ARDB-43 David Pritzkau 4/17/97

Coupled Cavity-Waveguide Calculations

The purpose of this note is to derive the equation that determines the magnitude of the magnetic field inside a cavity that is coupled to an input waveguide. Most of the work done here is derived from David Whittum's notes on microwave electronics[1].

We first assume that the electric and magnetic fields are represented by a sum over all modes in a cavity. These modes are orthogonal and are normalized with respect to the cavity volume. That is

$$
\begin{aligned}
E(\vec{x},t) &= \sum_{\lambda} e_{\lambda}(t)E_{\lambda}(\vec{x}) \\
H(\vec{x},t) &= \sum_{\lambda} h_{\lambda}(t)H_{\lambda}(\vec{x}) \\
\int_V \left|E_{\lambda}(\vec{x})\right|^2 dV &= \int_V \left|H_{\lambda}(\vec{x})\right|^2 dV = 1
\end{aligned}
\tag{1}
$$

By Fourier transforming the fields and using Maxwell's equations to relate them we get from Whittum's notes[2]

$$j\varepsilon\omega\tilde{e}_{\lambda} = k_{\lambda}\tilde{h}_{\lambda} + \sum_a V_{a\lambda}I_a - \tilde{J}_{\lambda} \tag{2a}$$

$$-j\mu\omega\tilde{h}_{\lambda} = k_{\lambda}\tilde{e}_{\lambda} + \frac{\mu}{Q_w}\left(1 + j\,\mathrm{sgn}(\omega)\right)|\omega|\tilde{h}_{\lambda} + \sum_a V_a I_{a\lambda} \tag{2b}$$

$$k_{\lambda} = \omega_{\lambda}\sqrt{\mu\varepsilon} \tag{2c}$$

where ~ stands for a Fourier transform, ω is the frequency of the system, ω_{λ} is the unperturbed resonant frequency of the cavity for mode λ, J_{λ} is the external current, Q_w is the unloaded Q of the cavity, $V_{a\lambda}$ and $I_{a\lambda}$ are overlap integrals of the unperturbed cavity fields for mode λ with the waveguide modes a over the port between the cavity and the waveguide, and V_a and I_a are the expansion coefficients for the mode a in the waveguide.

We are considering the case that only one waveguide mode is coupled to one cavity mode, so the sum over a in eqs.(2a) and (2b) is reduced to a=1. If we place the port between the waveguide and the cavity at a plane called the "detuned short" then $I_{a\lambda} = 0$. There is also no external current so $\tilde{J}_{\lambda} = 0$. Eliminating $\tilde{e}_{\lambda}$ from eqs.(2a) and (2b) gives

$$\left[\omega^2 - \frac{j\omega|\omega|}{Q_w + 1} - \frac{\omega_{\lambda}^2}{1 + \frac{1}{Q_w}}\right]\tilde{h}_{\lambda} = \frac{1}{1 + \frac{1}{Q_w}}\frac{\omega_{\lambda}}{\sqrt{\varepsilon\mu}}V_{1\lambda}I_1 \tag{3}$$

Since $Q_w >> 1$ we can approximate $|\omega| = \omega_{\lambda}$.

---

[1]D. Whittum, "Ch.2 Microwave Electronics", 1997, to be published

[2]D. Whittum, p.2-69

From Whittum's notes[3] we have that

$$V_{1\lambda} = \sqrt{\frac{Z_{c1}\varepsilon\omega_\lambda}{Q_{e\lambda}}} \tag{4}$$

$$I_1 = \frac{V_1^+ - V_1^-}{Z_{c1}} \tag{5}$$

where Z_{c1} is the characteristic impedance of the waveguide for mode 1, $Q_{e\lambda}$ is the external Q, and V_1^+ and V_1^- are the incident and reflected voltages in the waveguide respectively. Using continuity of the electric field at the port we get[4]

$$V_1^+ + V_1^- = V_{1\lambda}\tilde{e}_\lambda \tag{6}$$

Using eqs.(2b), (5) and (6), eq.(3) becomes

$$\left[\omega^2 - \frac{j\omega\omega_\lambda}{Q_w} - \frac{\omega_\lambda^2}{1+\frac{1}{Q_w}}\right]\tilde{h}_\lambda = \frac{1}{1+\frac{1}{Q_w}}\frac{\omega_\lambda}{\sqrt{\mu\varepsilon}}\frac{2V_{1\lambda}V_1^+}{Z_{c1}} + \frac{j\omega\omega_\lambda}{Q_{e\lambda}\left(1+\frac{1}{Q_w}\right)}\tilde{h}_\lambda + \frac{\omega_\lambda|\omega|(1+j\,\mathrm{sgn}(\omega))}{Q_w Q_{e\lambda}\left(1+\frac{1}{Q_w}\right)}\tilde{h}_\lambda \tag{7}$$

Usually $Q_w >> 1$ and $Q_{e\lambda} >> 1$, so we can neglect the last term on the right hand side of eq.(7). For the same reason we can replace the denominator in the second term with $Q_{e\lambda}$.

The incident voltage is related to the incident power as

$$V_1^+ = \sqrt{2P_+Z_{c1}}\exp(j\omega_d t) \tag{9}$$

where ω_d is the drive frequency in the waveguide and P_+ is the incident power. If we replace $j\omega$ by d/dt and use eqs.(4) and (9) we may rewrite eq.(7) using the definition of the loaded Q

$$\frac{1}{Q_L} = \frac{1}{Q_w} + \frac{1}{Q_{e\lambda}} \tag{10}$$

as

$$\left[\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_L}\frac{d}{dt} + \frac{\omega_\lambda^2}{1+\frac{1}{Q_w}}\right]h_\lambda(t) = -\frac{1}{1+\frac{1}{Q_w}}\sqrt{\frac{8P_+\omega_\lambda^3}{\mu Q_{e\lambda}}}\exp(j\omega_d t) \tag{11}$$

---

[3]D. Whittum, pp.2-72, 2-73

[4]D. Whittum, p.2-72

Eq.(11) gives the expansion coefficient of the magnetic field inside the cavity as a function of time. The cavity knows about the coupling of the input waveguide through the external Q. Eq.(11) is only applicable to the case in which one waveguide mode is driving one cavity mode (or vice versa). It should be noted that the cavity resonant frequency is

$$\omega_0 = \frac{\omega_\lambda}{\sqrt{1+\frac{1}{Q_w}}} \tag{12}$$

The reason why the resonany frequency of the cavity is affected by the Q can be seen from a geometrical standpoint[5]. By using the average value of the magnetic field in eq.(16) as a crude approximation we find that the Q is the ratio of the volume of the cavity to a volume of a shell with thickness $\delta/2$ where δ is the skin depth of the metal. The change in resonant frequency is due to the effective change in the volume of the cavity.

It will serve as an example to solve eq.(11) given that all of the parameters are constant. Let us use the case of a waveguide driving a cavity initially empty of energy. Initially there are no fields inside the cavity, so the initial conditions are $h_\lambda = 0$ and $dh_\lambda/dt = 0$ by eq.(2b). By considering the case where $Q_w >> 1$ and $Q_L >> 1$ so that $\omega_0 = \omega_\lambda$ and driving the waveguide at $\omega_d = \omega_\lambda$ one can show that

$$h_\lambda(t) = jQ_L\sqrt{\frac{8P_+}{\mu Q_{e\lambda}\omega_\lambda}}\left[1-\exp\left(-\frac{\omega_\lambda t}{2Q_L}\right)\right]\exp\left(j\omega_\lambda t\right) \tag{13}$$

It is interesting to understand eq.(13) in terms of the dissipated power on the cavity surface. The local dissipated power per unit area of surface is

$$\frac{dP_d}{dA} = \frac{1}{2}R_s|H|^2 = \frac{1}{2}R_s|h_\lambda|^2|H_\lambda|^2 \tag{14}$$

where P_d is dissipated power and R_s is the surface resistance of the material. The local incident power per unit area can be written as

$$\frac{dP_+}{dA} = P_+\frac{|H_\lambda|^2}{\int_S |H_\lambda|^2 dA} \tag{15}$$

and the definition of the unloaded Q of the cavity is[6]

$$\frac{1}{Q_w} = \frac{R_s}{\mu\omega_\lambda}\int_S |H_\lambda|^2 dA \tag{16}$$

---

[5]Slater, Microwave Electronics, 1957, p.70

[6]D. Whittum, p.2-67

Using eqs.(13), (15), and (16) in eq.(14) gives

$$\frac{dP_d}{dA} = \frac{4Q_L^2}{Q_0 Q_{e\lambda}} \frac{dP_+}{dA} \left[1 - \exp\left(-\frac{\omega_\lambda t}{2Q_L}\right)\right]^2 \tag{17}$$

If we use the definition of the coupling coefficient between the waveguide and the cavity

$$\beta = \frac{Q_w}{Q_{e\lambda}} \tag{18}$$

and the definition of the fill-time of the cavity

$$\tau = \frac{\omega_\lambda}{Q_L} \tag{19}$$

we may integrate eq.(17) and use eq.(10) to get

$$P_d = \frac{4\beta}{(\beta + 1)^2} P_+ \left[1 - \exp\left(-\frac{t}{2\tau}\right)\right]^2 \tag{20}$$

Eq.(20) is a well known relation involving the dissipated power in the cavity due to an input waveguide[7].

---

[7]Padamsee and Knobloch, "Issues in Superconducting RF Technology", USPAS 1/96