Physics of Free Electron Lasers - Homework 4
Space-Charge, Focusing, Emittance - FEL Design

You have been put in charge of the equipment used in the ELF experiment (recall handout). Your mission is to produce a high-power rf pulse at 91.392GHz. You have identified an EEVMG5335 magnetron as a suitable source; it is rated at 2.5kW, but you can only get about 1kW due to some difficulty in matching into the overmoded 3cm x 10cm waveguide.

(1) Neglecting space-charge effects, and consulting the article from homework 2, determine the beam energy and wiggler field settings which will provide maximum power at saturation. Feel free to employ scalings, simulation or both in your optimization. What is the maximum power your FEL can achieve (without tapering)?

(2) For subsequent work you will want to estimate the beam cross-section $\Sigma_{sc} = \pi r_b^2$. Assume quadrupolar focussing is available, and focusing is equal in both planes. Use an rms normalized emittance ε_n˜0.47cm-rad.

You may find it helpful to recall that with quadrupolar focusing in addition to the vertical focusing provided by the wiggler, one has the relation

$$k_{\beta x}^2 + k_{\beta y}^2 = \frac{1}{2}\left(\frac{k_w a_w}{\gamma\beta}\right)^2 .$$

Also rms normalized emittance is related to the beam radius by $\varepsilon_n = \gamma k_\beta r_b^2$.

(3) Check on the importance of space-charge by means of a simple estimate. Recall that this involves comparison of the longitudinal plasma wavenumber, $k_{b\|}$

$$k_{b\|}^2 \approx 4\pi \frac{I/I_0}{\gamma\gamma_\|^2 \Sigma_{sc}} ,$$

with the growth rate, $k_{b\|} << \mathrm{Re}(\Gamma)$, where $\mathrm{Re}(\Gamma) \approx 3^{1/2}\rho k_w$.

(4) Confirm this estimate by revising “code2” to a new version, “code3”, that includes space-charge. The modification to the FEL equations to incorporate space-charge appears in the equation for particle energy, reflecting the correction due to the axial electric field from space-charge,

$$\frac{d\gamma}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta}[JJ]\mathrm{Im}\left[a\exp(i\theta)\right] - \frac{8\pi}{\Sigma_{sc}}\left(\frac{c}{\omega}\right)\left(\frac{I}{I_0}\right)\sum_{n=1}^{\infty}\frac{\eta_n}{n}\mathrm{Im}\left\{\exp(in\theta)\left\langle\exp(-in\theta)\right\rangle\right\}.$$

For your simulation use only the n=1 term in this sum, with $\eta_1=1$. Check the validity of the assumption $\eta_1=1$.

(5) Estimate the spread in detuning due to emittance. Could emittance have a significant effect on gain in this experiment? Recall that

$$(\Delta k)_\varepsilon \approx \frac{\omega}{c}\theta^2,$$

with θ the beam opening angle, and that one wants

$$(\Delta k)_\varepsilon < \mathrm{Re}(\Gamma),$$

to avoid a reduction in gain. With a 1D code, can you simulate the effect of emittance , insofar as it contributes to a distribution in detuning?

Physics of Free Electron Lasers - Homework 4 -Solutions
Space-Charge, Focusing, Emittance - FEL Design

In this homework we redesign the ELF experiment to operate at 91.392Ghz, using a 1kW input signal, and operating in 3cm x 10cm waveguide.

(1) First we estimate the beam energy and wiggler field settings which will provide maximum power at saturation. Our analytic estimate for saturation power is $P_{sat} = \rho\gamma I\, mc^2/e$ where the gain or Pierce parameter is

$$\rho \approx \frac{1}{\gamma}\left\{\frac{\pi}{8}\frac{1}{\Sigma k_w^2}\left(\frac{I}{I_0}\right)\left(a_w[JJ]\right)^2\right\}^{1/3}.$$

For fixed current, clearly the parameters to adjust are γ (through the beam voltage) and a_w (through the wiggler field setting). On the other hand one also infers that the result for absolute power is only weakly dependent on beam energy. This suggests increasing the wiggler field to the maximum value attainable for ELF, 5kG. Two issues arise, however. First, is this value consistent with resonance, given that the maximum beam voltage is 4.5MeV? Second, as energy is increased, to match resonance at this higher wiggler field, ρ will actually decrease, so that the length for saturation will increase. So we will need to check that saturation occurs within the available wiggler length.

Realizing things may get complicated, let's proceed blithely along. Maximum, wiggler parameter is

$$a_w = \frac{B_w}{1.7kG - cm}\frac{\lambda_w}{2\pi} = \frac{5kG}{1.7kG - cm}\frac{9.8}{2\pi} = 4.56.$$

Maximum γ is

$$\gamma = 1 + \frac{4.5MeV}{0.511MeV} = 9.81.$$

Let's now check the resonance requirement. The resonance condition is

$$\Delta k = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + \frac{a_w^2}{2}\right\} \approx 0\,.$$

Where $\delta k = \omega / c - k_z$. Recall that for the TE_{01} mode,

$$k_z = \left\{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)\right\}^{1/2},$$

and here b=3cm, and $\omega/c=19.154cm^{-1}$, so that $k_z=19.126cm^{-1}$ and $\delta k=0.028cm^{-1}$. (The waveguide correction is so small that one might as well use the free-space resonance condition.) Using these numbers and $k_w = 0.641cm^{-1\text{-}1}$, we can determine the maximum wiggler parameter consistent with resonance,

$$\begin{aligned} a_w &= 2^{1/2}\left\{2\gamma^2\frac{k_w - \delta k}{\omega / c} - 1\right\}^{1/2} \\ &\approx 2^{1/2}\left\{2(9.81)^2\frac{0.641 - 0.028}{19.154} - 1\right\}^{1/2} \\ &\approx 3.21 \end{aligned}$$

This corresponds to resonance at 3.5kG and is attainable with this wiggler. We can also write the resonance condition simply as

$$\beta = \frac{\omega / c}{k_w + k_w} = \frac{19.154}{0.641 + 19.126} \approx 0.969.$$

Let's check next to see if saturation occurs within the length of wiggler available. First we need to compute the gain length, and this requires ρ. Recall that the Bessel function factor, $[JJ] = J_0(\xi) - J_1(\xi)$, where

$$\xi = \frac{1}{8}\frac{\omega / c}{k_w}\left(\frac{a_w}{\gamma}\right)^2 \approx \frac{1}{8}\frac{19.154}{0.641}\left(\frac{3.21}{9.81}\right)^2 = 0.4,$$

and

$$[JJ] \approx 1 - \frac{1}{2}\xi - \frac{1}{4}\xi^2 \approx 0.77.$$

We may then compute

$$\begin{aligned}\rho &\approx \frac{1}{\gamma}\left\{\frac{\pi}{8}\frac{1}{\Sigma k_w^2}\left(\frac{I}{I_0}\right)(a_w[JJ])^2\right\}^{1/3} \\ &\approx \frac{1}{9.81}\left\{\frac{\pi}{8}\frac{1}{\left(\frac{3cmx10cm}{2}\right)(0.641cm^{-1})^2}\left(\frac{0.8kA}{17kA}\right)(3.21x[0.77])^2\right\}^{1/3} \\ &\approx 0.027\end{aligned}$$

Next we compute the gain length,

$$L_g = \frac{\lambda_w}{2\pi 3^{1/2}\rho} = \frac{9.8}{2\pi 3^{1/2} 0.027} = 33.3cm,$$

or $G \approx 8.7dB / L_g \approx 26dB / m$. As we begin to get worried at how low gain is, we check the power at saturation, using

$$\begin{aligned}P_{sat} &\approx \rho mc^2\gamma I \\ &\approx 0.027x\,0.51MeV\,x9.81x\,800A \\ &\approx 110MW\end{aligned}$$

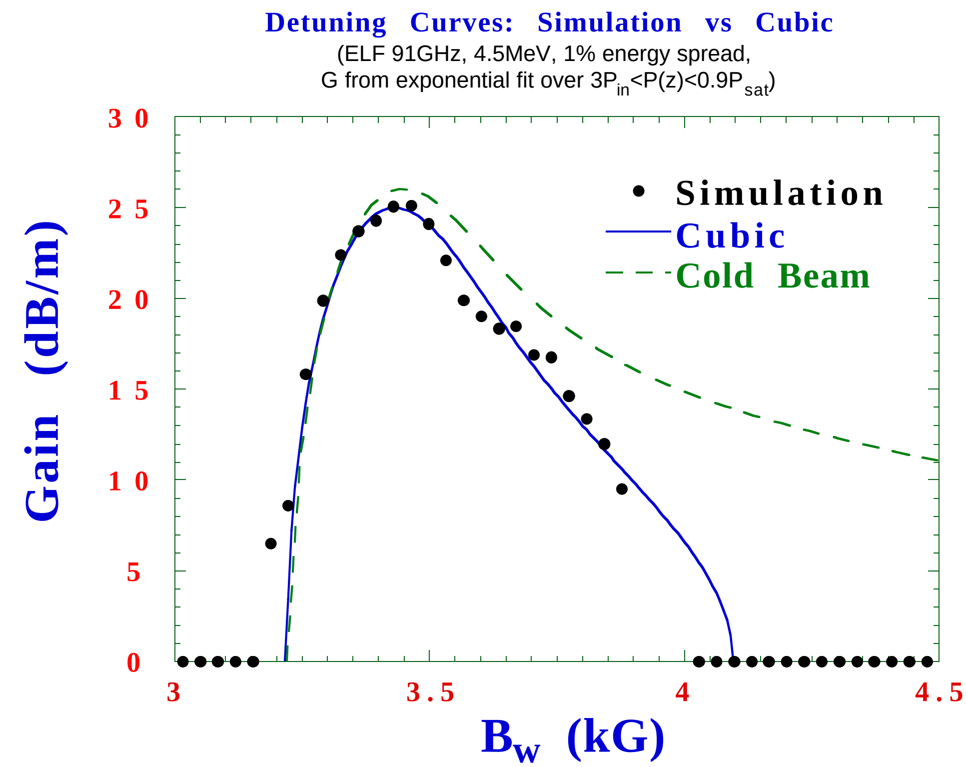
Since power grows asymptotically as

$$P \approx \frac{1}{9} P_{in} \exp\left(\frac{2z}{L_g}\right),$$

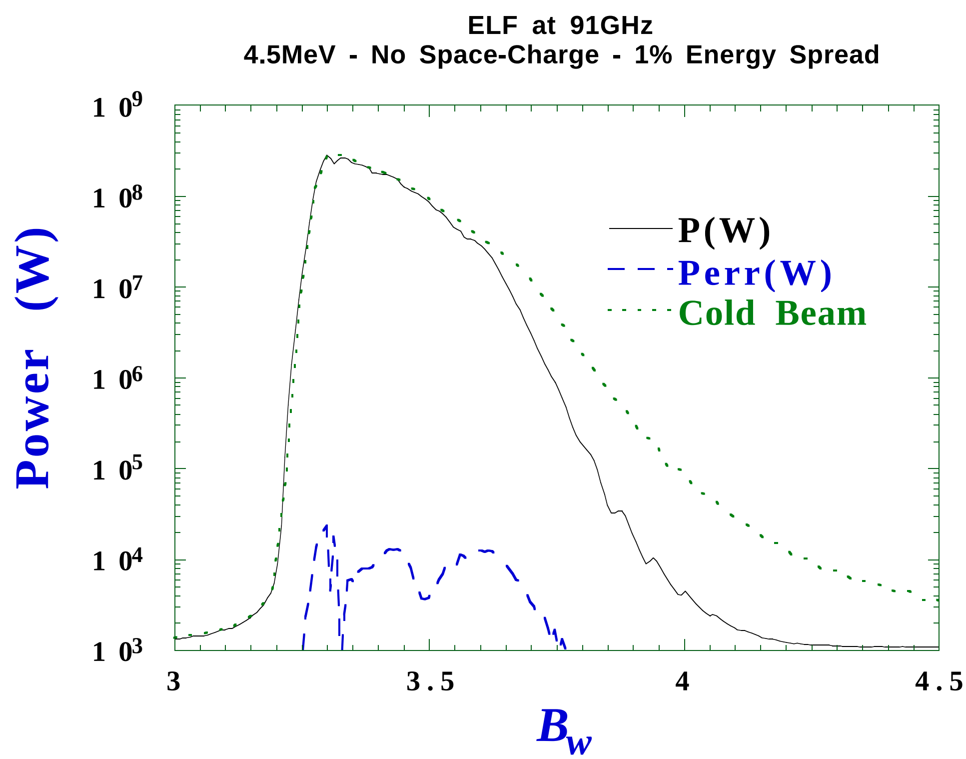
the length to reach saturation is just

$$L_{sat} = \frac{L_g}{2} \ln\left(\frac{9P_{sat}}{P_{in}}\right) = \frac{33.3cm}{2} \ln\left(\frac{9x110MW}{0.001MW}\right) = 230cm$$

where we make use of the value 1kW for the input signal. This is rather uncomfortably close to the wiggler length, as we will see. This is “uncomfortably close” since not only is the full wiggler length not available for interaction (see reference), but this is the “cold beam” estimate, excluding all the deleterious effects such as space-charge and emittance. Notice also that every 3dB more in input signal one can get reduce the required wiggler length by 1/3 of a gain length, so one does regret difficulties in getting a larger input signal.



Detuning Curves: Simulation vs Cubic
(ELF 91GHz, 4.5MeV, 1% energy spread,
G from exponential fit over 3P_in<P(z)<0.9P_sat)
Gain (dB/m)
Simulation
Cubic
Cold Beam
B_w (kG)



ELF at 91GHz
4.5MeV - No Space-Charge - 1% Energy Spread
Power (W)
P(W)
Perr(W)
Cold Beam
B_w

In any case, estimates in hand, we are ready to perform some numerical studies (still no space-charge included), to gauge performance. The detuning curves looks something like the plots above for 1% energy spread. The curves labelled "cubic" are simply solutions of the cubic dispersion relation,

$$\zeta^3 + 2\delta\zeta^2 + \left(\delta^2 - \delta_\sigma^2\right)\zeta + \rho^3 = 0,$$

where, recall,

$$\Gamma = 2ik_w\zeta,$$

gives the growth rate according to $\mathrm{Re}\,\Gamma = 1/L_g$, and the center-detuning (of the assumed flat distribution in detuning) is

$$\delta = \frac{\Delta k}{2k_w}$$

and the distribution in detuning is over $\pm 2k_w\delta_\sigma$ about the mean. Thus, for a flat-distribution in γ, over $\pm\Delta\gamma$,

$$\delta_\sigma = \left(\frac{\omega}{2k_w c}\right)\left(1 + \frac{1}{2}a_w^2\right)\frac{\Delta\gamma}{\gamma^3} \approx \frac{\Delta\gamma}{\gamma}.$$

Based on these results one can see that if space-charge is indeed negligible, one could achieve on the order of 100MW, and better than 20dB/m gain. Before getting too excited about these estimates though, we should consider space-charge.

(2) Assuming equal focusing in both planes, we have

$$k_\beta = \frac{a_w k_w}{2\gamma\beta} = \frac{3.21\,x\,0.641cm^{-1}}{2\,x\,9.81\,x\,0.969} \approx 0.108cm^{-1},$$

or

$$\lambda_\beta = \frac{2\pi}{k_\beta} \approx 58.2cm.$$

This is reasonably longer than the wiggler period so that the assumptions behind our treatment of focusing are satisfied for the present work. We can determine the matched beam radius from

$$r_b = \left(\frac{\varepsilon_n}{\gamma k_\beta}\right)^{1/2} = \left(\frac{0.47cm-rad}{9.81\,x\,0.108cm^{-1}}\right)^{1/2} \approx 0.67cm$$

and the beam cross-section $\Sigma_{sc} = \pi r_b^2 \approx 1.41cm^2$.

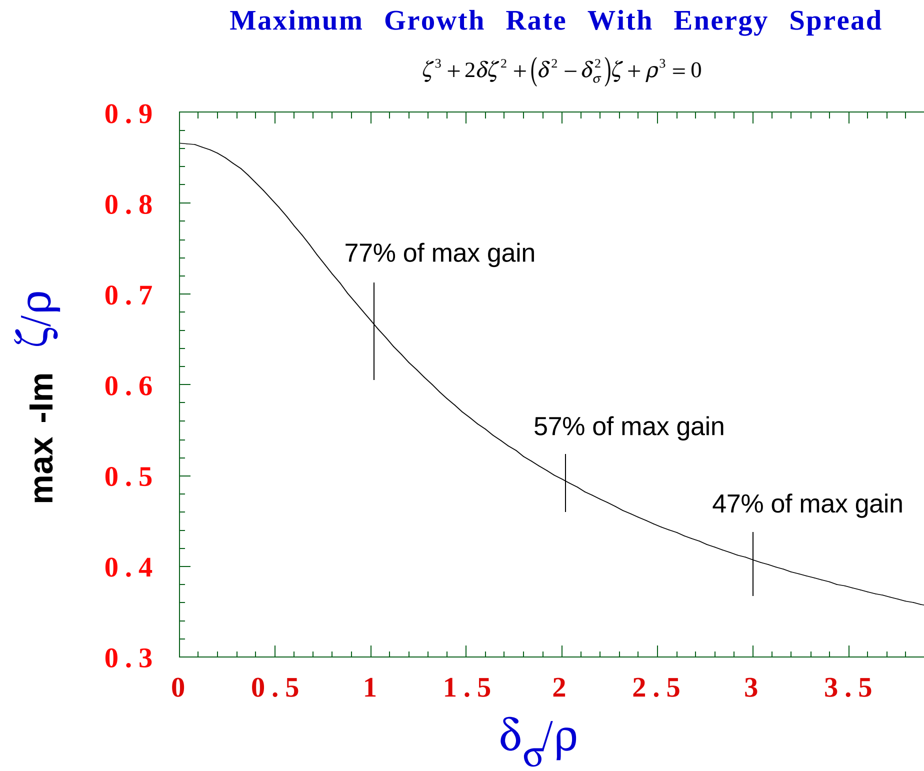
(3) We check on the importance of longitudinal space-charge by comparing the growth rate

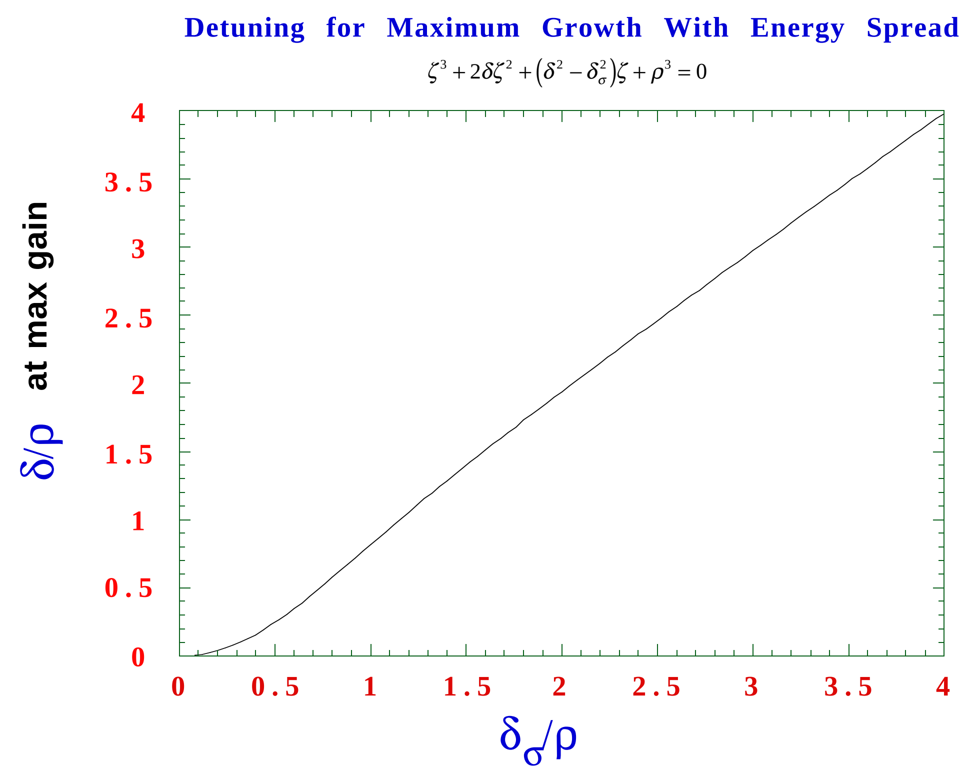
$$\mathrm{Re}(\Gamma) = 3^{1/2}\rho k_w \approx 3^{1/2}\,x\,0.027\,x\,0.641cm^{-1} \approx 0.03cm^{-1}$$

with the longitudinal plasma wavenumber, $k_{b\|}$

$$\begin{aligned} k_{b\|} &= \left\{4\pi \frac{I/I_0}{\gamma\gamma_\|^2\Sigma_{sc}}\right\}^{1/2} \\ &= \left\{4\pi \frac{I/I_0}{\gamma^3\Sigma_{sc}}\left(1+a_w^2/2\right)\right\}^{1/2} \\ &= \left\{4\pi \frac{0.8/17}{(9.81)^3 1.41cm^2}\left(1+(3.21)^2/2\right)\right\}^{1/2} \\ &= 0.052cm^{-1} \end{aligned}$$

This estimate indicates that space-charge is a potentially very significant effect.





To confirm this we can resort to the cubic dispersion relation, using

$$\delta_c = \frac{k_{b\|}}{2k_w} \approx 0.041 \approx 1.5\rho .$$

(in general this term adds in quadrature with that due to energy spread). The general solution of the cubic takes the form indicated in the plots and can be fit roughly by

$$G(\delta_c) \approx \frac{G(0)}{1+0.4\delta_\sigma/\rho} \approx 0.63G(0) \approx 16dB/m .$$

The plot of shift in detuning for peak gain indicates that the optimal wiggler field setting will be shifted by a detuning of about δ_σ , corresponding to $\Delta k \approx 2k_w\delta_\sigma$ or

$$\begin{aligned}\frac{a_w}{a_{wr}} &= \left\{1-4\delta_\sigma \frac{1+a_{wr}^2/2}{a_{wr}^2}\right\}^{1/2} \\ &\approx \left\{1-4x0.041\frac{1+(3.21)^2/2}{(3.21)^2}\right\}^{1/2} \\ &\approx 0.95\end{aligned}$$

Thus one expects peak gain at $B_w \approx 0.95B_{wr} \approx 3.3kG$.

(4) We can check these estimates by revising “code2” to a new version, “code3”, that includes space-charge. The modification to the FEL equations to incorporate space-charge appears in the equation for particle energy, reflecting the correction due to the axial electric field from space-charge. Our equations take the form,

$$\frac{d\theta}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1+\frac{a_w^2}{2} - a_w[JJ]\mathrm{Re}\left[a\exp(i\theta)\right]\right\}$$

$$\frac{d\gamma}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta}[JJ]\mathrm{Im}\left[a\exp(i\theta)\right] + \frac{8\pi}{\Sigma_{sc}}\left(\frac{c}{\omega}\right)\left(\frac{I}{I_0}\right)\sum_{n=1}^{\infty}\frac{\eta_n}{n}\mathrm{Im}\left\{\exp(in\theta)\langle\exp(-in\theta)\rangle\right\},$$

$$\left(\frac{d}{dz}+\frac{2\pi i}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle\right)a=\frac{2\pi i}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle,$$

with

$$\beta=1-\frac{1}{2\gamma^2}\left(1+\frac{a_w^2}{2}\right).$$

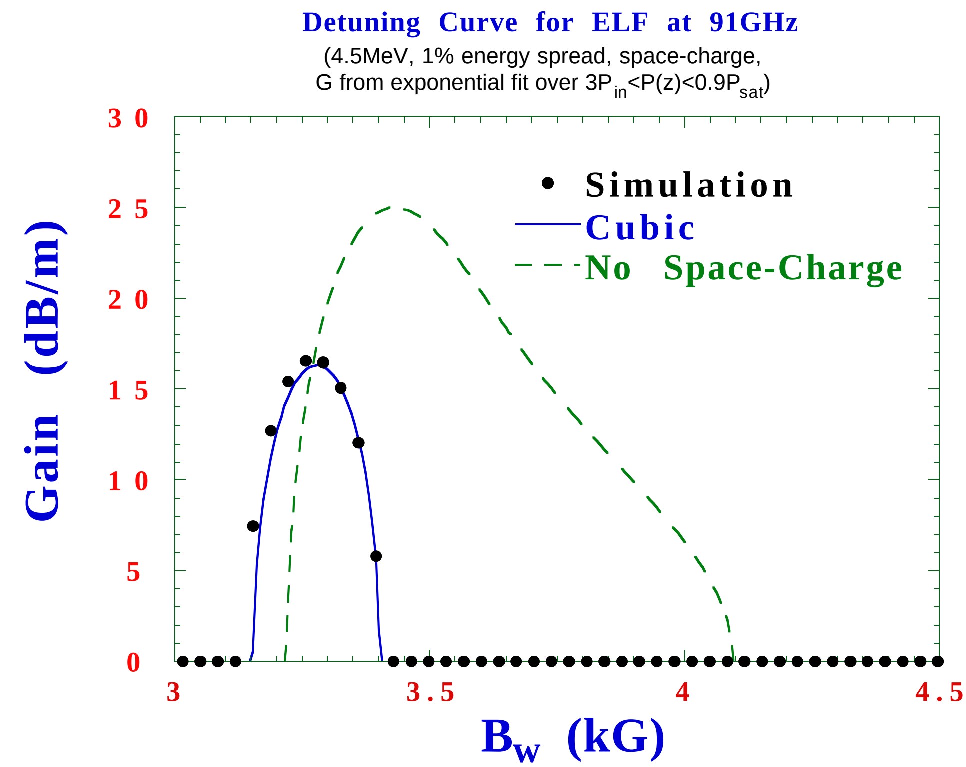
For the simulation we use only the n=1 term in this sum, with η_1=1. (By the way, neglect of higher harmonics does not contribute an error to the result for exponential gain; it can however modify results for saturation, and effect the optimal taper). The assumption η_1=1 requires

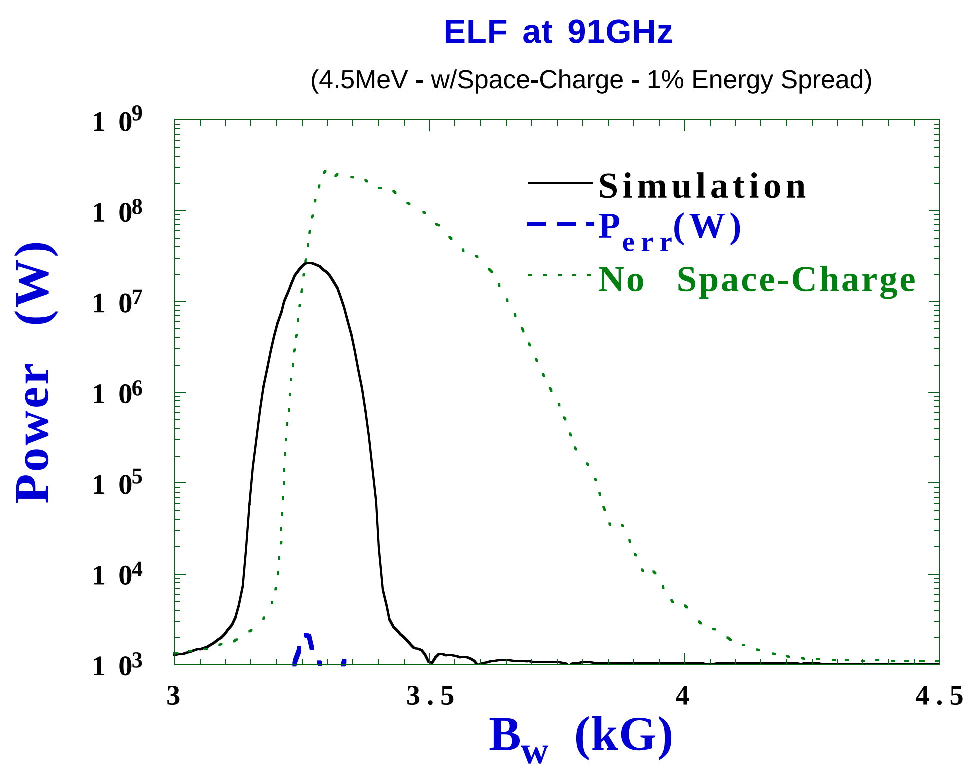
$$1<<\frac{\omega r_b/c}{\gamma_{\|}\beta}\approx\frac{19.154\,x\,0.67}{3.96\,x\,0.969}\approx 3.3,$$

which will do for the time being. The simplified form for the γ-equation takes the form

$$\frac{d\gamma}{dz}=-\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta}[JJ]\,\mathrm{Im}\left[a\exp(i\theta)\right]+\frac{8\pi}{\Sigma_{sc}}\left(\frac{c}{\omega}\right)\left(\frac{I}{I_0}\right)\left\{\sin\theta\langle\cos\theta\rangle-\cos\theta\langle\sin\theta\rangle\right\}$$

Results are indicated below (see “code3” at the class website).


Detuning Curve for ELF at 91GHz
(4.5MeV, 1% energy spread, space-charge,
G from exponential fit over 3P_in<P(z)<0.9P_sat)
Simulation
Cubic
No Space-Charge
Gain (dB/m)
0
5
10
15
20
25
30
3
3.5
4
4.5
B_w (kG)


ELF at 91GHz
(4.5MeV - w/Space-Charge - 1% Energy Spread)
Simulation
P_err(W)
No Space-Charge
Power (W)
10^3
10^4
10^5
10^6
10^7
10^8
10^9
3
3.5
4
4.5
B_w (kG)

Peak gain is 16.6dB/m, and occurs at 3.27kG, quite close to our estimates based on the cubic dispersion relation.

(5) Next we estimate the spread in detuning due to emittance. Recall that

$$(\Delta k)_\varepsilon \approx \frac{\omega}{c}\theta^2,$$

with θ the beam opening angle, and that one wants

$$(\Delta k)_\varepsilon < \mathrm{Re}(\Gamma),$$

to avoid a reduction in gain. More explicitly, the effective width in detuning arising from emittance is

$$\delta_\sigma \approx \frac{\omega / c}{2k_w}\left(\frac{\varepsilon_n}{\gamma r_b}\right)^2 \approx \frac{1}{1+a_w^2/2}\left(\frac{\varepsilon_n}{r_b}\right)^2,$$

and this can be expressed equivalently as

$$\delta_\sigma \approx \frac{\gamma k_\beta \varepsilon_n}{1+a_w^2/2} \approx \frac{a_w/2}{1+a_w^2/2}k_w\varepsilon_n$$

Note the resulting constraint on emittance implied by $\delta_\sigma < \rho$,

$$k_\beta \varepsilon_n < \frac{\rho}{\gamma}\left(1+a_w^2/2\right).$$

With a 1D code, we can simulate the effect of emittance, insofar as it contributes to a distribution in detuning, by adding the corresponding "effective energy spread" in quadrature in the numerical particle energy distribution,

$$\Delta\gamma_{eff} \approx \gamma\delta_\sigma \approx \frac{\gamma}{1+a_w^2/2}\left(\frac{\varepsilon_n}{r_b}\right)^2 .$$

This simple estimate yields an 8% effective energy spread for the present parameters, enough to reduce gain to nil.

Both space-charge and the effect of emittance favor weaker focusing. Lower energy tends to reduce the importance of emittance, but aggravates the effect of space-charge. For this example, the two effects are so large that they dominate the FEL design. Moreover, the wiggler is too short.

After this sobering look at the effects of emittance and space-charge it is intstructive to consider 3.5MeV, a longer 4 meter wiggler, a higher current of 1000A and (normalized) brightness of $2\text{x}10^4$A/(cm-rad)^2, corresponding to a lower normalized emittance of 0.2cm-rad. You can check that this provides gain in the range of 15dB/m range with output power of order 100MW (not saturated). Additional improvement can be obtained by relaxing the assumption of equal focusing in both planes.