Physics of Free Electron Lasers - Homework 3 - Gain & Saturation

For this homework, you will first want to revise your FEL simulator ("code1" from homework 2) to include the Bessel function factor and the corrections due to the effect of a_s on the transverse motion. The FEL equations you will want to solve in this new "code2", will take the revised form

$$\frac{d\theta}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + \frac{a_w^2}{2} - a_w[JJ]\mathrm{Re}\left[a\exp(i\theta)\right]\right\},$$

$$\frac{d\gamma}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta}[JJ]\mathrm{Im}\left[a\exp(i\theta)\right],$$

$$\left(\frac{d}{dz} + \frac{2\pi i}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle\right)a = \frac{2\pi i}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle$$

where the average axial speed normalized by c is,

$$\beta = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{a_w^2}{2}\right)$$

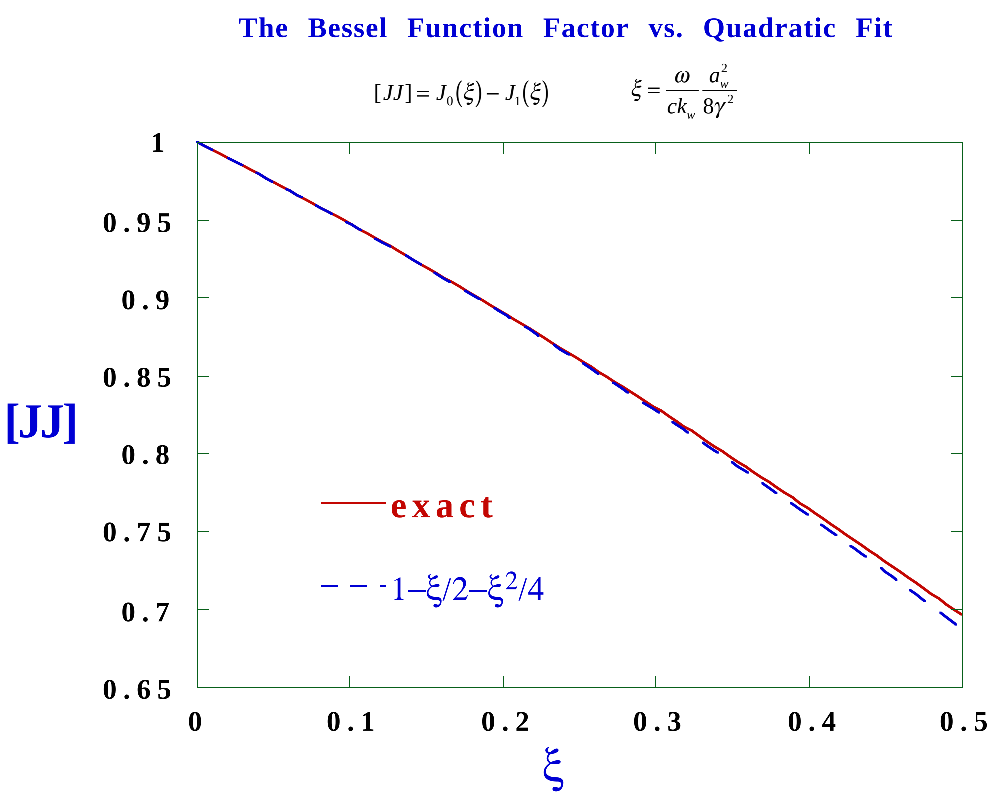
Recall the Bessel function factor, $[JJ] = J_0(\xi) - J_1(\xi)$, where

$$\xi = \frac{1}{8}\frac{\omega}{ck_w}\frac{a_w^2}{\gamma^2}.$$

This factor is well approximated by

$$[JJ] \approx \left(1 - \frac{1}{4}\xi^2 + \frac{1}{64}\xi^4\right) - \left(\frac{1}{2}\xi - \frac{1}{16}\xi^3\right) \approx 1 - \frac{1}{2}\xi - \frac{1}{4}\xi^2,$$

as can be seen in the following plot.



You may wish to check code2, by turning off the corrections and confirming that it reproduces the results of code1. The following questions pertain to the parameters of homework#2, for the ELF experiment.

(1)Shift in Resonance: Due to the non-resonant portion of the beam susceptibility, the effective axial wavenumber of the eikonal is shifted by an amount

$$\nu = \frac{2\pi}{k_z \Sigma}\frac{I}{I_0}\left\langle \frac{1}{\gamma\beta} \right\rangle.$$

Because of this, the precise resonance condition takes the form

$$\frac{\omega / c}{k_w + k_z - \nu} = \beta = 1 - \frac{1}{2\gamma^2}\left(1 + a_w^2 / 2\right)$$

Compute a_w at resonance, a_{wr}, with this correction term. Why do almost all treatments of FELs neglect this term?

(2)Effect of Higher Order Corrections: How does the maximum power (at resonance) predicted by code1 compare with that from code2 (at resonance)? And the saturation length? Is it reasonable that numerous FEL works omit the [JJ] correction, and the two a_s corrections?

(3)Gain Parameter & Efficiency: Compute the gain parameter

$$\rho = \left\{ \frac{\pi}{8} \frac{a_w^2 [JJ]^2}{k_w^2 \Sigma} \frac{I}{\gamma^3 I_0} \right\}^{1/3}.$$

Predict the FEL efficiency from analytic theory, and compare it to the result of code2.

(4)Exponential Gain: Predict the gain length on resonance, L_g in cm,

$$L_g = \frac{\lambda_w}{2\pi 3^{1/2} \rho}$$

and the corresponding gain in dB/m.

$$G = \frac{20}{\ln 10} \frac{dB}{L_g} \approx \frac{8.7 dB}{L_g} \approx 95 dB \frac{\rho}{\lambda_w}$$

Fit the result of code2 to an exponential over the region in z where growth is exponential (e.g., $5P_{in} < P(z) < 0.5P_{sat}$) and infer the gain in dB/m. How does it compare with the predicted value?

(5) Detuning Curve: Run code2 for 50 (20 if your platform is very slow) values of a_w in the range $0.5a_{wr}<a_w<2a_w$, and plot maximum power versus the corresponding B_w.

Physics of Free Electron Lasers - Homework 3 - Solutions - Gain & Saturation

For this homework we examined exponential gain and saturation via simulation and compared the results to the FEL scalings derived in class. At the outset, we incorporated corrections due to the Bessel function factor and the corrections due to the effect of a_s on the transverse motion in the form of a new 1D code "code2". We compared results for ELF parameters, on resonance, from code2 and those from code1 (without the correction terms) to confirm the corrections were small.

The FEL equations in code2 take the form

$$\frac{d\theta}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + \frac{a_w^2}{2} - a_w[JJ]\mathrm{Re}\left[a\exp(i\theta)\right]\right\},$$

$$\frac{d\gamma}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta}[JJ]\mathrm{Im}\left[a\exp(i\theta)\right],$$

$$\left(\frac{d}{dz} + \frac{2\pi i}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle\right)a = \frac{2\pi i}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle$$

where the average axial speed normalized by c is,

$$\beta = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{a_w^2}{2}\right)$$

The Bessel function factor, $[JJ] = J_0(\xi) - J_1(\xi)$, where

$$\xi = \frac{1}{8}\frac{\omega}{ck_w}\frac{a_w^2}{\gamma^2} \approx 0.36,$$

and

$$[JJ] \approx 1 - \frac{1}{2}\xi - \frac{1}{4}\xi^2 \approx 0.79.$$

(1)Shift in Resonance: With the shift in the resonance due to the non-resonant portion of the beam susceptibility, **a_w˜3.98** is shifted in the fourth digit (3.977 shifted to 3.976) This is completely “in the noise” for this experiment.

(2)Effect of Higher Order Corrections: Code1 predictions are 209MW for power at saturation, and a saturation length of 108cm. Code2 predicts 180MW (lower by 0.6 dB) and 124cm. The difference in power is rather modest, somewhat larger than a typical measurement error. The saturation length is measurably different, about two wiggler periods. In this connection, note that the numerical resolution in z is on the order of a step-size, 2cm.

(3)Gain Parameter & Efficiency: The gain parameter

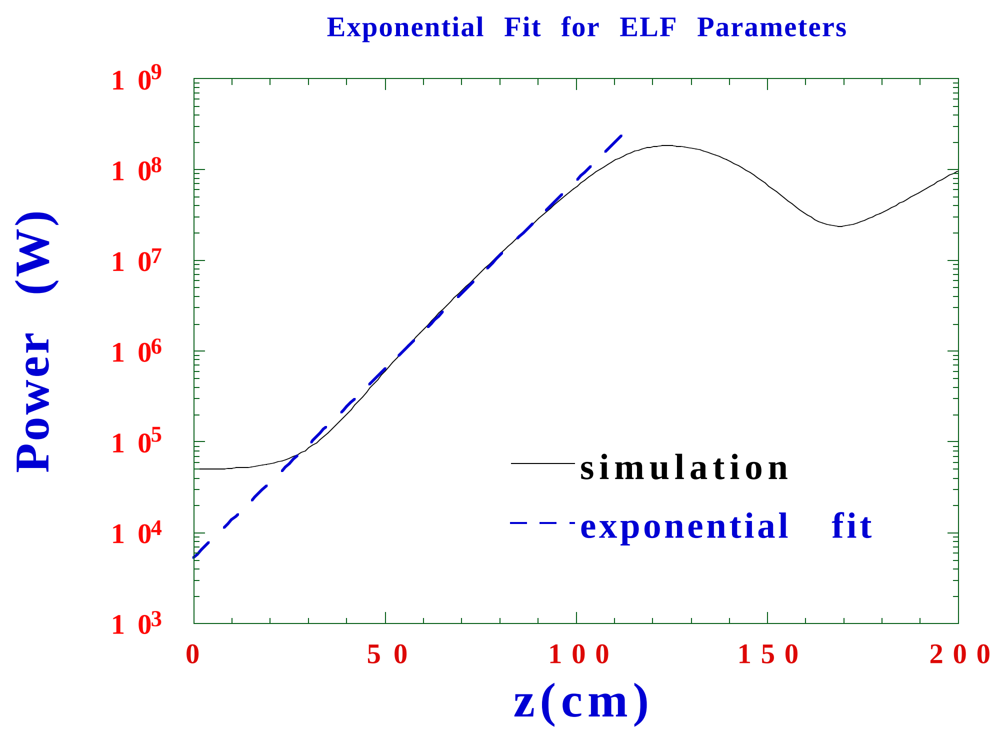
$$\rho = \left\{ \frac{\pi}{8} \frac{a_w^2 [JJ]^2}{k_w^2 \Sigma} \frac{I}{\gamma^3 I_0} \right\}^{1/3} \approx 3.9\%.$$

Corresponding to an FEL efficiency of

$$\eta \approx \rho \frac{\gamma}{\gamma - 1} \approx 4.5\%.$$

Code2 predicts 6.2%.This is typical of the accuracy of such efficiency estimates. Analytic work that is simplified enough to be tractable is usually only helpful as a guide and to gain confidence in and understanding of a simulation that itself is compared to experiment. In the present case, the simulation includes more accurately deviations from the relativistic approximation, and these are noticeable in the ELF example.

(4)Exponential Gain: An exponential fit to code2 (see plot) results in an intercept of 1/9 of 48kW, and a gain length of 20.9 cm or 42dB/m.



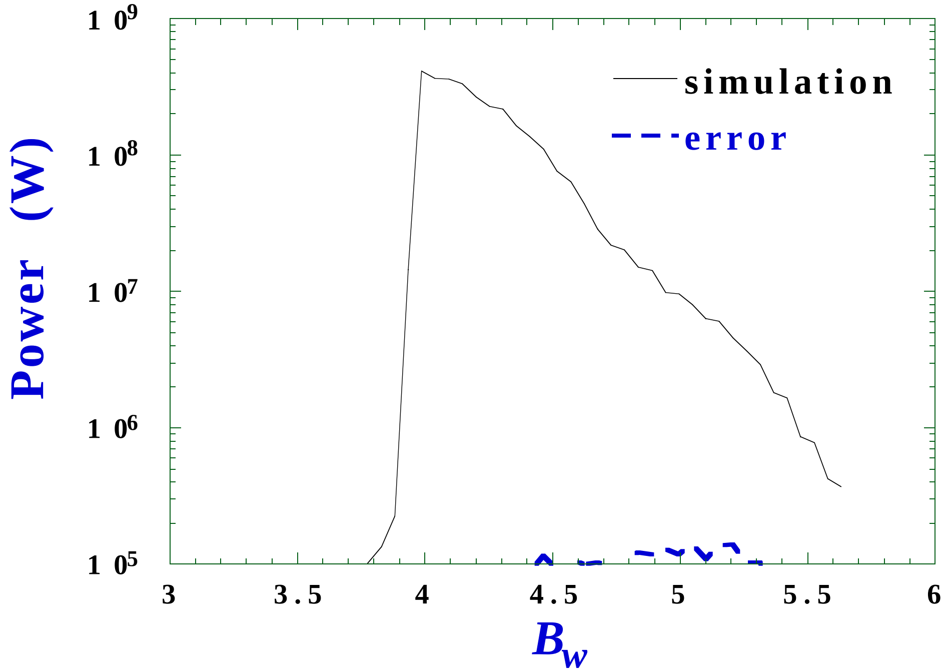
The gain length on resonance from the dispersion relation is

$$L_g = \frac{\lambda_w}{2\pi 3^{1/2}\rho} \approx 23cm$$

and the corresponding gain in dB/m.

$$G = \frac{20}{\ln 10}\frac{dB}{L_g} \approx \frac{8.7dB}{L_g} \approx 95dB\frac{\rho}{\lambda_w} = 38dB/m$$

(5) Detuning Curves: I ran code2 for 50 values of a_w in the range $0.5a_{wr}<a_w<2a_w$, and the result for maximum power versus B_w, is depicted below,



Another plot of interest is gain versus B_w, and this is readily obtained by performing a series of runs for different B_w, and performing an exponential fit over the region of exponential gain for each run. As a check of such a simulation result we need only solve the cubic dispersion relation for the root corresponding to growth (we ignore for now the small amount of energy spread we have):

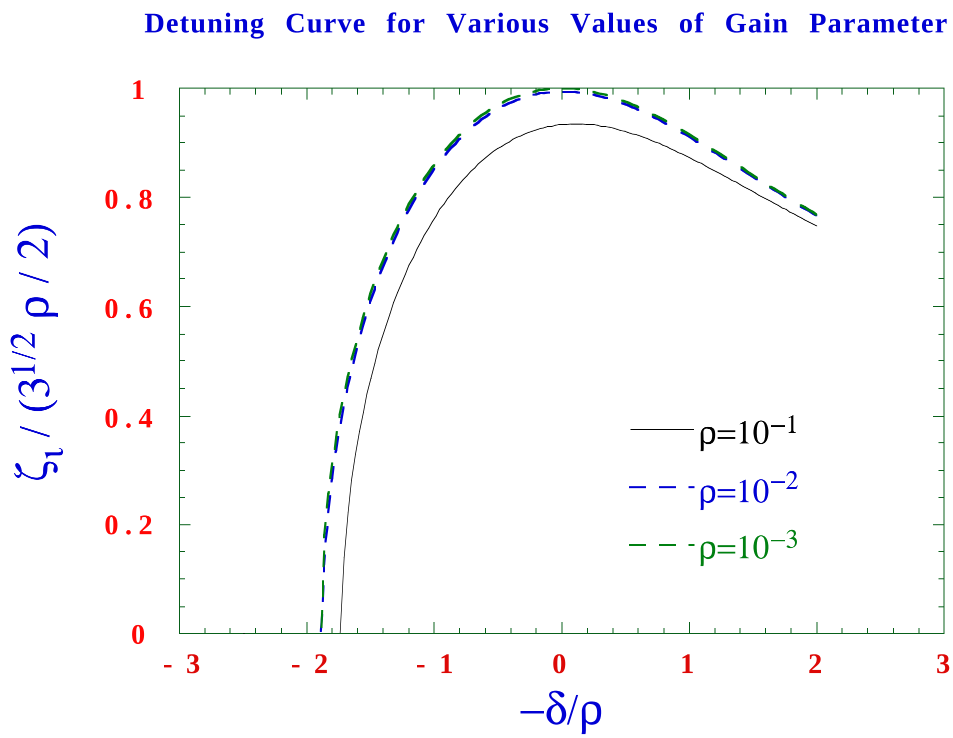
$$(\zeta+\delta)^2\zeta-\hat{\rho}^2(\zeta+\delta)=-\rho^3,$$

where, recall

$$\zeta=\frac{\Gamma+i\nu}{2ik_w},$$

$$\delta=\frac{\Delta k-\nu}{2k_w}.$$

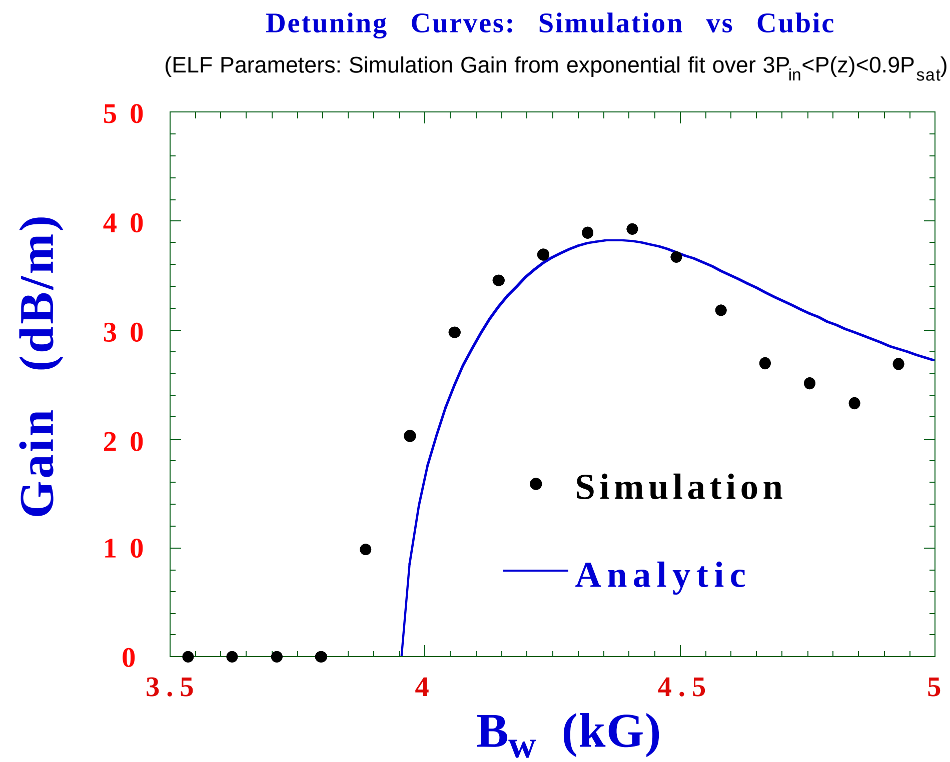
Typical results looks like something like this



Relating the detuning to B_w, via

$$\Delta k_i = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma_i^2}\left\{1+\frac{a_w^2}{2}\right\},$$

we can then compare simulation and the cubic as depicted below.



The differences at high B_w are due to the failure of the relativistic approximation. For this comparison, the exact expressions for $\hat{\rho}, \rho$ were used. The deviations in the smoothness of the simulation gain are due to the way the fitting is done, not really taking into account the variety in the shape of P(z) in the transient regime, when the FEL is well-off resonance.

This homework has given us a look at the more or less “ideal” FEL, not limited by the host of effects we will soon add: space-charge, emittance & focusing, energy spread, and diffraction. The effect of these corrections will be to lower gain and lower efficiency.