Physics of Free Electron Lasers

Thumbnail Guide to the 82 Most Popular Equations

from Lectures 1-4

http://beam.slac.stanford.edu/~whittum/ap453b

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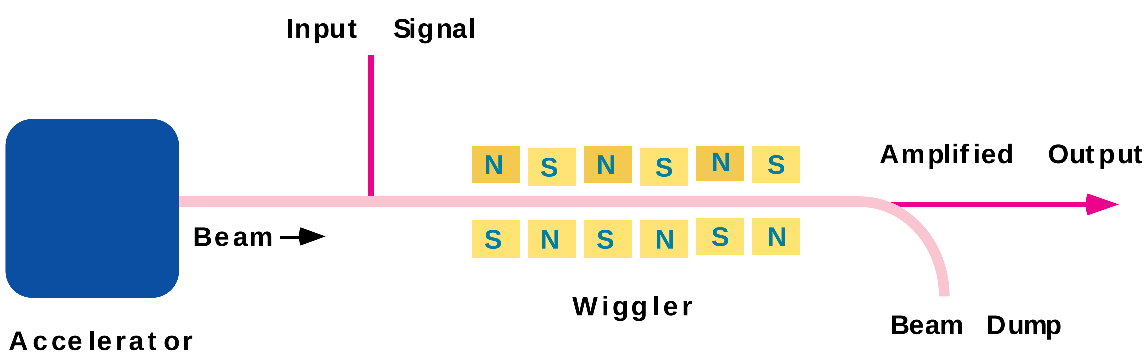


Maxwell's Equations & The Lorentz Force Law

In the first lecture we reviewed Maxwell's Equations in those units most favored by the latest Gallup polls (cgs),

$$\nabla \bullet \vec{E} = 4\pi\rho,$$

$$\nabla \bullet \vec{B} = 0,$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t},$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J},$$

and the expression in terms of the vector and scalar potentials,

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\varphi,$$

$$\vec{B} = \nabla \times \vec{A}.$$

We chose to work in the Lorentz Gauge, (being alert to the perils of lesser gauges),

$$\nabla \bullet \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0,$$

and we found that Maxwell's equations take the form of wave-equations,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{J},$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -4\pi\rho.$$

We also recalled the Lorentz Force Equation,

$$\frac{d\vec{p}}{dt} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right).$$

Later we showed that this can be rewritten in terms of potentials as

$$\frac{d}{dt}\left(p^i - \frac{eA^i}{c}\right) = -\frac{\partial}{\partial x^i} e\left(\frac{\vec{v}}{c} \bullet \vec{A} - \varphi\right).$$

We did this using the anti-symmetric tensor ε, such that $(A \times B)^i = \varepsilon^{ijk} A^j B^k$, and the identity $\varepsilon^{ijk}\varepsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$, where a repeated index is summed over, and δ^{ij} is the Kronecker delta, 1 if indices are equal, else 0.

We discussed accelerators, their basis in electrodynamics, the simplest accelerators - guns, and worked out the problem of a planar diode, deriving the notion of **perveance**. We also considered a beam drifting down a tube (e.g. from the anode plane) and derived the notion of **emittance**. Later, these notions of emittance and perveance will come back to haunt us.

Wave-Particle Interaction

In the second lecture, after some discussion of wave-particle interaction, TM vs. TE interactions, fast-wave vs slow-wave, and the limitations on slow-wave devices as we go to short wavelengths, we settled on the need for a device that would induce a periodic transverse oscillation in a beam to permit a fast-wave, TE resonance.

After mention of gyrotrons, cyclotron auto-resonant masers, laser-undulators, etc., we decided to concentrate for the time-being on the interaction induced by a static magnetic wiggler and the corresponding free-electron device, the free-electron laser.

We considered an idealized planar wiggler, consisting of a periodic array of alternating static dipoles, of infinite extent in *x* and *z*, with period λ_w. This treatment **neglected fringe field effects** at the wiggler entrance. The vector potential for this problem has only an *x*-component, and satisfies the Laplace equation

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right) A_w(y,z) = 0,$$

within the beamline. Since the source terms (the dipoles) are periodic in *z*, one suspects the solution will be also, and would be amenable to a Fourier series expansion,

$$A_w(y,z) = \sum_{n=-\infty}^{\infty} A_{wn}(y) \exp(ink_w z).$$

where $k_w = 2\pi / \lambda_w$. For an ideal wiggler only the $n=\pm1$ terms are non-zero, and their *y*-dependence is prescribed by the Laplace equation to be $\exp(\pm k_w z)$.The symmetry of the dipole magnets selects the form $\cosh(k_w y)$, with the result that for our idealized planar wiggler,

$$A_w = \frac{mc^2}{e} a_w \cosh(k_w y) \sin(k_w z).$$

The choice of $\sin(k_w z)$(as opposed to cosine or some other phase) amounts to a choice of origin in *z*. The factor in front, $mc^2 / e \approx 1.7 kG - cm$, insures that a_w, the wiggler parameter, is dimensionless. This vector potential corresponds to a spatially alternating magnetic field with *y* and *z* components,

$$B_{wy} = \frac{mc^2}{e} k_w a_w \cosh(k_w y) \cos(k_w z),$$

$$B_{wz} = -\frac{mc^2}{e} k_w a_w \sinh(k_w y) \sin(k_w z).$$

Eventually we neglected the *y*-dependence, assuming our beam was rather small, and that *y*-motion was not central to the problem. This amounts to a **neglect of focusing** and emittance. (Soon we will find the time to enjoy more fully the details of focussing and emittance.)

Physical Optics with Beam Interaction

Having decided that we could indeed induce the periodic oscillation we sought, we asked next, how to determine the radiation generated by such an undulating beam. Why not just compute the Green's function and solve the wave-equation directly? This reminded us of the central problem of such "collective phenomena", that one must solve for the particle motion in terms of the fields, and the fields in terms of the particle motion, all in one go, self-consistently. Deriving such self-consistent **FEL Equations**, is the work of the subsequent lectures.

Eikonal Equation

To begin the analysis, we analyzed the wave-equation first. We examined the propagation of an electromagnetic signal through the wiggler with the beam. **Neglecting space-charge effects** arising from ρ and J_z in the wave equations for the vector and scalar potentials, we kept only the *x*-component of the vector potential, taking the signal to be of the form

$$A_s = \frac{mc^2}{e} a_s \sin\left(k_z z - \omega t + \varphi_s\right) = \frac{mc^2}{e}\frac{1}{2}\left\{-ia\exp\left(ik_z z - i\omega t\right) + c.c.\right\},$$

where "*c.c*" denotes the complex conjugate, and $a = a_s exp(i\varphi_s)$. The eikonal *a* is a slowly varying phasor. This corresponds to an electric field,

$$\vec{E} \approx \hat{x}\left(\frac{\omega}{c}\right)\frac{mc^2}{e} a_s \cos\left(k_z z - \omega t + \varphi_s\right),$$

and a magnetic field,

$$\vec{B} \approx \hat{y} k_z \frac{mc^2}{e} a_s \cos\left(k_z z - \omega t + \varphi_s\right).$$

Maxwell's equations can then be rewritten in terms of *a*. To do this we first compute derivatives,

$$\frac{\partial A_s}{\partial t} = \frac{mc^2}{e}\frac{1}{2}\left\{-i\left(-i\omega a + \frac{\partial a}{\partial t}\right)\exp\left[i\left(k_z z - \omega t\right)\right] + c.c.\right\},$$

$$\frac{\partial^2 A_s}{\partial t^2} = \frac{mc^2}{e}\frac{1}{2}\left\{-i\left(-\omega^2 a - 2i\omega\frac{\partial a}{\partial t} + \frac{\partial^2 a}{\partial t^2}\right)\exp\left[i\left(k_z z - \omega t\right)\right] + c.c.\right\}.$$

We then implement the "eikonal approximation", namely that *a* varies slowly on the scale of the signal wavelength,

$$\left|\frac{\partial^2 a}{\partial t^2}\right| << \left|\omega\frac{\partial a}{\partial t}\right| << \left|\omega^2 a\right|.$$

In this case,

$$\frac{\partial^2 A_s}{\partial t^2} \approx \frac{mc^2}{e}\frac{1}{2}\left\{-i\left(-\omega^2 a - 2i\omega\frac{\partial a}{\partial t}\right)\exp\left[i\left(k_z z - \omega t\right)\right] + c.c.\right\},$$

and a similar calculation shows that

$$\frac{\partial^2 A_s}{\partial z^2} \approx \frac{mc^2}{e}\frac{1}{2}\left\{-i\left(-k_z^2 a + 2ik_z\frac{\partial a}{\partial z}\right)\exp\left[i\left(k_z z-\omega t\right)\right]+c.c.\right\},$$

provided,

$$\left|\frac{\partial^2 a}{\partial z^2}\right| << \left|k_z\frac{\partial a}{\partial z}\right| << \left|k_z^2 a\right|.$$

The wave equation then takes the form,

$$\begin{aligned} -\frac{4\pi}{c}J_x &= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_x \\ &= \frac{1}{2}\frac{mc^2}{e}\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{-ia\exp\left[i\left(k_z z-\omega t\right)\right]+c.c\right\} \\ &\approx -i\frac{1}{2}\frac{mc^2}{e}\exp\left[i\left(k_z z-\omega t\right)\right]\left(\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2 + 2i\left(k_z\frac{\partial}{\partial z}+\frac{\omega}{c^2}\frac{\partial}{\partial t}\right)\right)a + c.c \end{aligned}$$

Next we multiply both sides by $\exp\left[-i\left(k_z z-\omega t\right)\right]$ and perform an average over one signal period in *t*, $2\pi/\omega$ and one signal wavelength in *z*, $2\pi/k_z$. Only terms varying slowly on this time-scale remain,

$$\left(\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2 + 2i\left(k_z\frac{\partial}{\partial z}+\frac{\omega}{c^2}\frac{\partial}{\partial t}\right)\right)a \approx -\frac{8\pi i}{I_0}\left\langle \exp\left[-i\left(k_z z-\omega t\right)\right]J_x\right\rangle_\lambda,$$

where $I_0 = mc^3/e \approx 17.05kA$. The subscript on the brackets reminds us that this is an average over the signal period in *z* and in *t*. This result can be further simplified by changing variables from *(z,t)* to *(z,ζ),* where

$$\zeta = t - \frac{\omega z}{c^2 k_z} = t - \frac{z}{v_g},$$

in terms of which,

$$\left(\nabla_\perp^2+\left(\frac{\omega}{c}\right)^2-k_z^2+2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta\right)a\approx-\frac{8\pi i}{I_0}\left\langle\exp\left[-i\left(k_z z-\omega t\right)\right]J_x\right\rangle_\lambda.$$

The parentheses with subscript ζ will often be omitted. The distinction between the partial derivative and the ordinary derivative becomes important when “slippage” is significant, i.e., when the difference in the group velocity of the wave, and the particle beam velocity result in the wave slipping off the beam during the course of the transit through the wiggler. In free-space the condition for neglect of slippage is just $N_w\lambda_w$<beam pulse length.

Beam Motion in a Wiggler & an Electromagnetic Wave

To make further progress in reducing the problem to manageable form, we need to analyze the source term driving the eikonal equation. This requires treating the particle motion in the wiggler. In this connection, the equation of motion in terms of the potentials,

$$\frac{d}{dt}\left(p^i-\frac{eA^i}{c}\right)=-\frac{\partial}{\partial x^i}e\left(\frac{\vec{v}}{c}\bullet\vec{A}-\varphi\right),$$

is quite helpful, as it indicates that when the potentials are independent of x, as they are in the present case, that,

$$p_x=m\gamma v_x=\frac{eA_x}{c},$$

up to a constant that we will neglect for the time-being, consistent with previous neglect of emittance. The Lorentz factor is,

$$\gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-1/2} .$$

At this point we have determined the transverse motion in the wiggler,

$$\gamma \frac{v_x}{c} = \frac{eA_x}{mc^2} = a_w \sin(k_w z) + a_s \sin(k_z z - \omega t + \varphi_s),$$

$$\gamma \frac{v_y}{c} = 0 .$$

This involves some approximations as noted previously. In particular the actual motion in *x* and *y* involves the "design" trajectories above superimposed on which are betatron oscillations in *x,y*. We will return to these later. For the time being, our model of the FEL consists of a prescribed undulation in *x*, with a small, rapid jitter superimposed and some very crucial goings-on in the longitudinal direction. The jitter in *x* due to the signal field is not a big effect and oftentimes it is neglected. We keep it here, for the sake of clarity.

Let's consider in detail the longitudinal motion. The axial speed normalized by *c* is

$$\beta_z = \frac{v_z}{c} = \left(1 - \frac{1}{\gamma^2} - \frac{v_x^2}{c^2}\right)^{1/2} \approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2}\frac{v_x^2}{c^2},$$

making the relativistic approximation γ>>1 in the last line. Using our result for v_x this may be expressed as

$$\beta_z = 1 - \frac{1}{2\gamma^2} - \frac{1}{2\gamma^2}\left\{a_w^2 \sin^2(k_w z) + 2a_w a_s \sin(k_w z)\sin(k_z z - \omega t + \varphi_s) + a_s^2 \sin^2(k_z z - \omega t + \varphi_s)\right\}$$

$$\approx 1 - \frac{1}{2\gamma^2}\left\{1 + a_w^2 \sin^2(k_w z) + 2a_w a_s \sin(k_w z)\sin(k_z z - \omega t + \varphi_s)\right\}$$

The second order term a_s^2 is quite negligible. Meanwhile, the energy varies according to

$$\frac{d}{dt}mc^2\gamma = -ev_x E_x .$$

It will be convenient to use *z* as the independent variable, so we may write

$$\frac{d\gamma}{dz} = -\frac{e}{mc^2}\frac{v_x}{v_z}E_x ,$$

and substituting from our results for v_x and E_x we obtain,

$$\begin{aligned}\frac{d\gamma}{dz} &\approx -\left(\frac{\omega}{c}\right)\frac{1}{\gamma\beta_z}\left\{a_w \sin(k_w z) + a_s \sin(k_z z - \omega t + \varphi_s)\right\}a_s \cos(k_z z - \omega t + \varphi_s) \\ &\approx -\left(\frac{\omega}{c}\right)\frac{a_w a_s}{\gamma\beta_z}\sin(k_w z)\cos(k_z z - \omega t + \varphi_s)\end{aligned}$$

We have neglected a small term in the first line (in the expression for E) that amounts to a radiation pressure force arising when there is a gradient in the field. In the second line we have neglected a second order term in the signal (in fact it will shortly average to zero, and neglect here amounts simply to neglect of radiation pressure). We may also write this as

$$\frac{d\gamma}{dz} \approx -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w a_s}{\gamma\beta_z}\left\{\sin(\psi) - \sin(\psi - 2k_w z)\right\},$$

where

$$\psi = k_w z + k_z z - \omega t + \varphi_s .$$

A still more convenient form will be

$$\frac{d\gamma}{dz} \approx -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta_z}\,\mathrm{Im}\left\{a\left[\exp(i\theta) - \exp\left(i\theta - i2k_w z\right)\right]\right\},$$

with

$$\theta = k_w z + k_z z - \omega t.$$

Recalling the discussion of wave-particle interaction, we see that we have here the opportunity for resonant energy exchange with the field, taking place over many wiggler periods. This requires that θ vary slowly (resonance), in a sense to be made precise shortly. To perform the average over many wiggler periods, and determine what if any secular interaction occurs, we must know more about t viewed as a function of z ("arrival time at z"), or equivalently, θ.

The evolution in θ is described by

$$\begin{aligned}\frac{d\theta}{dz} &= k_w + k_z - \frac{\omega}{c}\frac{1}{\beta_z} \\ &= k_w + k_z - \frac{\omega}{c} - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + a_w^2 \sin^2\left(k_w z\right) + 2a_w a_s \sin\left(k_w z\right)\sin\left(k_z z - \omega t + \varphi_s\right)\right\}\end{aligned}$$

We will abbreviate

$$\delta k = \frac{\omega}{c} - k_z,$$

often referred to as the "waveguide correction", since it is zero in free-space, and non-zero in any waveguide worth its salt. We may then simplify a bit

$$\frac{d\theta}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + a_w^2 \sin^2(k_w z) + 2a_w a_s \sin(k_w z)\sin(k_z z - \omega t + \varphi_s)\right\}$$
$$= \Delta k + \frac{\omega}{c}\frac{1}{4\gamma^2}\left\{a_w^2 \cos(2k_w z) + 2a_w a_s\left[\cos(\psi) - \cos(\psi - 2k_w z)\right]\right\}$$
$$= \Delta k + \frac{\omega}{c}\frac{1}{4\gamma^2}\left\{a_w^2 \cos(2k_w z) + 2a_w \operatorname{Re}\left[a\left(\exp(i\theta) - \exp(i\theta - i2k_w z)\right)\right]\right\}$$

where we have introduced,

$$\Delta k = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + \frac{a_w^2}{2}\right\},$$

a measure of the detuning from resonance for each electron. Next let's integrate to obtain θ. We do this in two steps. First we integrate neglecting the a_s term,

$$\theta = \bar{\theta} + \frac{\omega}{c}\frac{a_w^2}{8\gamma^2}\sin(2k_w z) + (a - term)$$
$$= \bar{\theta} + \xi \sin(2k_w z) + (a - term) .$$

and we will abbreviate,

$$\xi = \frac{\omega}{ck_w}\frac{a_w^2}{8\gamma^2}.$$

Next to compute average value for the a-term we employ this result for θ. This is self-consistent since we are keeping only first order terms in *a*.

$$\frac{d}{dz}(a - term) = \frac{\omega}{c}\frac{1}{4\gamma^2}2a_w \operatorname{Re}\left[a\exp(i\bar{\theta})\left(\exp(i\xi\sin(2k_w z)) - \exp(i\xi\sin(2k_w z) - i2k_w z)\right)\right].$$

This we may average using the Bessel Function identity,

$$\exp\left(i\xi\sin(\delta)\right) = \sum_{n=-\infty}^{+\infty} J_n(\xi)\exp(in\delta),$$

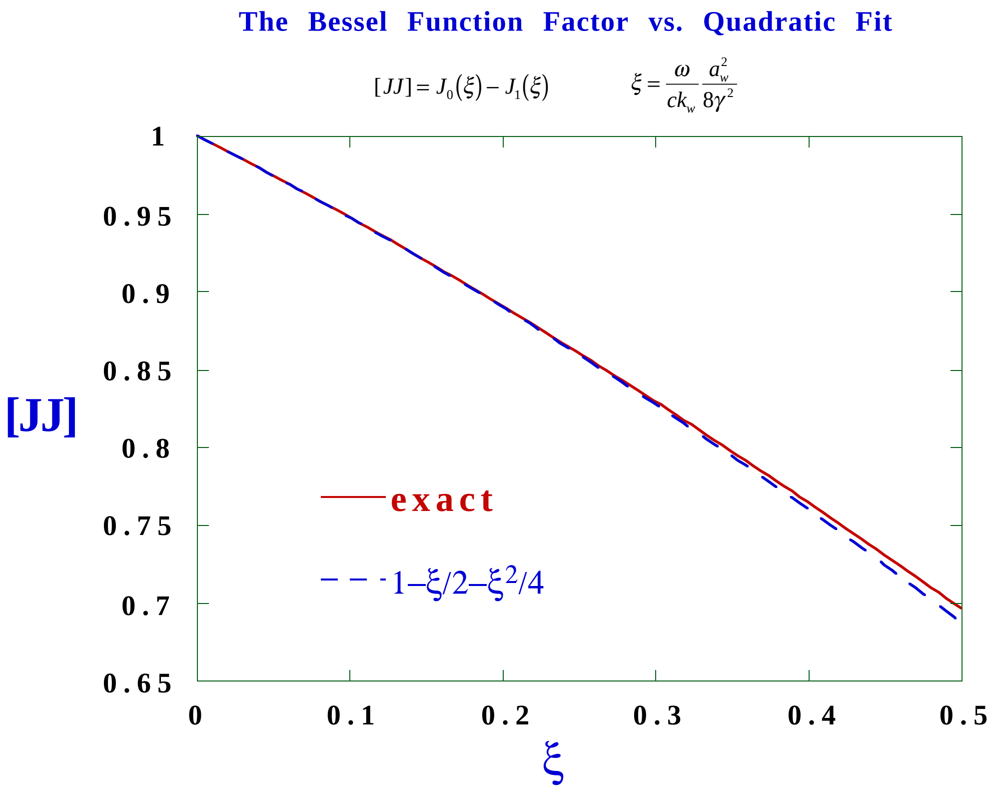
to obtain,

$$\overline{\frac{d}{dz}(a-term)} = \frac{\omega}{c}\frac{a_w}{2\gamma^2}[JJ]\mathrm{Re}\left[a\exp\left(i\overline{\theta}\right)\right],$$

where

$$\begin{aligned}[JJ] &= J_0(\xi) - J_1(\xi) \\ &\approx \left(1 - \frac{1}{4}\xi^2 + \frac{1}{64}\xi^4\right) - \left(\frac{1}{2}\xi - \frac{1}{16}\xi^3\right). \\ &\approx 1 - \frac{1}{2}\xi - \frac{1}{4}\xi^2\end{aligned}$$

The validity of the quadratic approximation can be checked by consulting the following plot



Finally we arrive at

$$\frac{d\bar{\theta}}{dz} = \Delta k + \frac{\omega}{c}\frac{a_w[JJ]}{2\gamma^2}\mathrm{Re}\left[a\exp(i\theta)\right]$$
$$= k_w + \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1+\frac{a_w^2}{2} - a_w[JJ]\mathrm{Re}\left[a\exp\left(i\bar{\theta}\right)\right]\right\}.$$

A similar calculation shows that

$$\frac{d\bar{\gamma}}{dz} \approx -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\bar{\gamma}\bar{\beta}_z}[JJ]\mathrm{Im}\left[a\exp\left(i\bar{\theta}\right)\right],$$

with the additional approximation that the jitter in axial velocity is small in absolute terms in the denominator. This is just the large γ approximation.

Evidently, when

$$\Delta k = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1+\frac{a_w^2}{2}\right\} \approx 0\,,$$

an electron maintains on average a constant phase-relation with the transverse *E* field, and can give up energy to the wave (or gain energy). This is just the FEL resonance condition. In free-space, it may be rewritten simply as

$$\lambda \approx \frac{\lambda_w}{2\gamma^2}\left(1+\frac{a_w^2}{2}\right).$$

Eikonal Equation Revisited

At this point we are in a position to reduce the eikonal equation to a very manageable form. Abbreviating

$$\alpha = k_z z - \omega t = \theta - k_w z\,,$$

the eikonal equation takes the form

$$\left(\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2 + 2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta\right)a \approx -\frac{8\pi i}{I_0}\left\langle \exp(-i\alpha)J_x\right\rangle_\lambda .$$

The current density from the undulating beam is just

$$J_x = \sum_{all\,e-} -ev_x\delta(x-x_i)\delta(y-y_i)\delta(z-z_i)\,,$$

a sum over all electrons in the beam, of the current density associated with each electron. We have already computed the transverse velocity, v_x, and applying this result we have

$$
\begin{aligned}
J_x &= \sum_{all\, e-} -\frac{ec}{\gamma}\delta(x-x_i)\delta(y-y_i)\delta(z-z_i)\left\{a_w \sin(k_w z) + a_s \sin(\alpha+\varphi_s)\right\} \\
&= J_w + J_s
\end{aligned}.
$$

Thus there are two components to the response of the beam, one due to the wiggler, and one due to the electric field of the wave itself.

The first term on the right-side of the eikonal equation is

$$
-\frac{8\pi i}{I_0}\langle \exp(-i\alpha) J_w\rangle_\lambda = -\frac{8\pi i}{I_0}\left\langle \sum_{all\, e-} -\frac{ec}{\gamma}\delta^3(\vec{r}-\vec{r}_i)\exp(-i\alpha) a_w \sin(k_w z) \right\rangle_\lambda ,
$$

and the second term on the right-side of the eikonal equation is

$$
-\frac{8\pi i}{I_0}\langle \exp(-i\alpha) J_s\rangle_\lambda = -\frac{8\pi i}{I_0}\left\langle \sum_{all\, e-} -\frac{ec}{\gamma}\delta^3(\vec{r}-\vec{r}_i)\exp(-i\alpha) a_s \sin(\alpha+\varphi_s) \right\rangle_\lambda .
$$

It is helpful to note that an average over the signal period may be expressed as,

$$
\begin{aligned}
\left\langle \sum_{all\, e-} \delta^3(\vec{r}-\vec{r}_i) f(\vec{r},t) \right\rangle_\lambda &= \left\langle \frac{k_z}{2\pi} \int_{z-\pi/k_z}^{z+\pi/k_z} dz \sum_{all\, e-} \delta^2(\vec{r}_\perp - \vec{r}_{\perp i})\delta(z-z_i) f(\vec{r},t) \right\rangle_{2\pi/\omega} \\
&= \left\langle \frac{k_z}{2\pi} \underbrace{\sum_{e-\, in\, one\, \lambda} \delta^2(\vec{r}_\perp - \vec{r}_{\perp i}) f(\vec{r}_i,t)}_{sum\ over\ e-\ within\ a\ wavelength} \right\rangle_{2\pi/\omega} \\
&= \left\langle \frac{k_z}{2\pi}\left[N_{e-\, in\, one\, \lambda}\right] \underbrace{\left\langle \delta^2(\vec{r}_\perp - \vec{r}_{\perp i}) f(\vec{r}_i,t)\right\rangle_{e-\, in\, one\, \lambda}}_{average\ over\ e-\ within\ a\ wavelength} \right\rangle_{2\pi/\omega}
\end{aligned}
$$

where the number of electrons in one wavelength is

$$N_{e-\,in\,one\,\lambda} = \frac{I}{ec\langle\beta_z\rangle}\frac{2\pi}{k_z},$$

and *I* is the time-averaged or “DC” current (with positive sign), and $\langle\beta_z\rangle$ is the beam axial speed normalized by *c*, and averaged over one signal wavelength. Thus,

$$\left\langle \sum_{all\,e-} \delta^3(\vec{r}-\vec{r}_i) f(\vec{r},t) \right\rangle_\lambda = \frac{I}{ec\langle\beta_z\rangle}\left\langle \delta^2(\vec{r}_\perp - \vec{r}_{\perp i}) f(\vec{r}_i,t)\right\rangle,$$

where we adopt the notation that an unsubscripted bracket indicates an average over electrons within the signal spatial period, and an average over the temporal period. In the final expressions that result, this will amount to an average over electrons with θ in some interval of width 2π, an interval which we will often take as a matter of convention, to be $-\pi<\theta<\pi$.

The first driving term then takes the form,

$$\begin{aligned}
-\frac{8\pi i}{I_0}\langle \exp(-i\alpha) J_w\rangle_\lambda &= 8\pi i\left(\frac{I}{I_0}\right)\left\langle\frac{1}{\beta_z}\right\rangle\left\langle \frac{1}{\gamma}\delta^2(\vec{r}_\perp - \vec{r}_{\perp i})\exp(-i\alpha) a_w \sin(k_w z) \right\rangle \\
&= -8\pi i a_w\left(\frac{I}{I_0}\right)\left\langle \frac{1}{\gamma\beta_z}\delta^2(\vec{r}_\perp - \vec{r}_{\perp i})\frac{\exp(-i\theta) - \exp(-i\theta + 2ik_w z)}{2i} \right\rangle \\
&= -4\pi a_w [JJ]\left(\frac{I}{I_0}\right)\left\langle \frac{\exp(-i\bar{\theta})}{\gamma\bar{\beta}_z}\delta^2(\vec{r}_\perp - \vec{r}_{\perp i})\right\rangle + [jitter\ terms]
\end{aligned}$$

where, in going to the second line, we made use of the large γ approximation to replace $\langle\beta_z\rangle$ with the individual electron β_z. In the third line we have expressed the exponentials, using the Bessel function expansion, to isolate those terms that will remain after a wiggle-average.

The second driving term takes the form,

$$-\frac{8\pi i}{I_0}\left\langle \exp(-i\alpha) J_s \right\rangle_\lambda = 8\pi i \frac{I}{I_0}\left\langle \ \frac{1}{\gamma\beta_z}\delta^2\left(\vec{r}_\perp - \vec{r}_{\perp i}\right)\exp(-i\alpha) a_s \sin\left(\alpha + \varphi_s\right) \ \right\rangle$$

$$= 8\pi i a_s \frac{I}{I_0}\left\langle \ \frac{1}{\gamma\beta_z}\delta^2\left(\vec{r}_\perp - \vec{r}_{\perp i}\right)\frac{\exp\left(i\varphi_s\right) - \exp\left(-2i\alpha + i\varphi_s\right)}{2i} \ \right\rangle$$

$$= 4\pi a \frac{I}{I_0}\left\langle \ \frac{1}{\gamma\beta_z}\delta^2\left(\vec{r}_\perp - \vec{r}_{\perp i}\right) \ \right\rangle$$

where in the last line we have used $a = a_s \exp\left(i\varphi_s\right)$.

Finally the eikonal equation takes the form

$$\left(\nabla_\perp^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2 - 4\pi \frac{I}{I_0}\left\langle \frac{\delta^2\left(\vec{r}_\perp - \vec{r}_{\perp i}\right)}{\gamma\bar{\beta}_z}\right\rangle + 2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta\right) a = -4\pi a_w [JJ]\left(\frac{I}{I_0}\right)\left\langle \frac{\exp\left(-i\bar{\theta}\right)}{\gamma\bar{\beta}_z}\delta^2\left(\vec{r}_\perp - \vec{r}_{\perp i}\right)\right\rangle$$

This form of the eikonal equation is quite useful as is, without further reduction, and will eventually be employed for simulation work in 2 and 3 dimensions.

Note that the δ-function term on the left can be related simply to the average density of electrons in the beam n_b,

$$\frac{4\pi n_b e^2}{mc^2\gamma} = 4\pi \frac{I}{I_0}\left\langle \frac{\delta^2\left(\vec{r}_\perp - \vec{r}_{\perp i}\right)}{\gamma\bar{\beta}_z}\right\rangle = \frac{\omega_{b\perp}^2}{c^2},$$

and in the last line we have expressed it in terms of the beam angular frequency for transverse perturbations. For example, considering a plane-wave propagating through a uniform beam, with no wiggler, the dispersion relation is evidently just

$$\left(\frac{\omega}{c}\right)^2 = k_z^2 + k_\perp^2 + \frac{\omega_{b\perp}^2}{c^2},$$

as one finds for a plasma. This can also be expressed as

$$\varepsilon\left(\frac{\omega}{c}\right)^2 = k_z^2 + k_\perp^2,$$

where

$$\varepsilon = 1 - \frac{\omega_{b\perp}^2}{\omega^2} = 1 + 4\pi\chi,$$

is the dielectric constant for the beam, with χ the susceptibility. Eventually in the course of our analysis of the FEL driving term, we will find that it too can be described in terms of a susceptibility, and as for an optical fiber, can provide some guiding. In the meantime, we are interested in deriving a simpler one-dimensional model.

Reduction to One-Dimension

Additional simplification results when the transverse profile for the signal can resolved into a dominant mode. In class we worked out the case of an FEL operating in the TE_{01} mode of a waveguide. In such a case the vector potential can be represented as a sum over orthogonal modes. Taking the scalar product of the mode profile with each side of the wave equation and integrating over the guide cross-section, one arrives at

$$\left(2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta - \frac{4\pi}{\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\bar{\beta}_z}\right\rangle\right)a = -\frac{4\pi}{\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp\left(-i\bar{\theta}\right)}{\gamma\bar{\beta}_z}\right\rangle,$$

where Σ is the "mode-area", $\Sigma = ab/2$ for the TE_{01} mode of a waveguide of dimensions *a x b*. In this result we have taken,

$$\left(\frac{\omega}{c}\right)^2 = k_z^2 + k_c^2,$$

where k_c is the cut-off wavenumber for the mode of interest. For the TE_{01} mode, $k_c = \pi / b$, with the notation *a>b*.

The FEL Equations

To summarize, our one-dimensional ("1D") Model of the FEL consists of a complex phasor, *a*, describing the electromagnetic signal amplitude and phase, propagating with some *N* particles distributed over 2π in phase θ. Particle motion is described by this phase-angle θ and $\gamma = 1 + eV/mc^2$, with *V* the "beam voltage", or *eV* the beam kinetic energy.

$$\frac{d\theta}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + \frac{a_w^2}{2} - a_w[JJ]\mathrm{Re}\left[a\exp(i\theta)\right]\right\},$$

$$\frac{d\gamma}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w}{\gamma\beta}[JJ]\mathrm{Im}\left[a\exp(i\theta)\right],$$

and

$$\beta = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{a_w^2}{2}\right),$$

is the average axial speed, up to a term of order *a*. The θ-equation tells us how particles slip back and forth longitudinally, as they execute more or less prescribed transverse motions. This longitudinal motion depends on the energy-variable, γ, and the γ-equation describes the work being done on the particle by the wave.

The eikonal evolves according to

$$\left(2ik_z\left(\frac{\partial}{\partial z}\right)_\zeta - \frac{4\pi}{\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle\right)a = -\frac{4\pi}{\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle,$$

and the use of β in the denominator requires again the relativistic approximation (since there is a term in β of first order in *a* that otherwise might not be negligible). When "slippage" is neglected, and the partial derivative in *z* is replaced with an ordinary derivative, the reduction to a one-dimensional system is complete,

$$\left(\frac{d}{dz} + \frac{2\pi i}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle\right)a = \frac{2\pi i}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\exp(-i\theta)}{\gamma\beta}\right\rangle.$$

This result describes the amplification (or absorption or simply phase-shifting) of the amplitude of excitation in a particular, dominant waveguide mode.

Note that we have suppressed all the bars indicating wiggler - averages, since they clutter up the place.

While the forms above are my personal favorites, oftentimes one will see them written as follows,

$$\frac{d\psi}{dz} = k_w - \delta k - \frac{\omega}{c}\frac{1}{2\gamma^2}\left\{1 + \frac{a_w^2}{2} - a_w a_s[JJ]\cos\psi\right\} + \frac{d\varphi_s}{dz},$$

$$\frac{d\gamma}{dz} = -\frac{1}{2}\left(\frac{\omega}{c}\right)\frac{a_w a_s}{\gamma\beta}[JJ]\sin\psi,$$

$$\frac{da_s}{dz} = \frac{2\pi}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\sin\psi}{\gamma\beta}\right\rangle,$$

$$\frac{d\varphi_s}{dz} + \frac{2\pi}{k_z\Sigma}\frac{I}{I_0}\left\langle\frac{1}{\gamma\beta}\right\rangle = \frac{1}{a_s}\frac{2\pi}{k_z\Sigma}a_w[JJ]\left(\frac{I}{I_0}\right)\left\langle\frac{\cos\psi}{\gamma\beta}\right\rangle.$$

This form is not the best for numerical work, but it does have a certain appeal to folks who don't like complex numbers. It also aids a bit in the interpretation of the equations. One can see that the vacuum (non-wiggler) contribution to the beam susceptibility contributes only to phase-shift (refraction) not gain; the same can be said of the $\langle \cos\psi \rangle$ term. Meanwhile non-zero $\langle \sin\psi \rangle$ corresponds to an in-phase rf component on the beam, and gain. (Recall the first homework, where we worried a bit about noise in macroparticle models.) Tune in next week, when we ask you to derive the FEL equations for a helical wiggler!