Linearized FEL Equations & Gain

To charactize amplification in the FEL, we linearize the FEL equations in perturbations due to the electromagnetic signal. Recall, the FEL equations are the equations of particle motion,

$$\frac{d\theta_i}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma_i^2}\left\{1+\frac{a_w^2}{2} - a_w[JJ]\mathrm{Re}\left[a\exp(i\theta_i)\right]\right\},$$

$$\equiv "k_w + k_z - \frac{\omega}{v_z}"$$

$$\frac{d\gamma_i}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma_i\beta_i}[JJ]\mathrm{Im}\left[a\exp(i\theta_i)\right],$$

$$\equiv "\vec{v}_\perp \bullet \vec{E}_\perp"$$

$$\beta_i = 1 - \frac{1}{2\gamma_i^2}\left(1+\frac{a_w^2}{2}\right),$$

where *i* is the particle index, and the eikonal equation,

$$\left(\frac{d}{dz} + i\underbrace{\frac{2\pi}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle}_{\text{"non-resonant dielectric effect"}}\right)a = i\underbrace{\frac{2\pi}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle}_{\text{"rf component on beam} \times \text{wiggler induced } v_\perp\text{"}}.$$

We will describe the particle motion by small perturbations to the “ballistic” or a=0 motion,

$$\theta_i = \theta_{i0} + \theta_{i1},$$

$$\gamma_i = \gamma_{i0} + \gamma_{i1},$$

where the unperturbed motion is just

$$\theta_{i0}(z) = \theta_i(0) + \Delta k_i z,$$
$$\gamma_{i0}(z) = \gamma_{i0}(0).$$

The initial detuning is

$$\Delta k_i = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma_i^2}\left\{1 + \frac{a_w^2}{2}\right\}.$$

The equations for the perturbations take the form

$$\begin{aligned}\frac{d\theta_{i1}}{dz} &= \Delta k' \gamma_{1i} + q_i \frac{\beta_{i0}}{\gamma_{i0}} \mathrm{Re}\left[a \exp(i\theta_{i0})\right] \\ &= \Delta k' \gamma_{1i} + q_i \frac{\beta_{i0}}{\gamma_{i0}} \mathrm{Im}\left[ia \exp(i\theta_{i0})\right]\end{aligned},$$
$$\frac{d\gamma_{i1}}{dz} = -q_i \,\mathrm{Im}\left[a \exp(i\theta_{i0})\right]$$

where we abbreviate

$$q_i = \frac{1}{2}\frac{\omega}{c}\frac{a_w[JJ]}{\gamma_{i0}\beta_{i0}},$$
$$\Delta k_i' = \frac{\partial \Delta k_i}{\partial \gamma_i} = \frac{\omega}{c}\frac{1}{\gamma_i^3}\left\{1 + \frac{a_w^2}{2}\right\}.$$

The eikonal equation takes the form

$$
\begin{aligned}
\left(\frac{d}{dz}+i\nu\right)a &= iQ\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle \\
&\approx iQ\left\langle\frac{-i\theta_{i1}\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}-\frac{\exp(-i\theta_{i0})}{(\gamma_{i0}\beta_{i0})^2}\frac{\partial(\gamma_{i0}\beta_{i0})}{\partial\gamma_{i0}}\gamma_{i1}\right\rangle \\
&\approx iQ\left\langle\frac{-i\theta_{i1}\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}-\frac{\exp(-i\theta_{i0})}{\gamma_{i0}^2\beta_{i0}^3}\gamma_{i1}\right\rangle
\end{aligned}
$$

where we abbreviate

$$
\nu=\frac{2\pi}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle,
$$

$$
Q=\frac{2\pi}{k_z\Sigma}\left(\frac{I}{I_0}\right)a_w[JJ],
$$

and have used

$$
\begin{aligned}
\frac{\partial(\gamma_{i0}\beta_{i0})}{\partial\gamma_{i0}} &= \frac{\partial}{\partial\gamma_{i0}}\gamma_{i0}\left\{1-\frac{1}{2\gamma_{i0}^2}\left(1+\frac{a_w^2}{2}\right)\right\} \\
&= 1+\frac{1}{2\gamma_{i0}^2}\left(1+\frac{a_w^2}{2}\right) \\
&\approx \frac{1}{\beta_{i0}}
\end{aligned}
$$

in the last line.

To solve this system we first represent the eikonal as a sum over exponentials

$$
a(z)=\sum_m a_m\exp(\Gamma_m z).
$$

Note the conditions on the coefficients,

$$a(0) = \sum_m a_m ,$$

and, assuming an initially unbunched beam,

$$\left(\frac{da}{dz} + i\nu a \right)_{z=0} = \sum_m \left(\Gamma_m + i\nu \right) a_m = 0 .$$

It is convenient to express the perturbed particle variables as

$$\gamma_{i1} = \frac{\tilde{\gamma}_{i1} - \tilde{\gamma}_{i1}^*}{2i},$$

$$\theta_{i1} = \frac{\tilde{\theta}_{i1} - \tilde{\theta}_{i1}^*}{2i},$$

where

$$\frac{d\tilde{\gamma}_{i1}}{dz} = -q_i a \exp\left(i\theta_{i0} \right),$$

$$\frac{d\tilde{\theta}_{i1}}{dz} = \Delta k' \tilde{\gamma}_{i1} + q_i \frac{\beta_{i0}}{\gamma_{i0}} ia \exp\left(i\theta_{i0} \right)$$

Integrating the first of these we obtain,

$$\tilde{\gamma}_{i1} = \int_0^z dz' \left[-q_i a \exp\left(i\theta_{i0} \right) \right]$$

$$= -q_i \sum_m a_m \int_0^z dz' \exp\left\{ i\theta_i(0) + \left(\Gamma_m + i\Delta k_i \right) z \right\}$$

$$= -q_i \sum_m \frac{a_m \exp\left\{ i\theta_i(0) \right\}}{\left(\Gamma_m + i\Delta k_i \right)} \left[\exp\left\{ \left(\Gamma_m + i\Delta k_i \right) z \right\} - 1 \right]$$

A similar calculation shows that

$$\tilde{\theta}_{i1} = -q_i \Delta k' \sum_m \frac{a_m \exp\left\{ i\theta_i(0) \right\}}{\left(\Gamma_m + i\Delta k_i \right)^2} \left[\exp\left\{ \left(\Gamma_m + i\Delta k_i \right) z \right\} - 1 - \left(\Gamma_m + i\Delta k_i \right) z \right]$$

$$+ q_i \frac{\beta_{i0}}{\gamma_{i0}} i \sum_m \frac{a_m \exp\left\{ i\theta_i(0) \right\}}{\left(\Gamma_m + i\Delta k_i \right)} \left[\exp\left\{ \left(\Gamma_m + i\Delta k_i \right) z \right\} - 1 \right]$$

Next, we substitute these results in the eikonal equation,

$$\left(\frac{d}{dz} + i\nu \right) a = \sum_m \left(\Gamma_m + i\nu \right) a_m \exp\left(\Gamma_m z \right)$$

$$= iQ \left\langle \frac{-i\theta_{i1} \exp\left(-i\theta_{i0} \right)}{\gamma_{i0} \beta_{i0}} - \frac{\exp\left(-i\theta_{i0} \right)}{\gamma_{i0}^2 \beta_{i0}^3} \gamma_{i1} \right\rangle$$

We will find that this amounts to an equation between two sums over exponentials and equating coefficients will then permit us to solve for the complex wavenumbers Γ_m. In the course of this, we will require that non-exponential terms vanish, and this imposes the conditions

$$\sum_m \frac{a_m}{\left(\Gamma_m + i\Delta k_i \right)} = 0,$$

$$\sum_m \frac{a_m}{\left(\Gamma_m + i\Delta k_i \right)^2} = 0.$$

(This last constraint will turn out to be redundant.) So in computing,

$$\begin{aligned}
\left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}(-i\theta_{i1}) \right\rangle &= \left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}\left(-i\frac{\tilde{\theta}_{i1}-\tilde{\theta}_{i1}^*}{2i}\right) \right\rangle \\
&= \left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}\left(-\frac{1}{2}\tilde{\theta}_{i1}\right) \right\rangle \\
&= \left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}\left(\frac{1}{2}q_i\Delta k'\right)\sum_m \frac{a_m\exp\{i\theta_i(0)\}}{(\Gamma_m+i\Delta k_i)^2}\left[\exp\{(\Gamma_m+i\Delta k_i)z\}-1-(\Gamma_m+i\Delta k_i)z\right] \right\rangle + \\
&\quad \left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}\beta_{i0}}\left(-\frac{i}{2}q_i\frac{\beta_{i0}}{\gamma_{i0}}\right)\sum_m \frac{a_m\exp\{i\theta_i(0)\}}{(\Gamma_m+i\Delta k_i)}\left[\exp\{(\Gamma_m+i\Delta k_i)z\}-1\right] \right\rangle \\
&= \left\langle \frac{q_i\Delta k'}{2\gamma_{i0}\beta_{i0}}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)^2}\left[1-(1+(\Gamma_m+i\Delta k_i)z)\exp\{-(\Gamma_m+i\Delta k_i)z\}\right] \right\rangle + \\
&\quad \left\langle -\frac{iq_i}{2\gamma_{i0}^2}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)}\left[1-\exp\{-(\Gamma_m+i\Delta k_i)z\}\right] \right\rangle \\
&= \left\langle \frac{q_i\Delta k'}{2\gamma_{i0}\beta_{i0}}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)^2} \right\rangle - \left\langle \frac{iq_i}{2\gamma_{i0}^2}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)} \right\rangle
\end{aligned}$$

we zero the non-exponential terms in the last line, likewise, we find

$$\begin{aligned}
\left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}^2\beta_{i0}^3}\gamma_{i1} \right\rangle &= \left\langle \frac{\exp(-i\theta_{i0})}{\gamma_{i0}^2\beta_{i0}^3}\frac{\tilde{\gamma}_{i1}}{2i} \right\rangle \\
&= \left\langle \frac{iq_i}{2\gamma_{i0}^2\beta_{i0}^3}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)} \right\rangle
\end{aligned}$$

With these results, the eikonal equation

$$\begin{aligned}
\sum_m(\Gamma_m+i\nu)a_m\exp(\Gamma_m z) = {} & iQ\left\langle \frac{q_i\Delta k'}{2\gamma_{i0}\beta_{i0}}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)^2} \right\rangle - iQ\left\langle \frac{iq_i}{2\gamma_{i0}^2}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)} \right\rangle + \\
& -iQ\left\langle \frac{iq_i}{2\gamma_{i0}^2\beta_{i0}^3}\sum_m \frac{a_m\exp(\Gamma_m z)}{(\Gamma_m+i\Delta k_i)} \right\rangle
\end{aligned}$$

is reduced to an algebraic relation determined by equating coefficients,

$$\left(\Gamma_m + i\nu\right) = iQ\left\langle \frac{q_i \Delta k'}{2\gamma_{i0}\beta_{i0}} \frac{1}{\left(\Gamma_m + i\Delta k_i\right)^2} \right\rangle - Q\left\langle \frac{q_i}{2\gamma_{i0}^2}\left(1+\frac{1}{\beta_{i0}^3}\right)\frac{1}{\left(\Gamma_m + i\Delta k_i\right)} \right\rangle.$$

At this point, the brackets amount to an average over the initial energy distribution of the electrons. The primary dependence on energy is through the resonant denominators, thus one may to a good approximation simply evaulate the algebraic factors at the average γ. This dispersion relation is most simply described in terms of dimensionless parameters, $\rho, \hat{\rho}, \delta$, where

$$\begin{aligned}
\left(2\rho k_w\right)^3 &= \frac{1}{2\gamma\beta} Q q \Delta k' \\
&= \frac{1}{2\gamma\beta} \underbrace{\frac{2\pi}{k_z \Sigma}\left(\frac{I}{I_0}\right) a_w[JJ]}_{Q} \underbrace{\frac{1}{2}\frac{\omega}{c}\frac{a_w[JJ]}{\gamma\ \beta}}_{q} \underbrace{\frac{\omega}{c}\frac{1}{\gamma^3}\left\{1+\frac{a_w^2}{2}\right\}}_{\Delta k'} \\
&= \frac{\pi}{2}\frac{(\omega/c)^2}{k_z\Sigma}\left(\frac{I}{I_0}\right)\frac{\left(a_w[JJ]\right)^2}{\gamma^5\beta^2}\left\{1+\frac{a_w^2}{2}\right\}
\end{aligned}$$

or

$$\begin{aligned}
\rho^3 &= \frac{\pi}{16}\frac{(\omega/c)^2}{k_z \Sigma k_w^3}\left(\frac{I}{I_0}\right)\frac{\left(a_w[JJ]\right)^2}{\gamma^5\beta^2}\left\{1+\frac{a_w^2}{2}\right\} \\
&\approx \frac{\pi}{8}\frac{1}{\Sigma k_w^2}\left(\frac{I}{\gamma^3 I_0}\right)\left(a_w[JJ]\right)^2
\end{aligned}$$

and in the last line we have evaluated ρ on resonance. The second dimensionless parameter, $\hat{\rho}$, arising from roughly equal contributions due to the relativistic mass effect, and the perturbation to the transverse motion due to signal-induced jitter, is

$$
\begin{aligned}
\left(2\hat{\rho}k_w\right)^2 &= \frac{Qq}{2\gamma^2}\left(1+\frac{1}{\beta^3}\right) \\
&= \frac{1}{2\gamma^2}\underbrace{\frac{2\pi}{k_z\Sigma}\left(\frac{I}{I_0}\right)a_w[JJ]}_{Q}\underbrace{\frac{1}{2}\frac{\omega}{c}\frac{a_w[JJ]}{\gamma\ \beta}}_{q}\left(1+\frac{1}{\beta^3}\right) \\
&= \frac{\pi}{2}\frac{(\omega/c)}{k_z\Sigma}\left(\frac{I}{I_0}\right)\frac{\left(a_w[JJ]\right)^2}{\gamma^3\beta}\left(1+\frac{1}{\beta^3}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
\hat{\rho}^2 &= \frac{\pi}{8}\frac{(\omega/c)}{k_z\Sigma k_w^2}\left(\frac{I}{I_0}\right)\frac{\left(a_w[JJ]\right)^2}{\gamma^3\beta}\left(1+\frac{1}{\beta^3}\right) \\
&\approx \frac{\pi}{4}\frac{1}{\Sigma k_w^2}\left(\frac{I}{\gamma^3 I_0}\right)\left(a_w[JJ]\right)^2 \\
&\approx 2\rho^3
\end{aligned}
$$

The dimensionless detuning is

$$\delta = \frac{\Delta k - \nu}{2k_w},$$

and this is in principle a particle - variable (a function of energy), but we will shortly specialize to the case of a mono-energetic (“cold”) beam. In terms of these variables, and the dimensionless complex wavenumber,

$$\zeta = \frac{\Gamma + i\nu}{2ik_w},$$

we arrive at the dispersion relation

$$\zeta = -\rho^3 \left\langle \frac{1}{(\zeta+\delta)^2} \right\rangle + \hat{\rho}^2 \left\langle \frac{1}{(\zeta+\delta)} \right\rangle,$$

or, for a cold beam, the cubic,

$$(\zeta+\delta)^2 \zeta - \hat{\rho}^2 (\zeta+\delta) = -\rho^3.$$

Making the approximation,

$$\hat{\rho}^2 \approx 2\rho^3,$$

this can be reduced to a two-parameter cubic equation. The three roots of this equation determine the actual complex wavenumbers Γ_m and they determine the mode coefficients according to

$$\sum_m a_m = a(0)$$
$$\sum_m \zeta_m a_m = 0,$$
$$\sum_m \frac{a_m}{\zeta_m} = 0.$$

Numerous useful results derive from analysis of this cubic, and we turn to this next.