ARDB Technical Note 40 - *Draft 10/6/97*

Beam Dynamics in a 2.5 TeV Planar Linac

David H. Whittum and Marc Hill

In this note we analyze single-bunch beam dynamics in a W-Band 2.5-TeV linac, for a 60-pC bunch with initial rms normalized emittance of $10^{-7}\ \mathrm{m-rad}$ in both the vertical and the horizontal, injected at 10 GeV, and accelerated to 2500 GeV at 1 GeV/m. We consider a bunch length 1/100th of an rf wavelength or 30 μm (0.1 ps). We consider beam dynamics first without focusing, and then with magnetic focusing. We go on to consider quadrupole requirements in light of the analysis of beam dynamics.

Beam Transport With No Wakefields, No Magnets

The first thing one might like to check is that magnets are actually useful for our parameters. We check this by considering first beam transport without wakefields, and then with wakefields.

Neglecting wakefields, consider the motion of a single electron described by energy $mc^2\gamma$, with m the electron mass, c the speed of light, and γ the Lorentz factor. Let the horizontal momentum be denoted mcp. We assume energy varies according to

$$\gamma = \gamma_0 + Gs,$$

with s the displacement down the beamline, γ_0 the initial Lorentz factor,

$$\gamma_0 \approx 1 + \frac{10\,\mathrm{GeV}}{0.511\times10^{-3}\,\mathrm{GeV}} \approx 1.96\times10^4$$

and G the gradient measured in m^{-1},

$$G \approx \frac{1\,\mathrm{GeV/m}}{0.511\times10^{-3}\,\mathrm{GeV}} \approx 1.96\times10^3\,\mathrm{m}^{-1}.$$

Neglecting transvserse forces (magnets, wakefields) the transverse motion is described by,

$$\frac{dp}{ds} = 0, \qquad \gamma\frac{dx}{ds} = p.$$

These equations we integrate to obtain,

$$x = x_0 + p_0\xi,$$

where ξ is the proper displacement down the linac

$$\xi(s) = \int_0^s \frac{ds}{\gamma} = \frac{1}{G}\ln\left(\frac{\gamma(s)}{\gamma(0)}\right) = \frac{1}{G}\ln\left(1+\frac{Gs}{\gamma_0}\right),$$

and varies from 0 to

$$\xi_{\max} = \frac{1}{G}\ln\left(\frac{\gamma_{\max}}{\gamma_0}\right) \approx 2.8\ \text{mm}.$$

The quantity ξ_{max} is just the length of the linac "seen" by a beam electron. At this gradient, proper length elapses at a rate of 1.2 mm for each decade in energy.

The first and second moments of the beam at any point ξ, may be expressed in terms of averages, denoted by brackets, over the initial position x_0 and momentum p_0,

$$\langle x\rangle = \langle x_0\rangle + \langle p_0\rangle\xi, \qquad \langle p\rangle = \langle p_0\rangle,$$
$$\langle x^2\rangle = \langle x_0^2\rangle + 2\langle x_0 p_0\rangle\xi + \langle p_0^2\rangle\xi^2, \qquad \langle p^2\rangle = \langle p_0^2\rangle,$$
$$\langle xp\rangle = \langle x_0 p_0\rangle + \langle p_0^2\rangle\xi.$$

The rms moments are

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \sigma_{p0}^2, \qquad \sigma_{xp} = \langle xp\rangle - \langle x\rangle\langle p\rangle = \sigma_{xp0} + \sigma_{p0}^2\xi,$$
$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \sigma_{x0}^2 + 2\sigma_{xp0}\xi + \sigma_{p0}^2\xi^2,$$

and the normalized edge emittance is ε_n is given by

$$\varepsilon_{en}^2 = 4\left(\sigma_x^2\sigma_p^2 - \sigma_{xp}^2\right),$$

and it will be convenient to make use of the rms normalized emittance

$$\varepsilon_n = \frac{1}{2}\varepsilon_{en} = 5\times10^{-8}\,\text{m}-\text{rad}.$$

It is straightforward to check that ε_n is constant (having neglected wakefields).

We will assume a a beam passing aperture of 0.3λ, with $\lambda \approx 3.3\,\text{mm}$ the rf wavelength; thus the full-width of the aperture is 1 mm. To avoid scraping of the beam we will require that the beam size be less than one-tenth the aperture, or 100 µm. We also ask that the beam not be steered into the aperture; for illustration we take this to mean that the beam centroid in the linac should not deviate by more than 1/100 of the aperture. This implies (having neglected wakefields) that the initial position should be controlled at the level of 10 µm, and the initial angle should be controlled to the level of $\langle p_0\rangle\xi_{\max} < 10\mu\text{m}$, or

$$\langle x_0'\rangle \approx \frac{\langle p_0\rangle}{\gamma_0} < 0.2\mu\,\text{rad}.$$

To check the requirement on beam-size we first note that the rms spot-size may be expressed as

$$\sigma_x^2 = \frac{\varepsilon_n^2}{\sigma_{p0}^2} + \sigma_{p0}^2\left(\xi + \frac{\sigma_{xp0}}{\sigma_{p0}^2}\right)^2,$$

a parabolic behavior. To minimize scraping, we select the waist position (a choice of launch condition) to minimize the maxima in the beam-size. This corresponds to a waist at

$$\xi_* = \frac{1}{2}\xi_{\max} \approx 1.4 \text{ mm},$$

or

$$s_* = \frac{\gamma_0}{G}\left(e^{G\xi_*} - 1\right) \approx 145\,\text{m}.$$

This requires $\sigma_{xp0} = -\xi_* \sigma_{p0}^2$, and permits us to express

$$\sigma_x^2 = \frac{\varepsilon_n^2}{\sigma_{p0}^2} + \sigma_{p0}^2 \left(\xi - \xi_*\right)^2.$$

This may also be expressed as

$$\sigma_x^2 = \sigma_{x*}^2 + \frac{\varepsilon_n^2}{\sigma_{x*}^2}\left(\xi - \xi_*\right)^2,$$

with

$$\sigma_{x*}^2 = \frac{\varepsilon_n^2}{\sigma_{p0}^2},$$

the minimum beam-size. With this choice for waist location, the maximum beam size occurs at the entrance and exit,

$$\sigma_{x-\max}^2 = \frac{\varepsilon_n^2}{\sigma_{p0}^2} + \sigma_{p0}^2 \xi_*^2,$$

and this is a minimum for

$$0 = \frac{\delta \sigma_{x-\max}^2}{\delta \sigma_{p0}^2} = \xi_*^2 - \frac{\varepsilon_n^2}{\sigma_{p0}^4},$$

or

$$\sigma_{p0}^2 = \frac{\varepsilon_n}{\xi_*}.$$

This corresponds to

$$\sigma_{xp0} = -\varepsilon_n, \qquad \sigma_{x-\max} = \left(2\xi_* \varepsilon_n\right)^{1/2}, \qquad \sigma_{x*} = \left(\xi_* \varepsilon_n\right)^{1/2}.$$

Thus for our parameters, the maximum beam-size is 12 μm and occurs at the entrance and exit to the linac. The minimum is 8 μm and occurs 145 m into the linac. The initial rms opening angle,

$$\sigma_{x'0} \approx \frac{1}{\gamma_0}\left(\frac{\varepsilon_n}{\xi_*}\right)^{1/2} \approx \frac{\sigma_{p0}}{\gamma_0} \approx 0.3\mu\text{rad},$$

and at the waist is 20 nrad. At the exit this is 1 nrad.

While we have assumed no external forces, other than the accelerating field, we could note the sensitivity to a constant perturbation,

$$\frac{dp}{ds} = \delta f ,$$

corresponding for example, to any unshielded component of the ambient magnetic field of the earth. With no mu-metal shielding, one expects,

$$\delta f \approx \frac{0.3\mathrm{G}}{17\mathrm{G-m}} \approx 2\times 10^{-2}\,\mathrm{m}^{-1}.$$

Such a term would produce an orbit perturbation given by

$$\delta x = \frac{\delta f}{G}\left\{ s - \frac{\gamma_0}{G}\ln\left(\frac{\gamma}{\gamma_0}\right)\right\} = \frac{\delta f}{G^2}\gamma_0\left\{ e^{G\xi} - 1 - G\xi \right\},$$

and at the linac exit, where $G\xi$˜5.5, this is

$$\delta x(\mathrm{m}) = 1.2\delta f\left(\mathrm{m}^{-1}\right),$$

and corresponds to $\delta x \approx 2.4\times 10^{-2}\,\mathrm{m}$. To hold such a perturbation down to the level of 10 μm would then require shielding the ambient magnetic field to the level of 40 ppm.

From this discussion one can see that magnets are not required in the limit of low charge (negligible wakefields) to confine the beam envelope to the aperture of a W-Band linac. On the other hand, one can see too that the system is quite sensitive to errors in injection angle, and to perturbations. Even for low charge it is clear that steering magnets would be useful.

Beam Transport With Wakefields, No Magnets

Next, we consider the effect of wakefields. Explicit results for the W-Band geometry are not yet available, so we will employ estimates. The analysis presented here sets down some important constraints on any geometry eventually selected.

Longitudinal Wakefield

The longitudinal wakefield may be characterized by the fundamental mode loss-factor at this wavelength, and a correction factor accounting for the higher-order mode losses. This correction is a function of beam aperture in units of the rf wavelength and is a significant factor, but will surely be less than 10 for reasonable beam apertures. For the fundamental mode, loss factor per cell scales inversely with wavelength, from 1 V/pC at S-Band to 32 V/pC at W-Band, assuming optimal *[R/Q]*˜221Ω. (Actual *[R/Q]* could be as low as 90 Ω). This would correspond to a fundamental mode wakefield amplitude of 32 V/pC for each 1/3 of a wavelength (taking a 2π/3 mode structure for illustration) or about 3×10^4 V/pC/m. If one scales the X-Band DDS wakefield

$$W_z \approx 1.4\times 10^3\,\frac{V}{pC-m}\times\underbrace{\left[\frac{W\ Band}{X\ Band}\right]^2}_{64}\times\underbrace{\exp\left(-1.16\left[8\,\frac{30\,\mu m}{1000\,\mu m}\right]^{0.55}\right)}_{0.6},$$

one obtains 5×10^4 V/pC/m. (For a 15μm bunch length this comes to 8x10[4] V/pC-m.) We can compare this figure to other, simpler estimates. For a transition in round pipe from radius *a* to *b*, with one transition every 1/3 wavelength, the longitudinal wakefield is

$$W_z \approx \frac{e^2}{\pi^{1/2}\sigma_z}\frac{1}{\lambda/3}\ln\left(\frac{b}{a}\right) \approx 1.5\times10^4\,\tfrac{V}{pC-m}\ln\left(\frac{b}{a}\right),$$

for a 30 μm bunch length. For a round resistive pipe of conductivity σ, and radius b, the longitudinal wakefield is[1]

$$W_z \approx \frac{e^2}{b}\left(\frac{c}{2\pi\sigma}\right)^{1/2}\frac{1}{\sigma_z^{3/2}}.$$

For ideal copper $\sigma \approx 5\times10^{17}\,\mathrm{sec}^{-1}$, and $b \approx 0.18\,\lambda$, this gives 1×10^2 V/pC/m.

Conservatively, at a level of 10^5 V/pC/m, and 60 pC, the single-bunch induced fields correspond to a decelerating gradient of 6 MV/m, or 0.6% of the no-load gradient. (1% of the no-load gradient for a 15μm bunch) The single-bunch fundamental-mode beam loading term will be about 0.2% with some reduction due to less than optimal [*R/Q*]. For 50 such bunches one expects a 10% loss in accelerating gradient, and thus to achieve a loaded 1 GeV/m, one would require a no-load gradient of 1.1 GeV/m. With $[R/Q] \approx 90\,\Omega$, this figure is reduced to 1.04 GeV/m. The conclusion from such estimates is that longitudinal wakes are not particularly severe at 60 pC per bunch.

Transverse Wakefield

Let us consider next transverse wakefields. In the notation of Chao,[2] particle deflection may be described by

$$\Delta\gamma\, x'\big|_{wakefield} = \frac{eQ}{mc^2}\int_z^\infty dz\;\rho(z')W_1(z-z')x(z'),$$

where on the left we have the net impulse received in one structure period, normalized by mc. On the right, we have a convolution quantifying the integrated Lorentz force due to beam-excited fields in the structure, with W_1 the wakepotential associated with one period. The coordinate z ("beam coordinate") is related to time t and displacement down the beamline s, according to $z = s - ct$. The integration extends from the beam head ($z \to \infty$) to the z-coordinate of the beam slice for which the impulse is being calculated. The beam charge is Q, and the line charge density associated with the beam is described by the function ρ, normalized according to

$$\int_{-\infty}^{\infty} dz\;\rho(z') = 1.$$

In general several kinds of terms contribute to the total wakefield; let us consider these. First we have the resonant "dipole modes" of the cavities. The contribution to the wakepotential from one such mode takes the form

$$W_1(z)\big|_{\text{mode }\lambda} = 2\frac{k_\lambda c}{\omega_\lambda}\sin\left(\frac{\omega_\lambda z}{c}\right).$$

---

[1] K. Bane and M. Sands, "Wakefields of Very Short Bunches" SLAC-PUB-4441 (1987). See also P. B. Wilson, "Introduction to Wakefields and Wake Potentials", SLAC-PUB-4547.

[2] Alexander Wu Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (John Wiley & Sons, New York, 1993).

Unlike the longitudinal problem, where feedback is negligible (particles move negligibly in the coordinate z), for the transverse motion, feedback is an essential aspect of the problem. Over the course of many periods, the motion is described by

$$\left(\frac{d}{ds}\gamma\frac{d}{ds}+K\right)x(s,z)=\frac{eQ}{mc^2L_p}\int_z^\infty dz\ \rho(z')W_1(z-z')x(s,z'),$$

and the mechanism for feedback (instability) is the multiplicative effect of beam slice offsets. To quantify this in a simple way, we take a "top-hat" line-charge density,

$$\rho(z')=\frac{1}{T}H(z')H(T-z'),$$

with H the step-function, and T the beam length in units of cm. We approximate $W_1(z)\approx z\,W'$, where the contribution from a single-mode to the slope is $W'\approx 2k_\lambda$. In terms of the normalized beam coordinate $\tau=-z/T$, we may express our equation of motion as

$$\left(\frac{d}{ds}\gamma\frac{d}{ds}+K\right)x(s,z)=\frac{eQT\,W'}{mc^2L_p}\int_0^\tau d\tau'\ (\tau-\tau')x(s,z')=\frac{1}{L^2}\int_0^\tau d\tau'\ (\tau-\tau')x(s,z').$$

Here we introduce a quantity L, with units of length,

$$\frac{1}{L^2}=\frac{eQT\,W'}{mc^2L_p}.$$

For the case of a pillbox (see Chao) one can show that $k_\lambda\propto\omega^2$ with ω the operating frequency, and from this one can see that $W'\propto\omega^2$. With $L_p\propto 1/\omega$, one has then

$$\frac{1}{L^2}\propto Q\left(\frac{T}{\lambda}\right)\frac{1}{\lambda^2},$$

with λ the operating wavelength. As a point of reference, for the SLAC S-Band structure ($\lambda\approx 10.5\ \text{cm}$), $W_1(-1\,\text{mm})=0.7\ \text{cm}^{-2}$, with $L_p=3.5\ \text{cm}$, $W'\approx 7\ \text{cm}^{-3}$. This can be converted to V/pC/cm using $1\ \text{V/pC}\approx 1.11\ \text{cm}^{-1}$, so that,

$$\begin{aligned}\frac{1}{L^2}&=\frac{eQT\,W'}{mc^2L_p}=\frac{Q(\text{pC})}{511\times10^3\ \text{V}}\left(\frac{T}{\lambda}\right)\frac{0.7\ \text{cm}^{-2}}{0.1\ \text{cm}}\frac{10.5\ \text{cm}}{3.5\ \text{cm}}\frac{1\ \text{V/pC}}{1.11\ \text{cm}^{-1}}\\&\approx 3.7\times10^{-5}\,\text{cm}^{-2}Q(\text{pC})\left(\frac{T}{\lambda}\right).\end{aligned}\qquad\text{(S-Band)}$$

Typically the bunch length in the linac is 1.1 mm so that $T/\lambda\approx 1.05\times10^{-2}$. With a 5 nC bunch then $L\approx 22.7\ \text{cm}$. Were we to scale the S-Band geometry to a linac operating at a wavelength 32 times smaller, with a 60 pC bunch, and $T/\lambda\approx 1\times10^{-2}$, the length scale L would then be given by

$$\frac{1}{L^2} = 3.7\times 10^{-5}\,\mathrm{cm}^{-2} Q(\mathrm{pC})\left(\frac{T}{\lambda}\right) 32^2, \qquad \text{(SLAC S-Band scaled to W-Band)}$$

or $L \approx 6.6\ \mathrm{cm}$.

The eventual outcome of W-Band structure design will no doubt offer more choices than a simple SLAC S-Band structure, and thus it is interesting to have some feeling for the dependence of this estimate on geometry. Consider then the DDS X-Band structure wakefields (a=iris radius=0.187λ, average), scaled to W-Band

$$W_x \approx 88 \tfrac{V}{pC-m-mm} \times \underbrace{\left[\tfrac{W\ Band}{X\ Band}\right]^2}_{64} \times \underbrace{\left\{1-\exp\left(-0.89\left[8\tfrac{30\,\mu m}{1000\,\mu m}\right]^{0.87}\right)\right\}}_{0.23},$$

in the notation of the ZDR.[3] The wakefield 30 μm after a point bunch passes is

$$W_x \approx 1.3\times 10^3 \tfrac{V}{pC-m-mm}.$$

(For a 15μm bunch length this comes to $W_x \approx 7.4\times 10^2\ \mathrm{V/pC-m-mm}$.) In our notation, this corresponds to

$$W_1(-30\,\mu\mathrm{m}) = L_p W_x \approx 1.3\times 10^1 \tfrac{V}{pC-cm-mm} \times \frac{3.28\,\mathrm{mm}}{3} \approx 1.4\times 10^1 \tfrac{V}{pC-cm},$$

or

$$W' = \frac{W_1(-30\,\mu\mathrm{m})}{30\,\mu\mathrm{m}} \approx 4.65\times 10^3 \tfrac{V}{pC-cm^2}.$$

This tells us that

$$\frac{1}{L^2} = \frac{eQT\,W'}{mc^2 L_p} = \frac{Q(\mathrm{pC})}{511\times 10^3\,\mathrm{V}}\left(\frac{T}{\lambda}\right)\times 4.65\times 10^3 \tfrac{V}{pC-cm^2}\times 3$$

$$\approx 2.73\times 10^{-2}\,\mathrm{cm}^{-2} Q(\mathrm{pC})\left(\frac{T}{\lambda}\right),$$

or $L \approx 7.8\,\mathrm{cm}$, for a 60 pC bunch, and $T/\lambda \approx 1\times 10^{-2}$. (We are taking $L_p/\lambda = 1/3$, as for a 2π/3 mode structure). (For a 15 μm bunch $L \approx 11.1\,\mathrm{cm}$). The number of significant digits kept here is for convenience only, as there are error bars in the 5-10% range on the calculated wakefields. Bane reports that these results are consistent with a slope at the origin of

$$W' \approx \frac{2Z_0 c}{\pi a^4} L_p = \frac{8/3}{(a/\lambda)^4}\frac{1}{\lambda^3} = 7.2\times 10^4\,\mathrm{cm}^{-3},$$

where in the last equality we have evaluated the result for our parameters (and used the cgs conversion, $Z_0 = 377\,\Omega = 4\pi/c$). Note that this result, arising from coherent radiation of a point bunch passing through the iris aperture (of radius a) is inappropriate to use for our entire bunch, as it implies a degree of coherence that is lacking in the longer, 15 or 30 μm bunches.

For purposes of estimates here, we will make use of the more conservative figure,

[3] *Zeroth-Order Design Report for the Next Linear Collider*, SLAC Report 474. See pp. 367-369 for short-range wakefield work.

$L \approx 6\ \text{cm}$.

In addition, we will want to consider the resistive wall wakepotential,

$$\frac{W_1}{L_p} = -\frac{2}{\pi a^3}\sqrt{\frac{c}{\sigma}}\frac{1}{z^{1/2}}.$$

In normalized form, dynamics with only a resistive-wall wake takes the form,

$$\left(\frac{d}{ds}\gamma\frac{d}{ds} + K\right)x(s,\tau) = \frac{1}{L_{rw}^2}\int_0^\tau d\tau' \ \frac{x(s,\tau')}{(\tau-\tau')^{1/2}},$$

where

$$\frac{1}{L_{rw}^2} = \frac{eQ}{mc^2}\frac{2}{\pi a^3}\sqrt{\frac{c}{\sigma T}},$$

or, in more practical units, for smooth copper, with $a/\lambda = 0.18$, and $\lambda \approx 0.328\ \text{cm}$, as for 91.392 GHz,

$$\frac{1}{L_{rw}^2} \approx 2.33\times 10^{-6}\,\text{cm}^{-2} Q(\text{pC})\sqrt{\frac{\lambda}{T}}.$$

For a 60 pC bunch, and $T/\lambda \approx 1\times 10^{-2}$, this gives $L_{rw} \approx 26.7\,\text{cm}$. Use of the power-law approximation to the wakefield requires that

$$\frac{T}{a} >> \chi^{1/3} = \left(\frac{c}{4\pi\sigma a}\right)^{1/3},$$

and this corresponds to

$$\frac{T}{a} \approx \frac{30\ \mu\text{m}}{590\,\mu\text{m}} \approx 5\times 10^{-2}, \qquad \chi^{1/3} \approx 4\times 10^{-3},$$

and is satisfied. This is just the small skin-depth approximation, and states that the frequencies driven by our bunch $1/T \approx 10\ \text{THz}$, diffuse to a depth $\delta \approx 0.02\mu\text{m}$ that is much smaller than the aperture.

Transverse Beam Break-Up, No Focusing

We consider next the evolution of the beam centroid in the absence of focusing, K=0. We consider in this section first the linear wake, and separately, the resistive wall wake. We assume a 60 pC bunch, with 30 μm bunch length.

Dynamics with the linear wake, are described by

$$\frac{d}{ds}\gamma\frac{d}{ds}x(s,\tau) = \frac{1}{L^2}\int_0^\tau d\tau'(\tau-\tau')x(s,\tau'),$$

with $L \approx 6\ \text{cm}$.

While it is not particularly accurate in this case, we will consider first, for the sake of "culture" the two-particle model, where the beam is treated as consisting of two particles, each of charge *Q/2*. The first or "head" particle satisfies,

$$\frac{d}{ds}\gamma\frac{d}{ds}x_1 = 0,$$

and for the case of no initial angle error, follows the trajectory x_1 =constant. The second particle satisfies,

$$\frac{d}{ds}\gamma\frac{d}{ds}x_2 = \frac{1}{2L^2}x_1.$$

This corresponds to the case considered previously of a constant perturbing force,

$$\delta f = \frac{1}{2L^2}x_1,$$

and the solution at the linac exit is then

$$\frac{x_2}{x_1} = \frac{1.2}{2L(m)^2} = 1.7\times 10^2.$$

This would suggest that, to keep the beam centroid oscillations to the level of 1/10th of the beam size (*i.e.*, 1 μm), one could require control of incoming centroid jitter in the linac at the level of 6 nm, quite a severe limit.

Next let us make a more rigorous estimate of the amplification of incoming orbit errors. The actual beam consists not of two-particles, but, to a better approximation, a continuum. We Laplace transform in the beam coordinate,

$$\tilde{x}(s,p) = \int_0^\infty d\tau\; x(s,\tau)e^{-p\tau},$$

in terms of which our equation takes the form,

$$\frac{d}{ds}\gamma\frac{d\tilde{x}}{ds} = \frac{\tilde{x}}{L^2p^2}.$$

Approximate eikonal solutions to this equation are

$$\tilde{x}_\pm = \left(\frac{\gamma_0}{\gamma}\right)^{1/4} e^{\pm\sigma},$$

where

$$\sigma = \frac{1}{Lp}\int_0^s \frac{ds}{\gamma^{1/2}} = \frac{2}{GLp}\left\{\gamma^{1/2} - \gamma_0^{1/2}\right\}.$$

Taking as an example, an initial error in beam position, x_0, with no angle error,

$$\tilde{x} \approx \frac{x_0}{p}\left(\frac{\gamma_0}{\gamma}\right)^{1/4}\cosh\sigma,$$

and, with an inverse Laplace transform, the solution is

$$x(s,\tau) = \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp \, \tilde{x}(s,p) e^{p\tau},$$

with the contour taken to the right of all poles in the integrand. Thus,

$$x(s,\tau) \approx x_0 \left(\frac{\gamma_0}{\gamma} \right)^{1/4} \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp \, \frac{1}{p} e^{p\tau} \cosh\sigma,$$

and we need to evaluate integrals of the form,

$$I_\pm = \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp \, \frac{1}{p} e^{p\tau \pm \sigma} = \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp \, \frac{1}{p} e^{\phi_\pm}.$$

We do this by the method of steepest descents. Abbreviating,

$$\sigma = \frac{2}{GLp} \left\{ \gamma^{1/2} - \gamma_0^{1/2} \right\} = \frac{\alpha}{p},$$

the phase in the exponent is

$$\phi_\pm = p\tau \pm \frac{\alpha}{p},$$

with stationary points such that

$$\frac{\partial \phi_\pm}{\partial p} = \tau \mp \frac{\alpha}{p^2} = 0,$$

or

$$p_\pm^2 = \pm \frac{\alpha}{\tau},$$

at which points,

$$\phi_\pm = 2 p_\pm \tau.$$

Points of interest are those which give rise to asymptotic growth, corresponding to $\Re\phi_\pm > 0$, and this occurs only for

$$p_+ = \left(\frac{\alpha}{\tau} \right)^{1/2},$$

with

$$\phi_+ = 2(\alpha\tau)^{1/2},$$

and

$$\frac{\partial^2 \phi_+}{\partial p^2} = 2 \frac{\alpha}{p^3} = \frac{4\tau^2}{\phi_+}.$$

This provides the estimate,

$$I_+ \approx \left(2\pi \frac{\partial^2 \phi_+}{\partial p^2}\right)^{-1/2} \frac{1}{p_+} e^{\phi_+} \approx \left(2\pi \phi_+\right)^{-1/2} e^{\phi_+},$$

or

$$\frac{x(s,\tau)}{x_0} \approx \left(\frac{\gamma_0}{\gamma}\right)^{1/4} (2\pi A)^{-1/2} e^{A},$$

with the exponent A given by

$$A = \left\{\frac{8\tau}{GL}\left(\gamma^{1/2} - \gamma_0^{1/2}\right)\right\}^{1/2}.$$

Evaluating this at the linac exit and the beam tail we have,

$$A = \left\{\frac{8}{1.96\times 10^3 m^{-1} 6\times 10^{-2} m}\left(250^{1/2} - 1\right)\left(1.96\times 10^4\right)^{1/2}\right\}^{1/2} \approx 11.9,$$

and this corresponds to growth by a factor of 4×10^3, including the factor of 0.25 for injection at 10 GeV. Comparison of this analytic result with a numerical solution of the equations of motion can be seen in Fig.1. Results from simulation for a variety of structure offsets can be seen in Fig.2

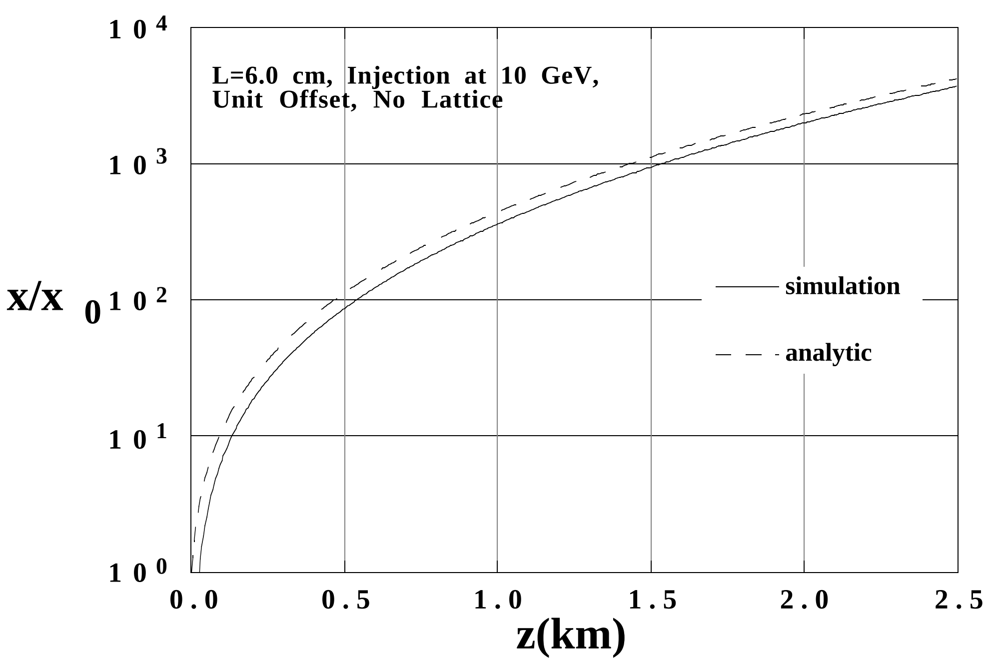

FIGURE 1. Comparison of the steepest-descents result with simulation for the case of a perfectly aligned linac, and an initial beam offset. No focusing lattice is present.

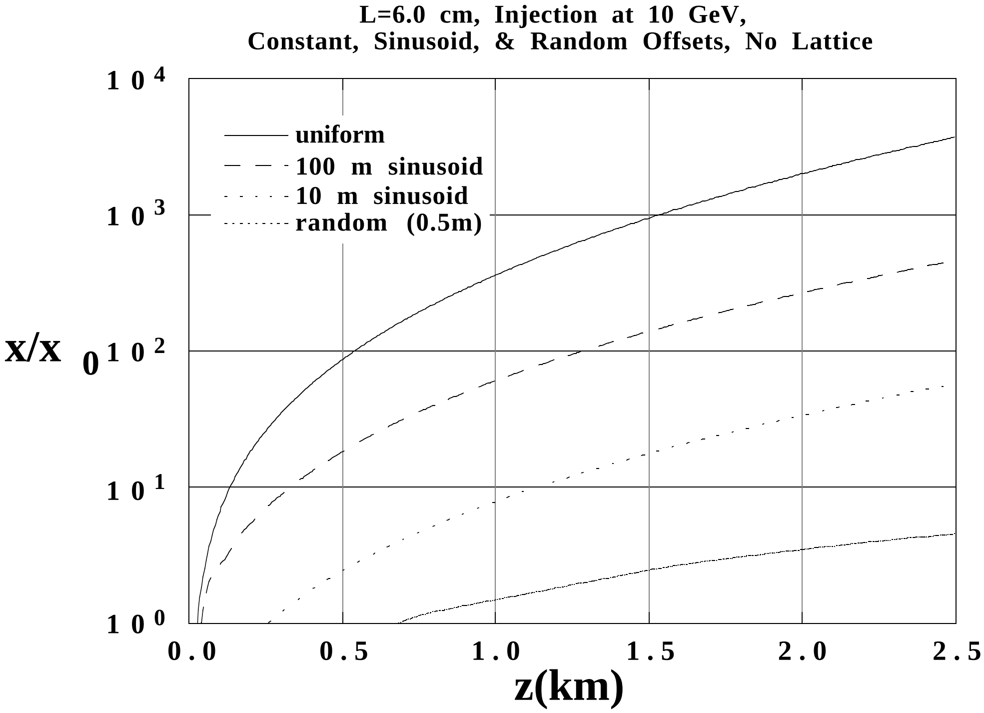

FIGURE 2. Results of simulation for a variety of linac structure misalignments: uniform offset by x_0, sinusoidal offsets of 100 m and 10 m wavelength with amplitude x_0, and random offsets every 0.5 m with amplitude x_0.

In passing we note the scaling in general takes the form,

$$A \approx \left(\frac{64\gamma}{G^2} \frac{eW_x}{mc^2} Q \right)^{1/4},$$

for $\gamma >> \gamma_0$. The conclusion from this analysis is that some magnetic focusing will be quite important to controlling transverse beam break-up.

Note that for fixed final energy, if gradient and charge are scaled with wavelength, the exponent is constant for constant geometrical aspect ratios. For comparison note that a scaled version of the SLC would have 32 x 20 MeV/m = 0.64 GeV/m and 5 nC / 32 = 160 pC. Our product G^2/Q is actually larger by a factor of than 6.5 this, and in this sense, our charge is "low", corresponding roughly to running the SLC at 5×10^9 particles per bunch. This comparison is a bit premature here though since we haven't yet included focusing.

Resistive Wall, No Focusing

Before moving on to a realistic lattice, let us consider the effect of the resistive wall wakefield. With a Laplace transform, our system takes the form,

$$\frac{d}{ds} \gamma \frac{d}{ds} \tilde{x}(s,p) = \frac{1}{L_{rw}^2} \frac{\pi^{1/2}}{2p^{1/2}} \tilde{x}(s,p).$$

Approximate eikonal solutions to this equation are

$$x_{\pm} = \left(\frac{\gamma_0}{\gamma}\right)^{1/4} e^{\pm\sigma},$$

where

$$\sigma = \frac{1}{L_{rw}} \frac{\pi^{1/4}}{2^{1/2} p^{1/4}} \int_0^s \frac{ds}{\gamma^{1/2}} = \frac{2^{1/2}\pi^{1/4}}{GL_{rw} p^{1/4}} \left\{\gamma^{1/2} - \gamma_0^{1/2}\right\}.$$

Taking as an example, an initial error in beam position, x_0, with no angle error,

$$\tilde{x} \approx \frac{x_0}{p}\left(\frac{\gamma_0}{\gamma}\right)^{1/4} \cosh\sigma,$$

and, with an inverse Laplace transform, the solution is

$$x(s,\tau) = \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp\, \tilde{x}(s,p) e^{p\tau},$$

with the contour taken to the right of all poles in the integrand. Thus,

$$x(s,\tau) \approx x_0\left(\frac{\gamma_0}{\gamma}\right)^{1/4} \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp\, \frac{1}{p} e^{p\tau} \cosh\sigma,$$

and we need to evaluate integrals of the form,

$$I_{\pm} = \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp\, \frac{1}{p} e^{p\tau\pm\sigma} = \frac{1}{2\pi j} \int_{-j\infty+}^{+j\infty+} dp\, \frac{1}{p} e^{\phi_{\pm}}.$$

We do this by the method of steepest descents. Abbreviating,

$$\sigma = \frac{2^{1/2}\pi^{1/4}}{GL_{rw} p^{1/4}} \left\{\gamma^{1/2} - \gamma_0^{1/2}\right\} = \frac{\alpha}{p^{1/4}},$$

the phase in the exponent is

$$\phi_{\pm} = p\tau \pm \frac{\alpha}{p^{1/4}},$$

with stationary points such that

$$\frac{\partial\phi_{\pm}}{\partial p} = \tau \mp \frac{\alpha}{4p^{5/4}} = 0,$$

or

$$p_{\pm}^{5/4} = \pm\frac{\alpha}{\tau},$$

at which points,

$$\phi_{\pm} = 2p_{\pm}\tau.$$

Points of interest are those which give rise to asymptotic growth, corresponding to $\Re\phi_{\pm} > 0$. The root with largest real exponent is

$$p_+ = \left(\frac{\alpha}{\tau}\right)^{4/5},$$

with

$$\phi_+ = 2\left(\alpha^4 \tau\right)^{1/5},$$

and

$$\frac{\partial^2 \phi_+}{\partial p^2} = \frac{5}{16}\frac{\alpha}{p^{9/4}} = \frac{5}{8}\frac{\tau^2}{\phi_+}.$$

This provides the estimate,

$$I_+ \approx \left(2\pi \frac{\partial^2 \phi_+}{\partial p^2}\right)^{-1/2} \frac{1}{p_+} e^{\phi_+} \approx \frac{2^{5/2}}{5^{1/2}} \left(2\pi \phi_+\right)^{-1/2} e^{\phi_+},$$

or

$$\frac{x(s,\tau)}{x_0} \approx \left(\frac{\gamma_0}{\gamma}\right)^{1/4} \frac{2}{5^{1/2}\pi^{1/2}A^{1/2}} e^A \approx \left(\frac{\gamma_0}{\gamma}\right)^{1/4} \frac{0.50}{A^{1/2}} e^A,$$

with the exponent A given by

$$A = 2^{7/5}\pi^{1/5}\tau^{1/5}\left(\frac{\gamma^{1/2} - \gamma_0^{1/2}}{GL_{rw}}\right)^{4/5}.$$

Evaluating this at the linac exit and the beam tail we have,

$$A = 3.318\left\{\frac{\left(250^{1/2} - 1\right)\left(1.96 \times 10^4\right)^{1/2}}{\left(1.96 \times 10^3 m^{-1} 2.67 \times 10^{-1} m\right)}\right\}^{4/5} \approx 10.0,$$

corresponding to growth by a factor of 9×10^2 including the factor of 0.25 due to adiabatic damping. For a 15 μm bunch, the length-scale L_{RW} is shorter by a factor of $2^{1/4}$, A is larger by a factor of $2^{1/5}$, and growth is 4×10^3, comparable to the structure wakefield induced growth for this bunch length.

Beam Transport With Wakefields, Arbitrary Lattice

Before analyzing an actual lattice, it is straightforward to set down the requirements we will impose on it. Our system is described by

$$\left(\frac{d}{ds}\gamma\frac{d}{ds} + K\right)x(s,\tau) = \frac{1}{L^2}\int_0^\tau d\tau'\left(\tau - \tau'\right)x\left(s,\tau'\right),$$

for some array of quadrupole gradients *K(s)*. We write this as

$$\gamma x'' + \gamma' x' + Kx = F,$$

and look for a solution

$$x(s,\tau) = \Re\left(\chi(s,\tau)B(s)\right),$$

where B is a phasor describing transport through this lattice in the absence of perturbations, and χ is a phasor (the eikonal) determined by the particular initial conditions, and evolving in a manner determined by the lattice and the perturbation. In conventional notation, we could write

$$B = \left(\frac{\beta}{\gamma}\right)^{1/2} e^{j\psi},$$

where γ is, as before, the Lorentz factor (at the particular location, s), β is the beta function, and ψ is the machine phase,

$$\psi(s) = \int_0^s ds' \frac{1}{\beta(s')}.$$

In terms of

$$A = \frac{B'}{B} = \frac{j}{\beta} + \frac{1}{2}\left(\frac{\beta'}{\beta} - \frac{\gamma'}{\gamma}\right),$$

and $a=\gamma A$, $b=\gamma B$, one can show that unperturbed transport in the lattice is governed by

$$a' + \frac{a^2}{\gamma} + K = 0.$$

The effect of perturbations may be expressed as,

$$\Re\left\{\left(\frac{ab}{\gamma} + b'\right)\chi'\right\} = F,$$

and we have neglected second derivatives in χ assuming that the effect of perturbations is gradual on the scale of β. We can make this result somewhat more explicit using,

$$\frac{ab}{\gamma} + b' = 2j\left(\frac{\gamma}{\beta}\right)^{1/2}(1 + j\alpha)e^{j\psi},$$

with $\alpha=-\beta'/2$. This gives,

$$\frac{d\chi}{ds}(s,\tau) = \left\langle \frac{\exp(-j\psi(s,\tau))}{2j\left(\frac{\gamma(s,\tau)}{\beta(s,\tau)}\right)^{1/2}(1 + j\alpha(s,\tau))} \frac{1}{L^2} \int_0^\tau d\tau'(\tau - \tau')\left(\frac{\gamma(s,\tau')}{\beta(s,\tau')}\right)^{1/2} \exp(j\psi(s,\tau'))\chi(s,\tau) \right\rangle$$

with the brackets denoting an average over the betatron period.

Neglecting variation in the zeroth-order optics or energy along the bunch (no BNS damping, *i.e.*, no τ-dependence in γ or β etc.) we have

$$\frac{d\chi}{ds}(s,\tau) = \frac{1}{2jL^2}\left\langle \frac{1}{(1 + j\alpha(s))} \frac{\beta(s)}{\gamma(s)} \right\rangle \int_0^\tau d\tau'(\tau - \tau')\chi(s,\tau).$$

We leave to a future note the evaluation of the bracketed term for various realizable lattices. In the meantime, we can straightforwardly evaluate asymptotic growth in general. For convenience let us denote,

$$\frac{1}{\gamma(s)k_\beta(s)}\left(1+j\varepsilon(s)\right)=\left\langle\ \frac{1}{\left(1+j\alpha(s)\right)}\frac{\beta(s)}{\gamma(s)}\ \right\rangle.$$

Performing a Laplace transform in τ, we have

$$\frac{d\tilde{\chi}}{ds}(s,p)=\frac{1}{2jL^2p^2}\frac{(1+j\varepsilon)}{\gamma\, k_\beta}\tilde{\chi}(s,p),$$

with p the Laplace transform variable. The solution is

$$\tilde{\chi}(s,p)=\tilde{\chi}(0,p)\exp\left(\frac{\eta(s)}{p^2}\right),$$

where

$$\eta(s)=\frac{1}{2jL^2}\int_0^s ds'\frac{(1+j\varepsilon)}{\gamma\, k_\beta}=\frac{1}{2jL^2}\int_0^s ds'\frac{1}{\left(1+j\alpha(s)\right)}\frac{\beta(s)}{\gamma(s)}.$$

Performing the inverse Laplace transform we have

$$\chi(s,\tau)=\frac{1}{2\pi j}\int_{-j\infty+}^{+j\infty+}dp\,\tilde{\chi}(0,p)\exp\left(\frac{\eta(s)}{p^2}+p\tau\right).$$

For example, for an initial offset uniform along the bunch, $\tilde{\chi}(0,p)=\chi(s=0)/p$. The stationary points of the phase in the exponent,

$$\phi=\frac{\eta}{p^2}+p\tau,$$

satisfy

$$p^3=\frac{2\eta}{\tau},$$

and the phase at such a point is

$$\phi=\frac{3}{2}p\tau,$$

with

$$\phi''=\frac{3\tau}{p}.$$

Asymptotic growth is then governed by the root p with the largest real part, and is given by

$$\chi\approx\frac{1}{2\pi^{1/2}\phi^{1/2}}e^{\phi}.$$

Making a choice of branch cut for the third root, such that $-\pi/3<\arg\ \eta^{1/3}<\pi/3$, the root giving growth may be expressed simply as $p=(2\eta/\tau)^{1/3}$, and

$$\phi=\frac{3}{2^{2/3}}\left(\eta\tau^2\right)^{1/3}.$$

As an illustrative example, if the lattice satisfies $\beta\propto\gamma$, then at the beam tail (τ=1), one has roughly

$$\eta\approx\frac{1}{2jL^2}\frac{s\beta_0}{\gamma_0}, \qquad\qquad (\beta\propto\gamma)$$

and

$$\phi=\frac{3}{2}\left(\frac{s\beta_0}{L^2\gamma_0}\right)^{1/3}e^{-j\pi/6}.$$

In terms of exponent

$$A=\frac{3^{3/2}}{4}\left(\frac{s\beta_0}{L^2\gamma_0}\right)^{1/3},$$

the result for the eikonal is

$$\chi\approx\frac{3^{1/4}}{2^{3/2}\pi^{1/2}}\frac{1}{A^{1/2}}\exp\left\{A\left(1-\tfrac{j}{3^{1/2}}\right)+j\tfrac{\pi}{12}\right\}.$$

The centroid of the beam tail normalized to the initial offset x_0, is given by

$$\frac{x}{x_0}\approx\frac{3^{1/4}}{2^{3/2}\pi^{1/2}}\frac{e^A}{A^{1/2}}\cos\left\{\psi-\tfrac{A}{3^{1/2}}+\tfrac{\pi}{12}\right\},$$

or $x/x_0\approx 0.26e^A/A^{1/2}$. For $A\approx 5$ this gives growth by a factor of 17, for A˜7 the factor is 108. We may express the required β_0 in terms of A according to

$$\beta_0=\left(\frac{4A}{3^{3/2}}\right)^3\frac{L^2\gamma_0}{s}=1.3\times10^{-2}A^3\mathrm{m},$$

using $L\approx 6$ cm as for a 30 μm bunch, $\gamma_0\approx 1.957\times10^4$ (injection at 10 GeV) and s=2490 m. Thus for $A\approx 5$, one requires a rather small $\beta_0\approx 1.6$ m for a 30 μm bunch. For $A\approx 7$, one requires $\beta_0\approx 4.4$ m for a 30 μm bunch. To achieve $A\approx 1.6$, corresponding to no amplification, one would require $\beta_0\approx 5$ cm. Note that for L=7.8 cm, $A\approx 5$ corresponds to $\beta_0\approx 2.7$ m.

Results for $\beta_0\approx 1.6$ m are depicted in Fig. 3, comparing the analytic result with the result of simulation.

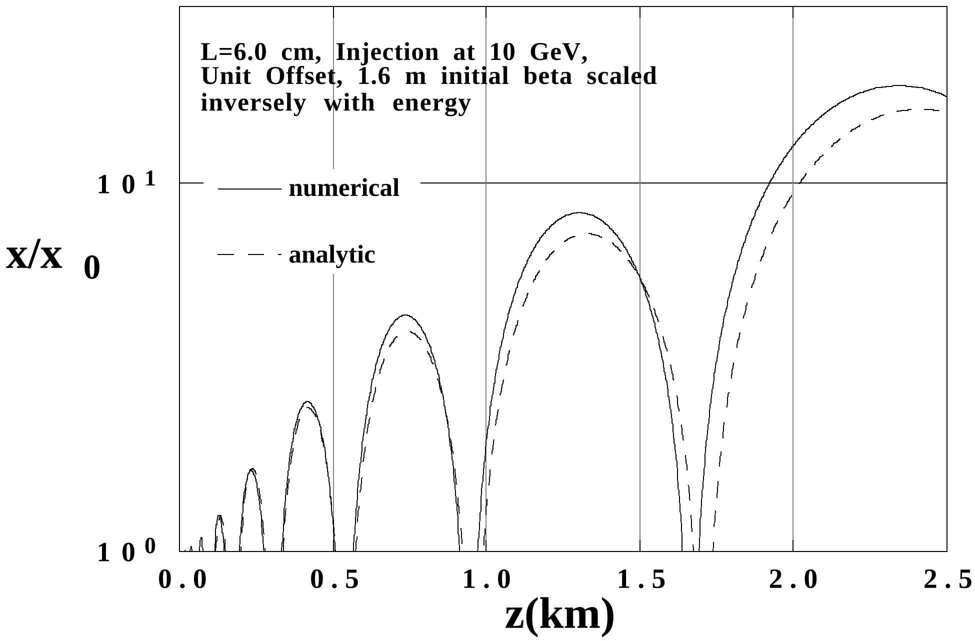

FIGURE 3. Results of simulation for a uniform beam offset in a perfectly aligned linac, with a perfectly aligned magnetic lattice corresponding to $\beta_0 \approx 1.6\ \mathrm{m}$. The analytic result from the steepest descents calculation is overlayed, showing fair agreement. Lattice scaling is $\beta \propto \gamma$.

Note also that if instead we scale $\beta \propto \gamma^{1/2}$, we obtain

$$\eta \approx \frac{1}{2jL^2}\frac{\beta_0}{2G}\left(\frac{\gamma}{\gamma_0}\right)^{1/2}, \qquad\qquad (\beta \propto \gamma^{1/2})$$

and our exponent takes the form

$$A = \frac{3^{3/2}}{4}\left(\frac{\beta_0}{2GL^2}\right)^{1/3}\left(\frac{\gamma}{\gamma_0}\right)^{1/6}.$$

In this case $\beta_0 \approx 1.6\ \mathrm{m}$ corresponds to $A \approx 1.6$, and no net amplification of an initial beam offset. We leave to a later note the task of evaluating these results with BNS damping, *i.e.*, taking into account a variation in focusing strength along the bunch.

Next we turn to the problem of devising a lattice and evaluating the lattice functions, particularly the quantity η (not to be confused with dispersion).

Simplest Lattice: Thin-Quad, FODO

To get a sense of the kind of quadrupole gradients we will require for 1m β functions at 10 GeV, let us first consider a FODO lattice with no acceleration. Let the quad thickness be l, and the quad separation be L. In terms of the focal length of 1/2 of a quad (thin lens approximation),[4]

$$\frac{1}{f} = \frac{1}{2} kl,$$

with the quadrupole strength k determined by the quadrupole gradient,

$$k\left(m^{-2}\right) = 0.3 \frac{g(T/m)}{E(GeV)},$$

the beta function varies from β_+ at a focusing quad center to β_- at a defocusing quad center, with

$$\beta_{\pm} = \kappa L \left(\frac{\kappa \pm 1}{\kappa \mp 1} \right)^{1/2},$$

and

$$\kappa = \frac{f}{L}.$$

The betatron phase-advance ϕ through one full period (a length $2L$) is given by

$$\sin\left(\frac{\phi}{2}\right) = \frac{1}{\kappa} = \frac{L}{f}.$$

Stable motion requires $L<f$. For example, for a 90° lattice,

$$\beta_+ \approx 3.4L, \qquad \beta_- \approx 0.6L, \qquad f \approx 1.4L.$$

If one asks for β_+˜0.5 m, the quad spacing should be 0.15 m. The corresponding focal length for one-half a quad would be 0.2 m. The quadrupole gradient at 10 GeV would be, in terms of the quad full thickness l,

$$g = \frac{6.7E(GeV)}{f(m)l(m)} \approx \frac{330\,T/m}{l(m)}.$$

This is a high quad gradient however with a mm-pole tip radius, and a 1 T pole-tip field, one can achieve 1000 T/m. On the other hand, even for such a gradient the quad length would required would approach the quad separation, suggesting that we avoid the thin lens assumption.

[4] Helmut Wiedemann, *Particle Accelerator Physics* (Springer-Verlag, Berlin, 1993)

Thick-Quad FODO Lattice

We consider next thick quads. Let the distance between quad edges be d, so that $L=d+l$ is the center-to center quad distance. Let each quad be of equal strength κ. Transport through this system may be computed from the equation of motion,

$$\frac{d}{ds}\gamma\frac{dx}{ds}+\begin{Bmatrix}+K\\0\\-K\end{Bmatrix}x=0,$$

where the terms in brackets correspond to (from top to bottom) the focusing quad, the drift, and the defocusing quad. We consider the case of no acceleration. Transport from the center of a focusing quad to the quad exit is given by the matrix

$$R_{1/2F}=\begin{bmatrix}\cos\theta & \frac{\sin\theta}{\kappa}\\ -\kappa\sin\theta & \cos\theta\end{bmatrix},$$

where we abbreviate

$$\kappa=\left(\frac{K}{\gamma}\right)^{1/2}=\left(0.3\frac{g(T/m)}{E(GeV)}\right)^{1/2}m^{-1},$$

[different from previous κ], and

$$\theta=\frac{1}{2}\kappa l.$$

The drift is described by the matrix,

$$R_{Drift}=\begin{bmatrix}1 & d\\ 0 & 1\end{bmatrix},$$

and transport from the defocusing quad edge to the defocusing quad center is governed by

$$R_{1/2D}=\begin{bmatrix}\cosh\theta & \frac{\sinh\theta}{\kappa}\\ \kappa\sinh\theta & \cosh\theta\end{bmatrix}.$$

Transport from quad center to quad center (1/2 period) is then governed by

$$R_{1/2}=R_{1/2D}R_{Drift}R_{1/2F}.$$

The beta functions may be obtained by computing this product of matrices and comparing to

$$R_{1/2}=\begin{bmatrix}\left(\frac{\beta_-}{\beta_+}\right)^{1/2}\cos\left(\frac{\phi}{2}\right) & \left(\beta_+\beta_-\right)^{1/2}\sin\left(\frac{\phi}{2}\right)\\ -\frac{1}{\left(\beta_+\beta_-\right)^{1/2}}\sin\left(\frac{\phi}{2}\right) & \left(\frac{\beta_+}{\beta_-}\right)^{1/2}\cos\left(\frac{\phi}{2}\right)\end{bmatrix}.$$

Considering the case d=0 one has

$$R_{1/2} = \begin{bmatrix} \cos\theta\cosh\theta - \sin\theta\sinh\theta & \frac{1}{\kappa}(\sin\theta\cosh\theta + \cos\theta\sinh\theta) \\ \kappa(\cos\theta\sinh\theta - \sin\theta\cosh\theta) & \sin\theta\sinh\theta + \cos\theta\cosh\theta \end{bmatrix}.$$

After some algebra one can check that

$$\cos\phi = \cos 2\theta \cosh 2\theta,$$

$$\beta_{\pm} = \frac{1}{\kappa}\left(\frac{\tan\theta + \tanh\theta}{\tan\theta - \tanh\theta}\right)^{1/2}\left(\frac{1 \pm \tan\theta\tanh\theta}{1 \mp \tan\theta\tanh\theta}\right)^{1/2}.$$

The motion is stable for θ<0.9375 rad. For example, a ϕ=90° lattice corresponds to θ=0.79, β_+=4.8/κ, β_-=1.0/κ. In this case g=30 T/m corresponds to $\kappa \approx 1\ \mathrm{m}^{-1}$. Quad length in this case would be $l = 2\theta/\kappa \approx 1.6\ \mathrm{m}$. The minimum β_+ corresponds to ϕ=53.8° (θ=0.627) β_+=4.225/κ, β_-=1.799/κ. Quad length in this case would be l=1.25 m. To reach a 1.7 m β_+ one would need $\kappa \approx 2.5\ \mathrm{m}^{-1}$ or g=185 T/m. Quad length in this case would be 0.5 m.

From this analysis, and that for beam dynamics, one concludes that quad gradients in the range of 200 T/m are needed, and that the quads will more or less fill the beamline. It is natural for such a lattice to consider scaling the lattice such that θ is constant and this corresponds to $\beta \propto \gamma^{1/2}$, with quad length scaling as $l \propto \gamma^{1/2}$. Quad length would start out in the 1 m range, and extend into the 15 m range at the linac exit.

Future Work

Important work to cover in future notes on this subject includes: magnet design, rf quadrupole compensation, and evaluation of wakefields for actual candidate structures. In addition, incorporation of BNS damping could be considered, although this is not be viewed with complete enthusiasm, as the downstream end has need of a beam with very low energy spread, and the additional complication associated with chromatic effects appears unnecessary for the time being. It would be natural however to cover this subject in the examination of rf quadrupole effects.