

Pulsed-Heating Calculations-Update of ARDB-31

Starting with eq.(1) and eq.(3) from ARDB-31, the temperature rise at the surface of a metal is governed by

$$\Delta T(\bar{x}, t) = \frac{1}{\sqrt{\pi\rho}} \int_0^t \frac{dt'}{\sqrt{t-t'}} \frac{1}{\sqrt{c_\epsilon(\Delta T + T_0)k(\Delta T + T_0)}} \frac{dP(\bar{x}, t', \Delta T + T_0)}{dA} \quad (1)$$

$$\frac{dP(\bar{x}, t, \Delta T + T_0)}{dA} = \frac{1}{2} R_s(\Delta T + T_0) |H_{\parallel}(\bar{x}, t, \Delta T + T_0)|^2 \quad (2)$$

where $\rho \equiv$ density, $c_\epsilon \equiv$ specific heat, $k \equiv$ thermal conductivity, $R_s \equiv$ surface resistance, $\bar{x} \equiv$ the position along the surface of the metal, and T_0 is the initial temperature. In ARDB-31, I ignored the temperature dependence of the specific heat, thermal conductivity, and surface resistance of the material. It is the purpose of this note to include this dependence to get a more accurate assessment of the local temperature rise inside the high-power test cavity that will be used in the pulsed-heating experiment.

In ARDB-43¹ I showed that the governing equation for the amplitude of the magnetic field inside a cavity driven by a coupling waveguide is

$$\left[\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_L} \frac{d}{dt} + \frac{\omega_\lambda^2}{1 + \frac{1}{Q_0}} \right] h_\lambda(t) = -\frac{1}{1 + \frac{1}{Q_0}} \sqrt{\frac{8P_+\omega_\lambda^3}{\mu Q_e}} \exp(j\omega_d t) \quad (3)$$

where $\omega_\lambda \equiv$ unperturbed cavity resonant frequency, $\omega_d \equiv$ drive frequency, $P_+ \equiv$ input peak power, $Q_0 \equiv$ unloaded Q of cavity, $Q_L \equiv$ loaded Q, $Q_e \equiv$ external Q, $h_\lambda \equiv$ amplitude of magnetic field. Eq.(3) assumes that the geometrical part of the square of the magnetic field has been normalized with respect to the cavity volume. The unloaded and loaded Q's of the cavity are temperature dependent since they depend on the surface resistance of the metal. They are given by

$$\frac{1}{Q_0(T)} = \frac{1}{\mu\omega} \int_S R_s(T) |H_{\parallel}(\bar{x})|^2 dS \quad (4)$$

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e} \quad (5)$$

where the magnetic field in eq.(4) is just the geometrical part of the field. The total magnetic field is found by multiplying the amplitude found in eq.(3) with the geometrical part of the field.

These equations are enough to determine the temperature rise at the surface of the cavity. They must be solved numerically. Since the Q of the cavity depends on the integral of the magnetic field over the surface of the cavity, we must find the temperature rise at each point along the

¹based on notes made by D. Whittum, "Microwave Electronics", 1997

surface. Once the magnetic field is known at one time step, the temperature-rise can be determined. The Q's are then updated and the magnetic field is found at the next time step. Therefore, one can know the change in temperature and the change in Q during the time of a pulse.

Eq.(3) must be modified in order to find the slowly varying part of the magnetic field. Initially, there are no fields in the cavity so the magnetic field can only be resonating at the drive frequency ω_d . Therefore, the magnetic field must be of the form

$$h_\lambda(t) = a(t) \exp(j\omega_d t) \quad (6)$$

where $a(t)$ is a complex-valued slowly varying function of time. Plugging eq.(6) into eq.(3) and using the eikonal approximation in which the second derivative of a is ignored gives us

$$\left(2j\omega_d + \frac{\omega_\lambda}{Q_L}\right) \frac{da}{dt} + \left(\frac{j\omega_d\omega_\lambda}{Q_L} + \frac{\omega_\lambda^2}{1 + \frac{1}{Q_0}}\right) a = -\frac{1}{1 + \frac{1}{Q_0}} \sqrt{\frac{8P_+ \omega_\lambda^3}{\mu Q_e}} \quad (7)$$

Keep in mind that the resonant frequency of the cavity is

$$\omega_0 = \frac{\omega_\lambda}{\sqrt{1 + \frac{1}{Q_0}}} \quad (8)$$

The resonant frequency of the cavity will change with temperature because the Q changes. Initially, the drive frequency will be the same as the cavity resonant frequency. However, as the temperature increases the cavity will be driven off resonance. Care must be taken to make sure the resonant frequency does not drift beyond the bandwidth of the mode being driven. If necessary, the source can always be made to sweep in frequency to prevent this problem from occurring.

Since eq.(7) must be numerically solved, we will rewrite it as two coupled equations involving the real and imaginary parts of a .

$$\begin{aligned} \frac{da_R}{dt} = & \left[1 + \frac{4Q_L^2}{1 + \frac{1}{Q_0(T_0)}}\right]^{-1} \left\{ -\omega_\lambda Q_L \left[\frac{1}{1 + \frac{1}{Q_0(\Delta T + T_0)}} + \frac{1}{1 + \frac{1}{Q_0(T_0)}} \right] a_R \right. \\ & \left. + \frac{\omega_\lambda}{\sqrt{1 + \frac{1}{Q_0(T_0)}}} \left[1 - 2Q_L^2 \left(\frac{1}{1 + \frac{1}{Q_0(\Delta T + T_0)}} - \frac{1}{1 + \frac{1}{Q_0(T_0)}} \right) \right] a_I - \frac{Q_L}{1 + \frac{1}{Q_0(\Delta T + T_0)}} \sqrt{\frac{8P_+ \omega_\lambda^3}{\mu Q_e}} \right\} \end{aligned} \quad (9a)$$

$$\frac{da_I}{dt} = -\frac{2Q_L}{\sqrt{1 + \frac{1}{Q_0(T_0)}}} \frac{da_R}{dt} - \frac{\omega_\lambda}{\sqrt{1 + \frac{1}{Q_0(T_0)}}} a_R - \omega_\lambda Q_L \left[\frac{1}{1 + \frac{1}{Q_0(\Delta T + T_0)}} - \frac{1}{1 + \frac{1}{Q_0(T_0)}} \right] a_I \quad (9b)$$

where a_R and a_I are the real and imaginary parts of a respectively. Eqs.(9a) and (9b) already include the fact that the drive frequency is equal to the cavity resonant frequency at zero temperature rise. Eqs.(9a) and (9b) can be numerically solved using the IMSL routine IVPRK with the initial conditions set to zero. These initial conditions are equivalent to $h_\lambda = 0$ and $dh_\lambda/dt = 0$.

Eq.(1) is an integral equation for the temperature rise ΔT . In the literature² it is specifically known as a Volterra integral equation of the second kind. This equation must be solved numerically using a product integration rule because of the singularity involving $(t-t')$. A product integration rule simply integrates the singular part of the equation first while summing over the other non-singular parts. Refer to Linz's book for more information. Since I do not need high accuracy for this problem, I elected to use a rectangular quadrature similar to Euler's method. Any higher accuracy method would involve a nonlinear equation for T , which requires a Jacobian to be calculated. Since the Jacobian cannot be given analytically, it must be solved numerically. This may destroy the higher accuracy being sought. If higher accuracy is desired, then Richardson's extrapolation can be employed³.

If we combine eqs.(1) and (2), we may rewrite them in the following way

$$\Delta T(\bar{x}, t) = \int_0^t \frac{K(\bar{x}, t, t', \Delta T + T_0)}{\sqrt{t-t'}} dt' \quad (10)$$

$$K(\bar{x}, t, t', \Delta T + T_0) = \frac{1}{2\sqrt{\pi\rho}} \frac{R_s(\Delta T + T_0) |h_\lambda(t', \Delta T + T_0) H_{||}(\bar{x})|^2}{\sqrt{c_\varepsilon(\Delta T + T_0) k(\Delta T + T_0)}} \quad (11)$$

Then the temperature rise can be solved at time t_n with⁴

$$\Delta T(\bar{x}, t_n) = \sum_{j=0}^{n-1} w_{nj} K(\bar{x}, t_n, t_j, \Delta T(\bar{x}, t_j) + T_0) \quad (12)$$

$$w_{nj} = 2 \left[\sqrt{t_n - t_j} - \sqrt{t_n - t_{j+1}} \right] \quad (13)$$

I've performed the integration in eq.(4) for the Q of the cavity using the Composite Simpson's Rule⁵. I approximated the specific heat as the specific heat at constant volume which is given by⁶

$$c_V = \frac{R}{m} \left[\frac{36}{x^3} \int_0^x \frac{y^3 dy}{e^y - 1} - \frac{9x}{e^x - 1} \right] \quad (14)$$

$$x = \frac{\theta_D}{T} \quad \theta_D = 343K$$

²P.Linz, Analytical and Numerical Methods for Volterra Equations, 1985

³P.Linz, pp. 133-135

⁴P.Linz, eqs.(8.15) and (8.16)

⁵Burden,R.L.and Faires,J.D., Numerical Analysis, 5th ed., 1993, p.186

⁶Furukawa, G.T., and Douglas, T.B., American Institute of Physics Handbook, 3rd ed., "Heat Capacities" p.4-113

where $R=8.3143$ J/mole-K and m is the molar weight of the metal(for Cu, $m=63.5 \times 10^{-3}$ kg/mole). θ_D is the Debye temperature for copper and T is the absolute temperature. The specific heat for copper is plotted in Fig.1. The integral in eq.(14) is performed with the IMSL routine QDAGS. I fit the data for the thermal conductivity of copper⁷ with a polynomial of order 5 given by

$$k(T) = 1.2830 \times 10^{-10} T^4 - 3.9524 \times 10^{-7} T^3 + 4.3231 \times 10^{-4} T^2 - 2.6485 \times 10^{-1} T + 451.27 \frac{W}{m \cdot K} \quad (15)$$

$200K \leq T \leq 1200K$

where T is the absolute temperature in °K. A plot of the data with the fit is shown in Fig.2. The surface resistance is given by

$$R_s(T) = \sqrt{\frac{\omega \mu \rho_{res}(T)}{2}} \quad (16)$$

where ρ_{res} is the resistivity, ω is the resonant cavity frequency (assumed constant), and T is the temperature. I fit the data for the resistivity of copper⁸ to a straight line given by

$$\rho_{res}(T) = 7.122 \times 10^{-11} T - 4.342 \times 10^{-9} \quad 10^{-8} \Omega - m \quad (17)$$

$293K \leq T \leq 900K$

where T is the absolute temperature in °K. A plot of the data with the fit is shown in Fig.3.

We now have all of the tools necessary to solve for the temperature rise in a cavity. My specific experiment involves driving two pillbox cavities in the TE_{011} mode at a frequency of 11.424GHz. The cavities will be attached to a magic tee. The source is an X-band klystron that will provide an input peak power of 20MW to each cavity with a pulse length of 1.5 μ s. At the same time, a CW low-power source will drive the cavity in a TE_{012} mode at a frequency of 17.8GHz from another waveguide that is cutoff to 11.424GHz. The change in Q will be determined from this second mode by measuring the change in reflected power given by

$$\frac{P_r}{P_i} = \left(\frac{Q_0/Q_e - 1}{Q_0/Q_e + 1} \right)^2 \quad (18)$$

where P_r is reflected power, P_i is incident power, Q_0 is the unloaded Q for the TE_{012} mode, and Q_e is the external Q for the TE_{012} mode. The external Q is assumed to change very little compared to the unloaded Q .

⁷D.R. Lide, editor, Handbook of Chemistry and Physics, 77th ed. 1996-1997, p.12-174

⁸D.R. Lide, editor, Handbook of Chemistry and Physics, 77th ed. 1996-1997, p.12-40

According to Fig.4, the maximum temperature rise occurs near the middle of the endcaps as expected. A plot of this temperature rise over the course of a pulse is given in Fig.5. The change in Q_0 for both modes is given in Fig.6. Fig.7 shows the resulting change in reflected power.

The phase change of reflected power due to the change in resonant frequency of the cavity can also be measured. Fig.8 shows the real and imaginary parts of the magnetic field given by eqs.(9a) and (9b). Initially the magnetic field is purely imaginary, but it takes on a real part during the course of a pulse. The change in phase can be determined from this for the magnetic field of the TE_{012} mode and it is shown in Fig.9.

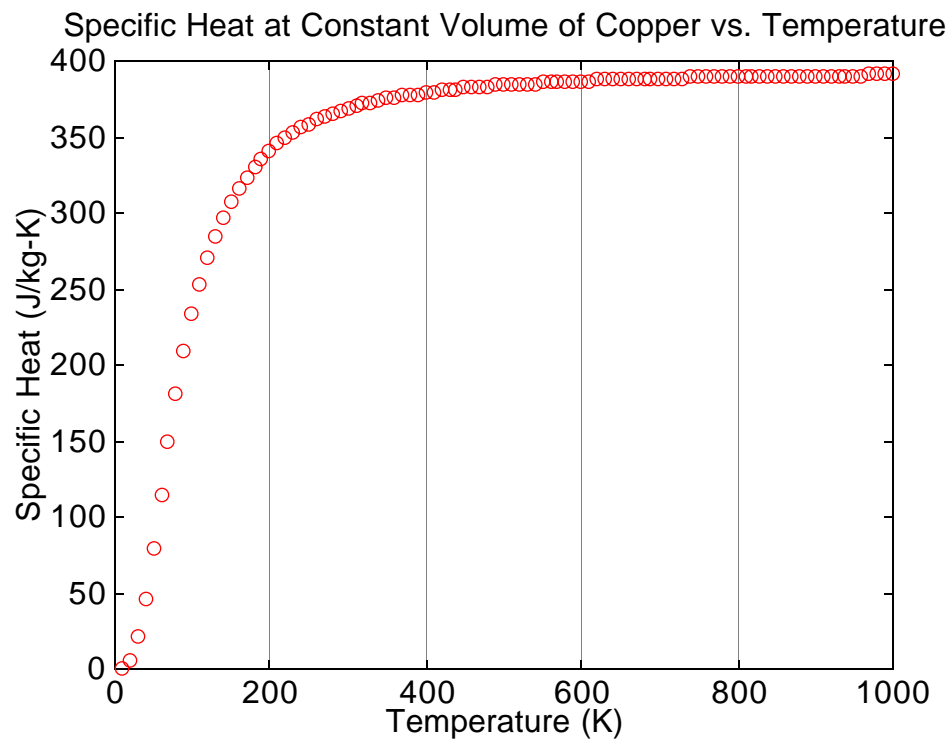


FIG 1

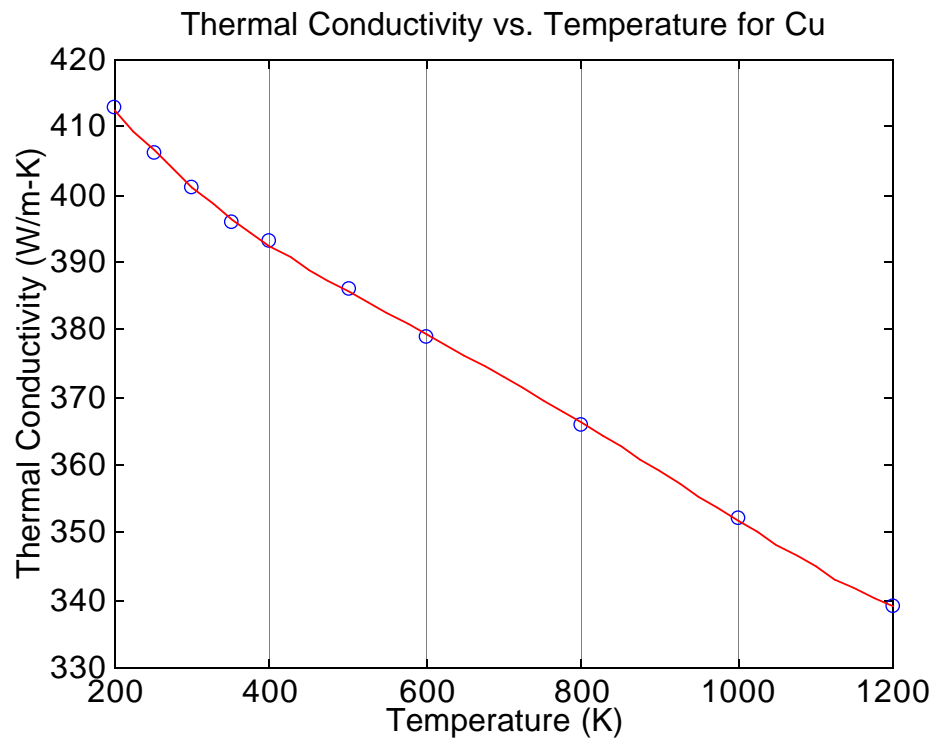


FIG 2

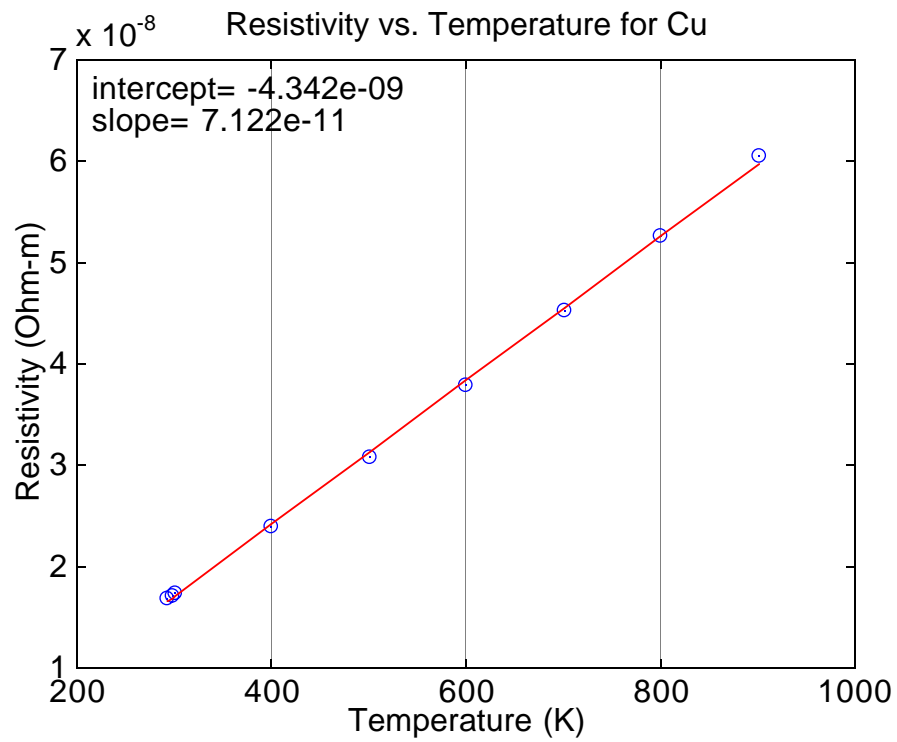


FIG 3

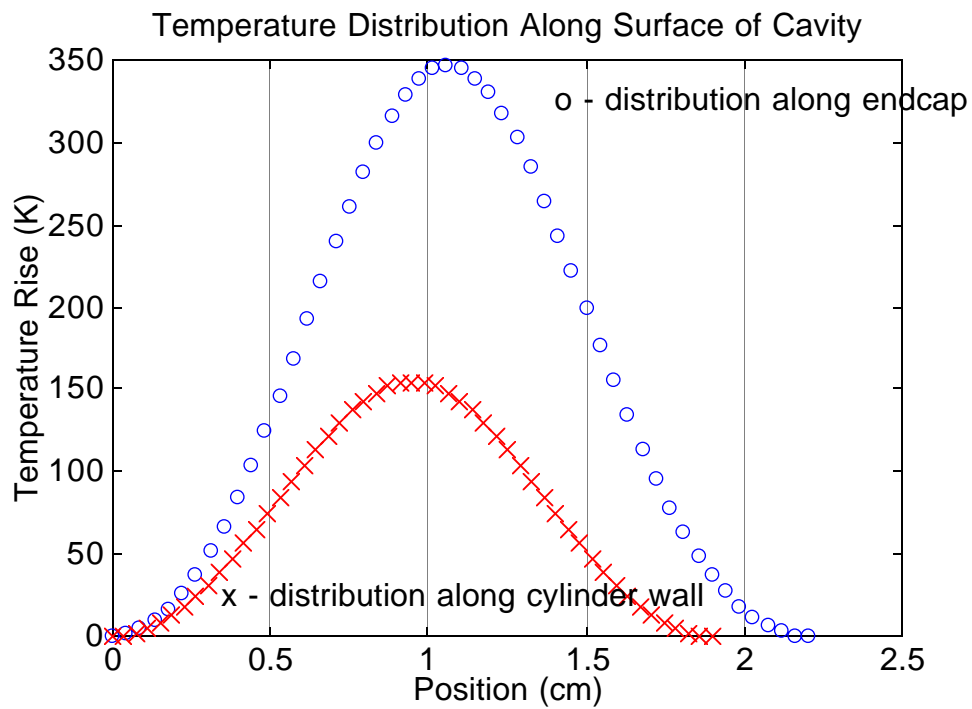


FIG 4

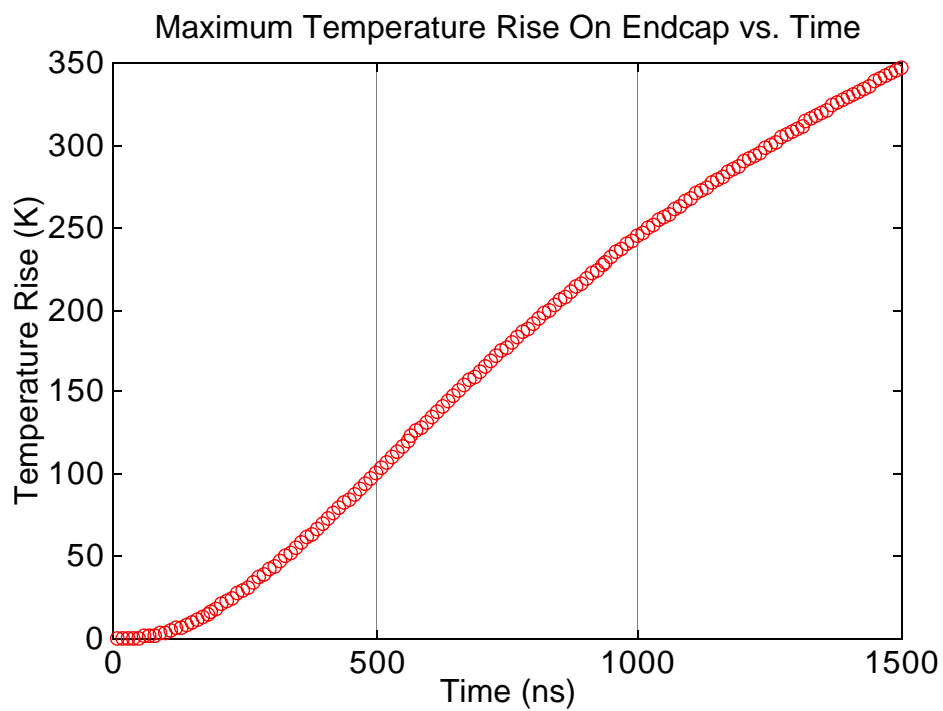


FIG 5

