$$

determine the polarization $\vec{\varepsilon}$ (a complex vector in the *x*-*y* plane) that couples to wiggler induced beam motions. Determine the resonance conditon in terms of $a_w = eB_w / mc^2 k_w$.

(3) In the limit of a small-emittance beam derive equations governing the longitudinal motion of electrons in (γ, θ - define θ) and the eikonal equation for *a*. Make any approximations you please, but do note them. By comparison with the planar wiggler equations, determine the gain parameter ρ.

(4) Determine the natural focusing in this wiggler ("k_β"), by considering a small perturbation to the zeroth-order motion you determined in the course of problem 3.

(5) Make a sketch of the beam motion in the *x*-*y* plane, and a sketch of the *E*-field vector. Then sketch the arrangement of magnets required to produce this field. Don't spend more than 5 minutes on this part of the midterm.

Physics of Free Electron Lasers - Midterm - Helical Wiggler FEL - Solutions

In this problem we analyze an FEL based on an ideal helical wiggler field,

$$\vec{B} = -\frac{2B_w}{k_w}\vec{\nabla}\left\{I_1(k_w r)\sin(k_w z - \phi)\right\},$$

where *r* is the radial coordinate, and

$$\vec{\nabla} = \hat{z}\frac{\partial}{\partial z} + \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi},$$

with polar coordinates in the transverse plane, $x=r\cos\phi$, $y=r\sin\phi$, or

$\hat{r} = (\cos\phi, \sin\phi)$, $\hat{\phi} = (-\sin\phi, \cos\phi)$.

(0) Let's check that this magnet field is consistent with Maxwell's Equations. Clearly $\nabla\times\vec{B}=0$ since curl of a gradient vanishes, to check that $\nabla\bullet\vec{B}=0$, we need only compute

$$\begin{aligned}\nabla^2\left\{I_1(k_w r)\exp i(k_w z-\phi)\right\} &= \left(\frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\left\{I_1(k_w r)\exp i(k_w z-\phi)\right\} \\ &= \exp i(k_w z-\phi)\left(-k_w^2 + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\right)I_1(k_w r)\end{aligned}$$

But I_1 satisfies (see the Bessel Function handout),

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2} - k_w^2\right)I_1(k_w r) = 0,$$

(1) Next, let's evaluate the wiggler field on-axis.

$$\begin{aligned}B_z &= -2B_w I_1(k_w r)\cos(k_w z-\phi) \\ &\approx -B_w k_w r\cos(k_w z-\phi) \\ &= -B_w k_w\left(x\cos(k_w z) + y\sin(k_w z)\right)\end{aligned},$$

$$\begin{aligned}B_r &= -2B_w I_1'(k_w r)\sin(k_w z-\phi) \\ &\approx -B_w\sin(k_w z-\phi)\end{aligned},$$

$$\begin{aligned}B_\phi &= 2B_w\frac{I_1(k_w r)}{k_w r}\cos(k_w z-\phi) \\ &\approx B_w\cos(k_w z-\phi)\end{aligned},$$

and we make use of $I_1(\xi) \approx \xi/2$. It is also helpful to have expressions in Cartesian coordinates,

$$\begin{aligned} B_x &= B_r \cos\phi - B_\phi \sin\phi \\ &\approx -B_w \sin(k_w z) \end{aligned},$$

$$\begin{aligned} B_y &= B_r \sin\phi + B_\phi \cos\phi \\ &\approx B_w \cos(k_w z) \end{aligned}.$$

(2) Considering an almost monochromatic electromagnetic signal,

$$\vec{A} = \frac{mc^2}{e} \mathrm{Im}\left\{\vec{\varepsilon} a \exp(ik_z z - i\omega t)\right\},$$

we are asked to determine the polarization $\vec{\varepsilon}$ (a complex vector in the *x*-*y* plane) that couples to wiggler induced beam motions. To do this, let's solve the equations of motion

$$\frac{dp_x}{dz} = -\frac{e}{cv_z}\left(v_y B_z - v_z B_y\right)$$

$$\frac{dp_y}{dz} = -\frac{e}{cv_z}\left(v_z B_x - v_x B_z\right)$$

$$\frac{dp_z}{dz} = -\frac{e}{cv_z}\left(v_x B_y - v_y B_x\right)$$

considering the zeroth order motion ("design orbit").

$$\frac{dp_{x0}}{dz} = \frac{e}{c} B_y = \frac{eB_w}{c} \cos(k_w z),$$

$$\frac{dp_{y0}}{dz} = -\frac{e}{c} B_x = \frac{eB_w}{c} \sin(k_w z).$$

These can be integrated to give,

$$v_{x0} = \frac{p_{x0}}{m\gamma} = c\frac{a_w}{\gamma} \sin(k_w z),$$

$$v_{y0} = \frac{p_{y0}}{m\gamma} = -c\frac{a_w}{\gamma} \cos(k_w z).$$

Here

$$a_w = \frac{eB_w}{mc^2 k_w}.$$

With these results we can also see that

$$\frac{dp_z}{dz} = -\frac{e}{cv_z}\left(c\frac{a_w}{\gamma}\sin(k_w z)B_w\cos(k_w z) - c\frac{a_w}{\gamma}\cos(k_w z)B_w\sin(k_w z)\right) = 0\,.$$

Thus p_z, γ and v_z are all constants of the zeroth-order motion. It will be helpful later to have the zeroth-order trajectory

$$x_0 = -\frac{a_w}{\gamma\beta_z k_w}\cos(k_w z),$$

$$y_0 = -\frac{a_w}{\gamma\beta_z k_w}\sin(k_w z).$$

Also useful to note that

$$\begin{aligned}\frac{v_z}{c} &\approx 1 - \frac{1}{2\gamma^2} - \frac{\beta_x^2}{2} - \frac{\beta_y^2}{2} \\ &= 1 - \frac{1}{2\gamma^2}\left(1 + a_w^2\right)\end{aligned}$$

Let's now determine the condition for resonant transfer of energy to the wave. The electric field is just

$$\begin{aligned}\vec{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\frac{mc^2}{e}\,\mathrm{Im}\left\{\vec{\varepsilon} a \exp(ik_z z - i\omega t)\right\} \\ &= \frac{mc^2}{e}\,\mathrm{Im}\left\{ i\frac{\omega}{c}\vec{\varepsilon} a \exp(ik_z z - i\omega t)\right\}\end{aligned},$$

and the rate of change of energy is governed by

$$\begin{aligned}\frac{d\gamma}{dz} &= -\frac{e}{mc^2 v_z}\vec{E}\bullet\vec{v} \\ &= -\mathrm{Im}\left\{ i\frac{\omega}{c} a \frac{\vec{v}}{v_z}\bullet\vec{\varepsilon}\exp(ik_z z - i\omega t)\right\} \\ &= -\mathrm{Im}\left\{ i\frac{\omega}{c}\frac{a a_w}{\gamma\beta}\left[\varepsilon_x \sin(k_w z) - \varepsilon_y \cos(k_w z)\right]\exp(ik_z z - i\omega t)\right\}\end{aligned}$$

We would like to choose polarization so that the particles can do work on the fields and the choice $(\varepsilon_x, \varepsilon_y) = (1, i)$, gives

$$\frac{d\gamma}{dz} = -\frac{\omega}{c}\frac{a_w}{\gamma\beta}\mathrm{Im}\left(ae^{i\theta}\right),$$

where $\theta = \left(k_z + k_w\right)z - \omega t$, and resonance corresponds to, $d\theta / dz \approx 0$, or

$$v_z = \frac{\omega}{\left(k_z + k_w\right)}.$$

Note that this choice of polarization corresponds to

$$\begin{aligned} A_x &= \frac{mc^2}{e}\mathrm{Im}\left\{a\exp(i\alpha)\right\} \\ &= \frac{mc^2}{e}a_s \sin\left(\alpha + \varphi_s\right) \\ A_y &= \frac{mc^2}{e}\mathrm{Im}\left\{ia\exp(i\alpha)\right\} \\ &= \frac{mc^2}{e}a_s \cos\left(\alpha + \varphi_s\right) \end{aligned}$$

and we abbreviate $\alpha = k_z z - \omega t$, $a = a_s \exp\left(i\varphi_s\right)$.

(3) In the limit of a small-emittance beam the equations governing the longitudinal motion of electrons in (γ, θ)are

$$\begin{aligned} \frac{d\theta}{dz} &= \left(k_z + k_w\right) - \frac{\omega}{v_z} \\ &= \left(k_z + k_w\right) - \frac{\omega}{c}\left(1 + \frac{1}{2\gamma^2}\left(1 + a_w^2\right)\right) \\ &= k_w - \delta k - \frac{\omega / c}{2\gamma^2}\left(1 + a_w^2\right) \end{aligned}$$

and

$$\frac{d\gamma}{dz} = -\frac{\omega}{c}\frac{a_w}{\gamma\beta}\mathrm{Im}\left(ae^{i\theta}\right).$$

The derivation of the eikonal equation for a follows that in the review notes. We express the vector potential as

$$\vec{A} = \frac{mc^2}{2e}\left[-ia\exp(i\alpha)\hat{x} + a\exp(i\alpha)\hat{y} + c.c.\right].$$

Maxwell's equations in the eikonal approximation take the form

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{1}{2}e^{i\alpha}\left(\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2 + 2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta\right)\left[-ia\hat{x} + a\hat{y}\right] + c.c.$$
$$= -\frac{4\pi}{c}\vec{J}$$

The current density takes the form

$$J_x = \sum_{all\,e-} -\frac{ec}{\gamma}\delta(x - x_i)\delta(y - y_i)\delta(z - z_i)\left\{a_w\sin(k_w z)\hat{x} - a_w\cos(k_w z)\hat{y} + a_s\sin(\alpha + \varphi_s)\hat{x} + a_s\cos(\alpha + \varphi_s)\hat{y}\right\}$$

performing averages over the signal wavelength and period, and making use of the "bucket average" (see notes),

$$\left\langle \sum_{all\,e-}\delta^3(\vec{r} - \vec{r}_i)f(\vec{r},t)\right\rangle_\lambda = \frac{I}{ec}\left\langle \beta_z^{-1}\delta^2(\vec{r}_\perp - \vec{r}_{\perp i})f(\vec{r}_i,t)\right\rangle$$

we arrive at

$$\frac{1}{2}\left(\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2 + 2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta\right)\left[-ia\hat{x} + a\hat{y}\right] = 4\pi\frac{I}{I_0}\left\langle \frac{\delta^2(\vec{r}_\perp - \vec{r}_{\perp i})}{\gamma\beta}\left[-\frac{a}{2i}\hat{x} - \frac{a}{2}\hat{y} - \frac{a_w e^{-i\theta}}{2i}\hat{x} - \frac{a_w e^{-i\theta}}{2}\hat{y}\right]\right\rangle,$$

which reduces after the usual approximations to

$$\left(\frac{d}{dz} + \frac{2\pi i}{k_z\Sigma}\frac{I}{I_0}\left\langle \frac{1}{\gamma\beta}\right\rangle\right)a = \frac{2\pi i}{k_z\Sigma}a_w\left(\frac{I}{I_0}\right)\left\langle \frac{\exp(-i\theta)}{\gamma\beta}\right\rangle$$

We can determine the gain parameter ρ by comparison with the planar wiggler FEL equations. The helical wiggler equations can be expressed as

$$\frac{d\theta}{dz} \approx \Delta k_0 + 2(k_w - \delta k)\left(\frac{\gamma-\gamma_0}{\gamma_0}\right),$$
$$\frac{d\gamma}{dz} = -\frac{\omega}{c}\frac{\hat{a}_w}{2\gamma\beta}\operatorname{Im}\left(\hat{a}e^{i\theta}\right),$$
$$\left(\frac{d}{dz}+i\nu\right)\hat{a} = \frac{2\pi i}{k_z\Sigma}\hat{a}_w\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle,$$

where $\hat{a}_w = a_w 2^{1/2}$, $\hat{a} = a2^{1/2}$, and

$$\Delta k_0 = k_w - \delta k - \frac{\omega/c}{2\gamma_0^2}\left(1+a_w^2\right).$$

In this form the equations are identical to those for the planar wiggler, and thus the Pierce parameter must be

$$\rho = \left(\frac{\pi}{8}\frac{1}{\Sigma k_w^2}\left(\frac{I}{\gamma^3 I_0}\right)\hat{a}_w^2\right)^{1/3}$$
$$= \left(\frac{\pi}{4}\frac{1}{\Sigma k_w^2}\left(\frac{I}{\gamma^3 I_0}\right)a_w^2\right)^{1/3}$$

Note that for a given value of a_w, the gain parameter for a helical wiggler can be as much as 60% more than that for a planar wiggler.

(4) Next we determine the natural focusing in this wiggler (“k_β”), by considering a small perturbation to the zeroth-order motion we examined above.

$$\frac{dp_{x1}}{dz} \approx -\frac{e}{cv_z}\left(v_{y0}+v_{y1}\right)B_z,$$

$$\frac{dp_{y1}}{dz} = \frac{e}{cv_z}\left(v_{x0}+v_{x1}\right)B_z$$

We can neglect 2nd order terms in B_x, B_y, as well as jitter in v_z in the limit $k_\beta << k_w$. In the same limit also, the v_{y1}, v_{x1} terms will average to zero over a wiggler period. In this limit, one is left with focusing due to the gradient in the solenoidal field,

$$\frac{dp_{x1}}{dz} \approx -\frac{e}{cv_z} v_{y0} B_z$$

$$= -\frac{e}{cv_z}\underbrace{\left(-c\frac{a_w}{\gamma}\cos\left(k_w z\right)\right)}_{v_{y0}}\underbrace{\left(-B_w k_w\right)\left(x\cos\left(k_w z\right)+y\sin\left(k_w z\right)\right)}_{B_z}$$

In this expression, we substitute

$$x = x_0 + x_1 = -\frac{a_w}{\gamma\beta_z k_w}\cos\left(k_w z\right)+x_1,$$

$$y = y_0 + y_1 = -\frac{a_w}{\gamma\beta_z k_w}\sin\left(k_w z\right)+y_1,$$

and average over a wiggler period,

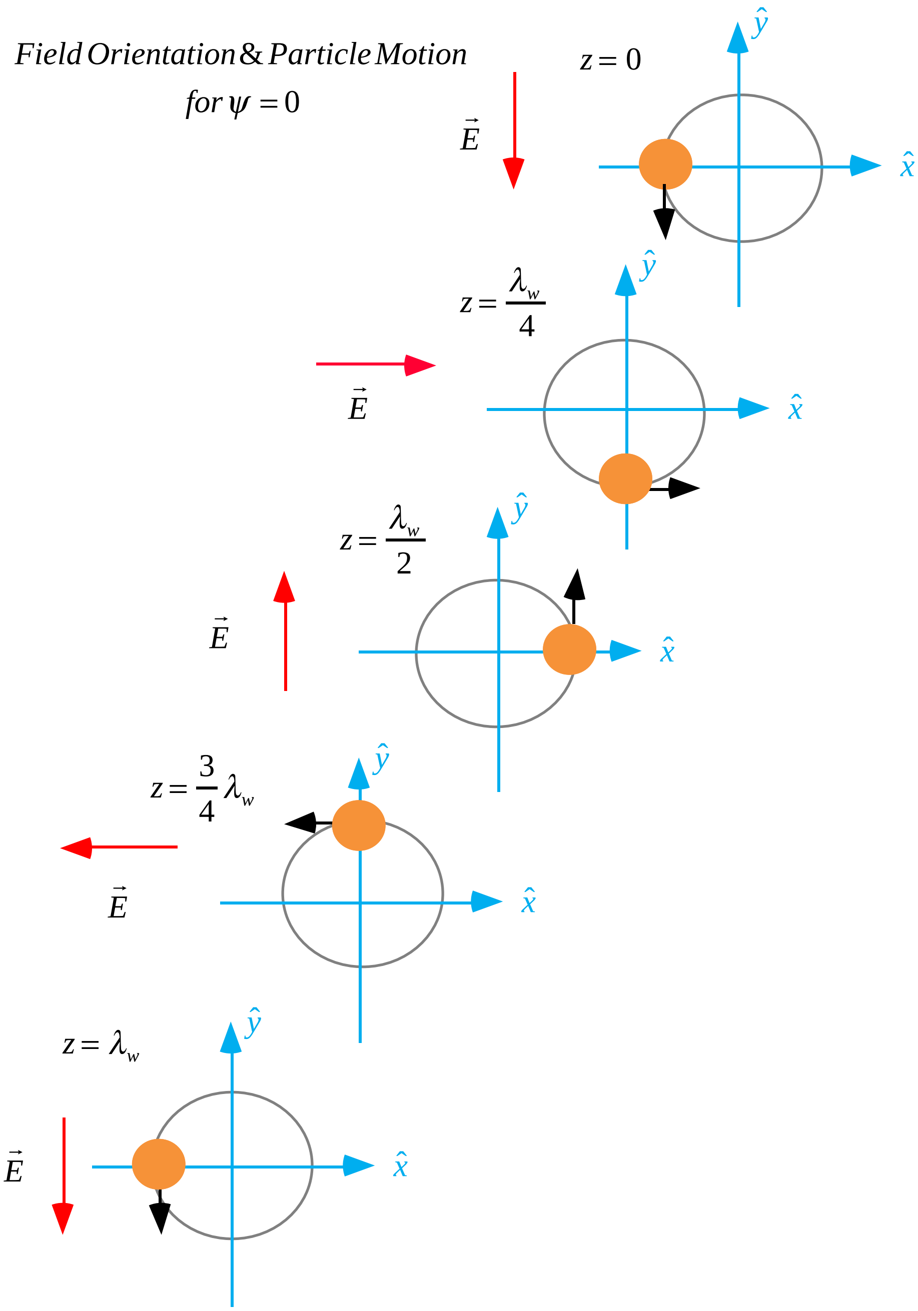
$$\frac{d^2 x_1}{dz^2} = \frac{1}{mc\gamma}\overline{\frac{dp_{x1}}{dz}} = -\frac{a_w^2 k_w^2}{\gamma^2 v_z}\overline{\cos^2\left(k_w z\right)}x_1 = -\frac{a_w^2 k_w^2}{2\gamma^2\beta}x_1 = -k_\beta^2 x_1,$$

with

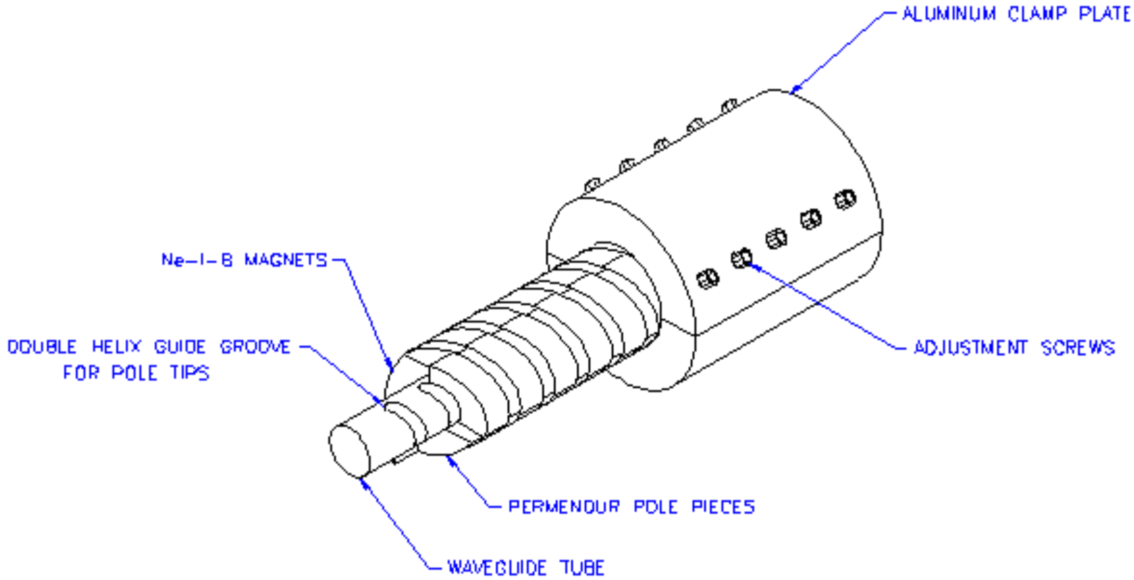
$$k_\beta = \frac{a_w k_w}{2^{1/2}\gamma\beta^{1/2}}.$$

Symmetry suggests that focusing should be equal in both planes, and this can be checked by a similar calculation.

(5) To illustrate, we make a sketch of the beam motion in the *x*-*y* plane and the co-moving *E*-field vector,



The arrangement of magnets required to produce this field might look like that for a proposed UCSB FEL (http://sbfel3.ucsb.edu/2mv/undulator.html):



or they might look like the twistor magnet (thanks to Roger Carr!) below or the Mitsubishi wiggler depicted in the attached brochure.