Space Charge Compensated Collisions of Uniform Beams

A computer simulation has been developed to model collisions of space charge compensated (neutral) beams. This system, shown below, is known to suffer from a charge separation instability, and the purpose of the simulation is to make estimates of tolerances for space charge compensated collisions. One would like to be able to work with a variety of beam distributions, but to check the simulation, comparisons are made with uniform beams of radius σ_r and total bunch length L. This case can be solved analytically, and, therefore, it serves as a check of the simulation.

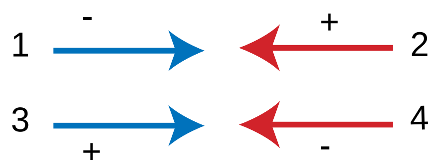

Figure 1: Space charge compensated collisions including definitions of beam numbers.

This note is divided into two major sections: Analytical Solution, and Comparison with Simulations.

I. ANALYTICAL SOLUTION

This problem was first posed and solved with the approximate solution given in section I.5 by D. Whittum in November, 1996. Most of section I is a repeat of the his original work.

I.1 Derivation of the Equations of Motion

The equation of motion for a single particle in a relativistic beam is

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) = 2q\vec{E}$$

where $\vec{E}$ and $\vec{B}$ are the transverse fields produced by the opposing beam(s). For a opposing beam with a uniform charge density

$$E = \frac{\rho}{2\varepsilon_0} r$$

where r is the distance of the particle from the center of the opposing beam and ρ is the charge density. This field points radially inward or outward depending on the relative charges; letting q and ρ both have signs

$$\frac{d^2\vec{r}}{dt^2} = \frac{2q}{\gamma m}\vec{E} = -\frac{q\rho}{\gamma m\varepsilon_0}\vec{r}$$

For two oppositely charged, equal charge density opposing beams

$$\frac{d^2\vec{r}}{dt^2} = -\frac{q\rho}{\gamma m\varepsilon_0}\left(\vec{r}_+ - \vec{r}_-\right)$$

where $\vec{r}_+$ is the distance from the particle to the center of the beam with the same sign and $\vec{r}_-$ is the distance to the center of the oppositely charged beam. With the vectors defined as in the picture below

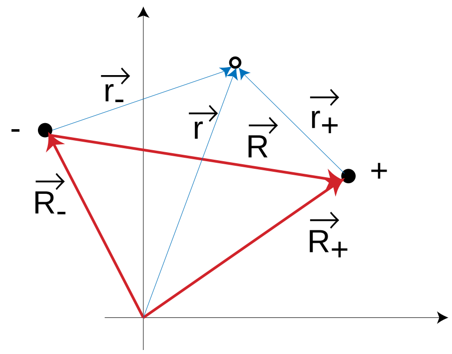

Figure 2: Vectors giving the location of positive and negative beams and particle location.

$$\frac{d^2\vec{r}}{dt^2} = -\frac{q\rho}{\gamma m\varepsilon_0}\left(\vec{r} - \vec{R}_+ - (\vec{r} - \vec{R}_-)\right) = -\frac{q\rho}{\gamma m\varepsilon_0}\left(\vec{R}_- - \vec{R}_+\right) = \frac{q\rho}{\gamma m\varepsilon_0}\vec{R}$$

The force on a particle depends only on the difference between the two centroids.

Let $\vec{R}_\ell$ denote the centroid difference of the pair of beams moving to the left and $\vec{R}_r$ denote the centroid for the pair of beams moving to the right. The equation of motion for the centroid of the positive particles (q = +e) moving to the right is

$$\frac{d^2\vec{R}_{r,+}}{dt^2} = \frac{d^2}{dt^2}\frac{1}{N}\sum_n \vec{r}_{n,+} = \frac{1}{N}\sum_n \frac{d^2\vec{r}_{n,+}}{dt^2} = \frac{e\rho}{\gamma m\varepsilon_0}\vec{R}_\ell$$

and the equation for the negative particles (q = -e) moving to the right is

$$\frac{d^2\vec{R}_{r,-}}{dt^2} = \frac{-e\rho}{\gamma m\varepsilon_0}\vec{R}_\ell$$

Adding these equations together gives

$$\frac{d^2\vec{R}_r}{dt^2} = \frac{d^2\left(\vec{R}_{r,+} - \vec{R}_{r,-}\right)}{dt^2} = \frac{2e\rho}{\gamma m\varepsilon_0}\vec{R}_\ell$$

Similarly

$$\frac{d^2\vec{R}_\ell}{dt^2} = \frac{2e\rho}{\gamma m\varepsilon_0}\vec{R}_r$$

These are the two coupled equations of motion that must be solved.

The plasma frequency that gives the time scale for exponential growth is given by

$$\Omega^2 = \frac{2e^2}{m\varepsilon_0}\frac{\rho_N}{\gamma} = 8\pi r_e c^2 \frac{\rho_N}{\gamma}$$

where $\rho_N = \rho/e$ is the number density. If the beams have radius σ_r and full length L and each beam has N particles

$$\rho_N = \frac{N}{\pi L \sigma_r^2}$$

The disruption parameter D is given by

$$D = \frac{r_e N L}{\gamma \sigma_r^2}$$

and the plasma frequency in terms of D is

$$\Omega^2 = \frac{8c^2 D}{L^2}$$

The final equations of motion are

$$\boxed{\frac{d^2\vec{R}_\ell}{dt^2} = \frac{8c^2D}{L^2}\vec{R}_r; \quad \frac{d^2\vec{R}_r}{dt^2} = \frac{8c^2D}{L^2}\vec{R}_\ell}$$

I.2 Solution of the Equations of Motion

Let L_ℓ and L_r be internal bunch coordinates in the left and right moving bunches respectively. Both are defined to be pointing in the +z direction. The figure below shows the picture at t = 0. The heads of the bunches are just touching at z = 0.

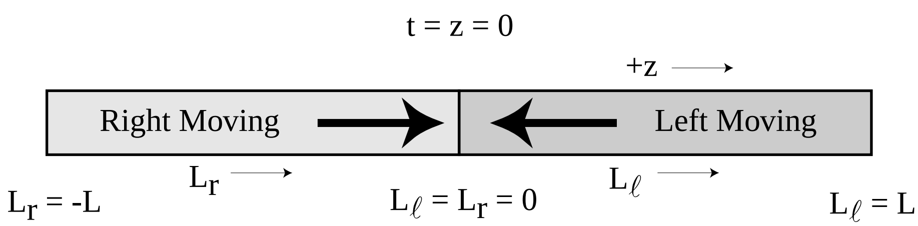

Figure 3: Configuration and coordinates. L_r and L_ℓ are right-handed coordinates that move with the beams. The figure is drawn at the start of the interaction.

The force on the particles located at L_r depend on the centroid of the opposing beam at coordinate $L_\ell = L_r + 2ct$, etc. The equations of motion are

$$\frac{d^2\vec{R}_\ell(t, L_\ell)}{dt^2} = \frac{8c^2D}{L^2}\vec{R}_r(t, L_r = L_\ell - 2ct)$$

$$\frac{d^2\vec{R}_r(t, L_r)}{dt^2} = \frac{8c^2D}{L^2}\vec{R}_\ell(t, L_\ell = L_r + 2ct)$$

Switch from time to L_ℓ and L_r as variables where R_ℓ is to be evaluated at a fixed position in the bunch (a fixed L_ℓ) and R_r is to be evaluated at a fixed L_r. For the right moving bunch z = ct + L_r, and for the left moving bunch z = ct - L_ℓ. Adding and subtracting these equations gives expressions for t and z. The time derivatives in the equations are the convective derivatives, so for the right moving beam

$$\begin{aligned}\frac{d}{dt} &= \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial z} \\ &= \frac{\partial L_r}{\partial t}\frac{\partial}{\partial L_r} + \frac{\partial L_\ell}{\partial t}\frac{\partial}{\partial L_\ell} + c\frac{\partial L_r}{\partial z}\frac{\partial}{\partial L_r} + c\frac{\partial L_\ell}{\partial z}\frac{\partial}{\partial L_\ell} \\ &= 2c\frac{\partial}{\partial L_\ell}\end{aligned}$$

Similarly, for the left moving beam

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial z} = 2c\frac{\partial}{\partial L_r}$$

Substituting these expressions into the equations of motion gives

$$\frac{\partial^2 \vec{R}_\ell(L_\ell, L_r)}{\partial L_r^2} = \frac{2D}{L^2}\vec{R}_r(L_\ell, L_r)$$

$$\frac{\partial^2 \vec{R}_r(L_\ell, L_r)}{\partial L_\ell^2} = \frac{2D}{L^2}\vec{R}_\ell(L_\ell, L_r)$$

The solution to the R_ℓ equation is

$$\vec{R}_\ell(L_\ell, L_r) = \vec{R}_\ell(L_\ell, 0) + \frac{\partial \vec{R}_\ell(L_\ell, 0)}{\partial L_r}L_r + \frac{2D}{L^2}\int_0^{L_r} dx_r (L_r - x_r)\vec{R}_r(L_\ell, x_r)$$

Substituting into the R_r equation gives an equation that depends on R_r and the initial value and slope of R_ℓ

$$\frac{\partial^2 \vec{R}_r(L_\ell, L_r)}{\partial L_\ell^2} = \frac{2D}{L^2}\left\{\vec{R}_\ell(L_\ell, 0) + \frac{\partial \vec{R}_\ell(L_\ell, 0)}{\partial L_r}L_r + \frac{2D}{L^2}\int_0^{L_r} dx_r (L_r - x_r)\vec{R}_r(L_\ell, x_r)\right\}$$

Let $\tilde{r}_r(L_\ell, p)$ denote the Laplace transform of $\vec{R}_r(L_\ell, L_r)$ with respect to L_r. The Laplace transform of the equation above is

$$\frac{\partial^2 \tilde{r}_r(L_\ell, p)}{\partial L_\ell^2} = \frac{2D}{L^2}\left\{\frac{\vec{R}_{\ell 0}}{p} + \frac{\vec{R}'_{\ell 0}}{p^2} + \frac{2D}{L^2 p^2}\tilde{r}_r(L_\ell, p)\right\}$$

where

$$\vec{R}_{\ell 0} = \vec{R}_\ell(L_\ell, 0), \text{ and } \vec{R}'_{\ell 0} = \frac{\partial \vec{R}_\ell(L_\ell, 0)}{\partial L_r}$$

The solution to this equation is

$$\tilde{r}_r(L_\ell,p)=\tilde{r}_{r0}\cosh\left(\frac{2DL_\ell}{L^2p}\right)+\frac{L^2p}{2D}\tilde{r}'_{r0}\sinh\left(\frac{2DL_\ell}{L^2p}\right)+\int_0^{L_\ell}dx_\ell\sinh\left(\frac{2D(L_\ell-x_\ell)}{L^2p}\right)\left\{\vec{R}_{\ell 0}+\frac{\vec{R}'_{\ell 0}}{p}\right\}$$

I.3 Specific Solution

Look at the specific case where $\vec{R}_{\ell 0}=\vec{R}'_{\ell 0}=\vec{R}'_{r0}=0;\vec{R}_{r0}=2\Delta$; that is the centroids of the right moving beams are displaced from each other by 2Δ in a single plane. Laplace transforming the right moving beam initial conditions gives $\tilde{r}'_{r0}=0;\tilde{r}_{r0}=2\Delta/p$. Substituting into the equation above

$$\tilde{r}_r(L_\ell,p)=\frac{2\Delta}{p}\cosh\left(\frac{2DL_\ell}{L^2p}\right)$$

Expanding the cosh in a Taylor series gives

$$\tilde{r}_r(L_\ell,p)=2\Delta\left\{\frac{1}{p}+\sum_{n=1}^{\infty}\frac{1}{(2n)!}\left(\frac{2DL_\ell}{L^2}\right)^{2n}\frac{1}{p^{2n+1}}\right\}$$

This inverse Laplace transform can be performed on this equation term-by term. The result is

$$\frac{R_r(L_\ell,L_r)}{2\Delta}=1+\sum_{n=1}^{\infty}\frac{1}{(2n)!}\frac{1}{(2n)!}\left(2D\frac{L_\ell}{L}\frac{L_r}{L}\right)^{2n}\qquad\textbf{(1)}$$

The separation of the left moving beams can be calculated by interchanging $\ell\leftrightarrow r$ in the general solution above and substituting the initial conditions

$$\tilde{r}_\ell(p,L_r)=2\Delta\int_0^{L_r}dx_r\sinh\left(\frac{2D(L_r-x_r)}{L^2p}\right)=-\frac{L^2p\Delta}{D}\left\{1-\cosh\left(\frac{2DL_r}{L^2p}\right)\right\}$$

Expanding the cosh and performing the inverse transformation. The result is

$$\frac{R_\ell(L_\ell,L_r)}{2\Delta}=\frac{1}{2D}\left(\frac{L}{L_\ell}\right)^2\sum_{n=1}^{\infty}\frac{1}{(2n)!}\frac{1}{(2n-2)!}\left(2D\frac{L_\ell}{L}\frac{L_r}{L}\right)^{2n}\qquad\textbf{(2)}$$

I.4 Discussion of Solution

For low values of D the dominant term in eq. (2) for the separation of the left moving beams is the n = 1 term. Keeping only that term

$$\frac{R_\ell(L_\ell,L_r)}{2\Delta}=D\left(\frac{L_r}{L}\right)^2$$

independent of L_ℓ. The beams 2 & 4 are uniformly displaced from each other after the collision. For the right moving beams

$$\frac{R_r(L_\ell, L_r)}{2\Delta} = 1 + \left(D \frac{L_\ell}{L} \frac{L_r}{L} \right)^2$$

The correction to the initial separation depends of the longitudinal location within the beam. This is the first place a regenerative interaction between the bunches is seen. The results for low disruption are plotted below.

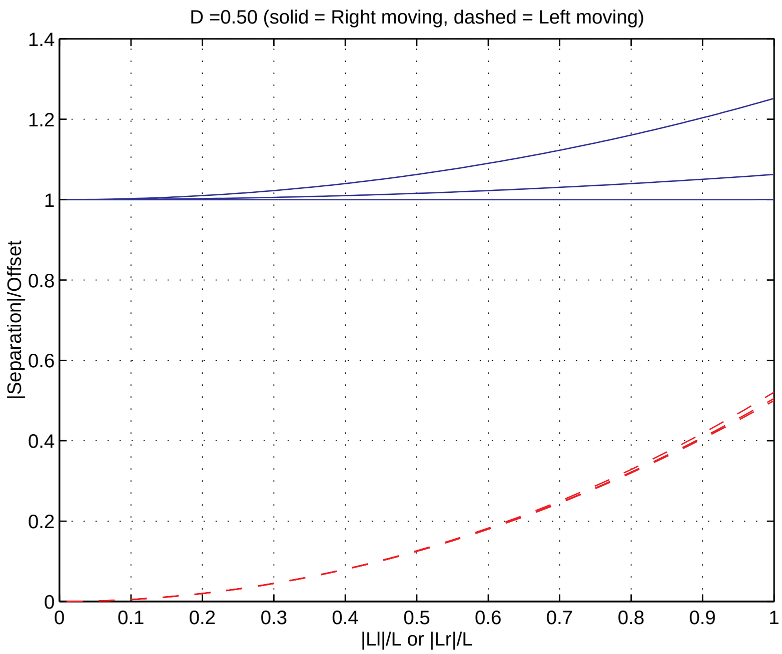

Figure 4: Separations for the head, middle, and tail for the left (dashed) and right (solid) moving beams. D = 0.5

As the disruption becomes larger, the forces become strong enough that separation between both the left and the right moving beams varies along the bunches. The beams bow away from each other forming bananas. The results for D = 5 follow.

Several interesting features are seen in this plot.

1) The separation of the heads of the right moving beams stays at roughly the initial value of the offset. That is because the left moving beam presents a highly neutralized target to it at the beginning of the collision giving a short path length for regenerative growth. However, the head of the left moving beam experiences the forces caused by the initial separation of the right moving beams throughout the collision, and it ends with a separation that is many times that of the right moving beam.

2) The middles of both beams have significant separation, but the middle of the left moving beam still has greater separation than the middle of the right moving beam even though the latter was offset initially.

3) At the end of the beams the right moving beam with the initial offset has greater separation than the left moving beam.

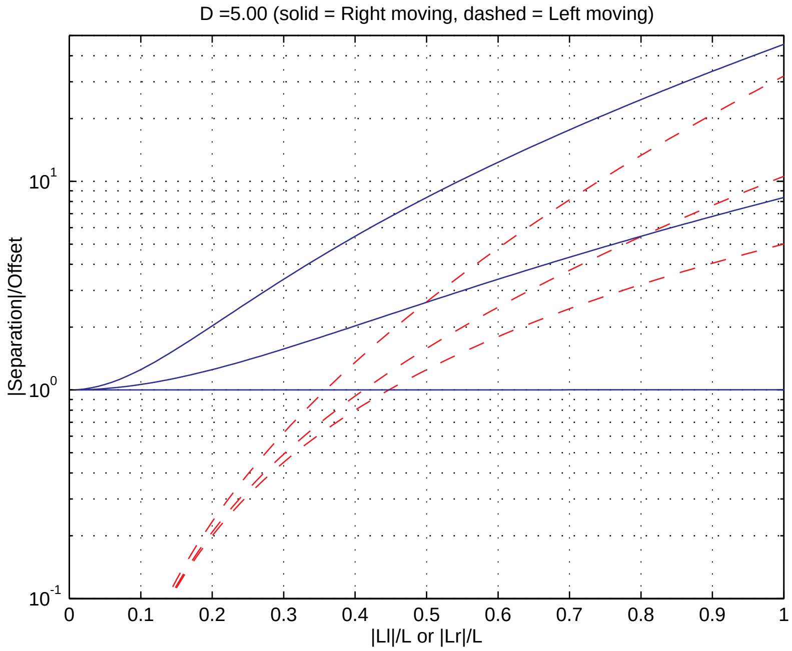

Figure 5: Separations for the head, middle, and tail for the left (dashed) and right (solid) moving beams. D = 5.

I.5 Approximate Specific Solution

It is possible to get an approximate solution by the method of steepest decent. Using the above expression for $\tilde{r}_\ell(p, L_r)$ and writing the inverse Laplace transform

$$R_\ell(L_\ell, L_r) = -\frac{1}{2\pi i}\frac{L^2\Delta}{D}\int_{-i\infty}^{i\infty} dp e^{pL_\ell} p\left\{1-\cosh\left(\frac{2DL_r}{L^2 p}\right)\right\} \approx \frac{1}{2\pi i}\frac{L^2\Delta}{D}\int_{-i\infty}^{i\infty} dp e^{pL_\ell} p\cosh\left(\frac{2DL_r}{L^2 p}\right)$$

The phase of the integrand is

$$\phi(p) = pL_\ell \pm \frac{2DL_r}{L^2 p}$$

Taking the first derivative and setting it equal to zero gives the saddle points

$$p_0^2 = \mp\frac{2DL_\ell}{L^2 L_r}$$

The solution will be exponentially growing when p_0 is real corresponding to the minus sign since L_r is always negative. Keeping only this solution gives

$$\phi''(p_0) = \left.\frac{d^2\phi}{dp^2}\right|_{p_0} = \sqrt{\frac{2L^2L_r^3}{DL_\ell}}$$

The approximate integral is

$$R_\ell(L_\ell, L_r) = \frac{1}{2\pi i}\frac{L^2\Delta}{D}\int_{-i\infty}^{i\infty} dp e^{pL_\ell} p \cosh\left(\frac{2DL_r}{L^2 p}\right)$$

$$= \frac{1}{2\pi i}\frac{L^2\Delta}{D}\sqrt{\frac{-2\pi}{\phi''(p_0)}}\frac{p_0}{2}\exp(\phi(p_0)) \qquad \textbf{(3)}$$

$$= \frac{\Delta}{\sqrt{2\pi}}\left[\frac{1}{8D}\frac{(|L_\ell|/L)^3}{(|L_r|/L)^5}\right]^{1/4} \exp\left(\sqrt{8D\frac{|L_r|}{L}\frac{|L_\ell|}{L}}\right)$$

This is the original solution Dave Whittum obtained.

The different solutions are plotted below for the tails of the bunches when $|L_\ell/L| = 1$ or $|L_r/L| = 1$.

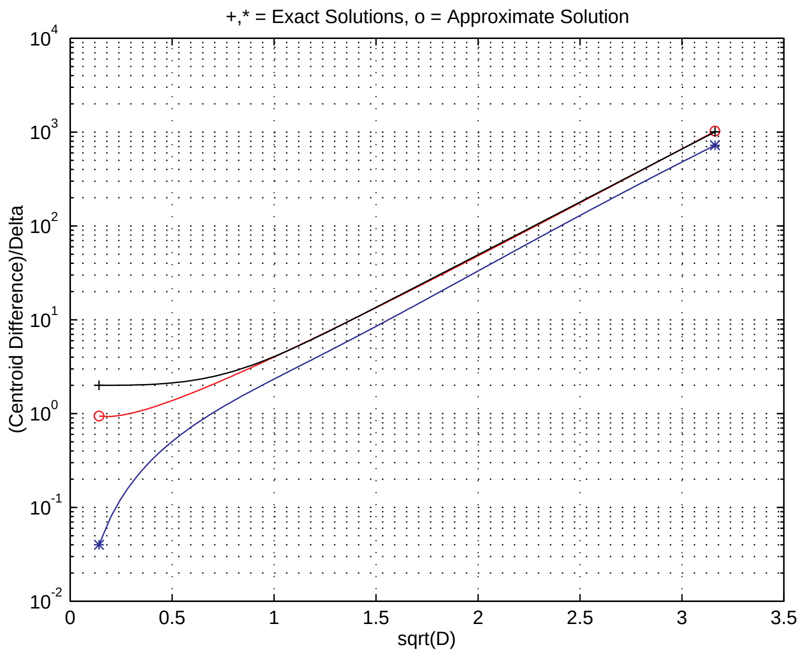

Figure 6: Separations at the end of the collision given by the exact solutions eqs. (1) and (2) and by the approximate solution eq. (3).

II. COMPARISON WITH SIMULATIONS

The theory for the separation instability in the collision between uniform, space-charge compensated beams has been developed in section I above. This section concerns the comparison with simulation. Beam numbers are defined in Fig. 1 with beams 1 & 3 being the right moving beams and beams 2 & 4 being the left moving beams. The exact solutions are given in eqs. 91) and (2) above which are repeated here for convenience.

$$\frac{R_r(L_\ell, L_r)}{2\Delta} = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{1}{(2n)!} \left(2D \frac{L_\ell}{L} \frac{L_r}{L} \right)^{2n} \tag{1}$$

$$\frac{R_\ell(L_\ell, L_r)}{2\Delta} = \frac{1}{2D} \left(\frac{L}{L_\ell} \right)^2 \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{1}{(2n-2)!} \left(2D \frac{L_\ell}{L} \frac{L_r}{L} \right)^{2n} \tag{2}$$

In these equations 2Δ is the initial offset between beams 1 & 3, L_r and L_ℓ are coordinates that move with the beams as shown in figure 3, L is the full length of the beams, and D is the disruption given by

$$D = \frac{r_e N L}{\gamma \sigma_r^2}$$

For these simulations the beam is assumed to be cold with $\beta_x = \beta_y = 4.5$ mm, and $\gamma\varepsilon_x = \gamma\varepsilon_y = 1\times10^{-8}$ m. The beam energy is E = 2.5 TeV ($\gamma = 4.9\times10^6$) giving $\sigma_r = 3.03$ nm. The full bunch length is L = 60 μm.

These simulations were performed with each beam being represented by 16 slices with 10^5 test particles/slice/beam. Collisions between slices were stepped through in the time order in which they occurred. Fields were calculated using the field solving algorithm of Krishnagopal, Podobedov and Siemann.[1] The simulation used 300 radial bins with a radial mesh size of $0.01\sigma_r$ and 16 azimuthal bins. The results presented are for $2\Delta/\sigma_r = 0.05$.[2]

Results are presented below for D = 1 and D = 6.9 and for slices 1, 8, and 16. In general the agreement is good with the largest discrepancy about a factor or 2 for large disruptions and the tails of the beams. Concentrate on understanding the large disruption case.

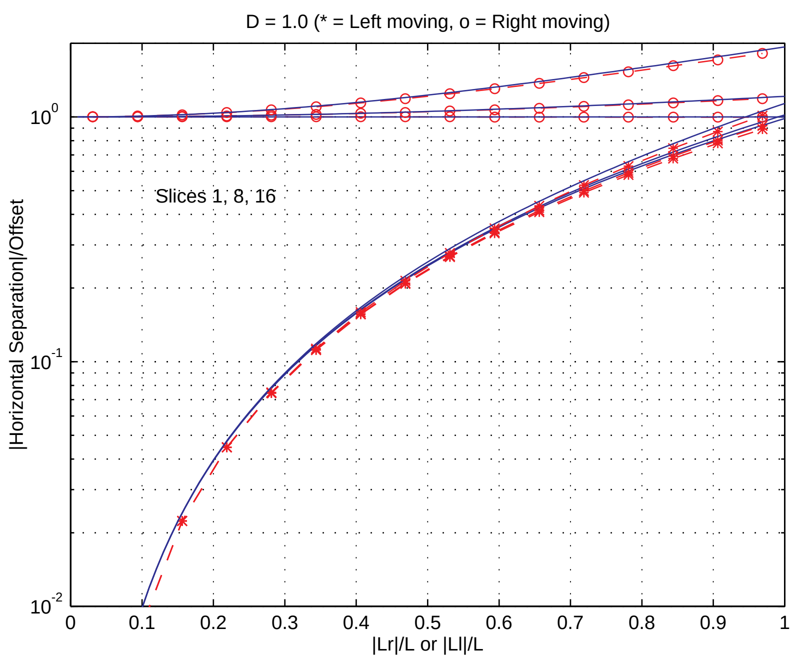

Figure 7: Comparison of simulation and analytical results given in eqs. (1) and (2) for D = 1 and slices 1, 8, and 16.

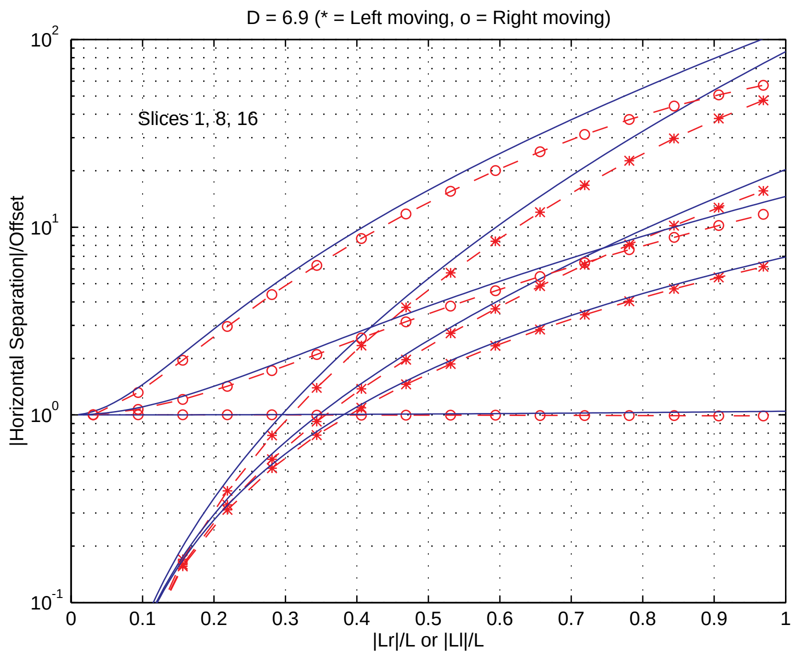

Figure 8: Comparison of simulation and analytical results given in eqs. (1) and (2) for D = 6.9 and slices 1, 8, and 16.

As the beams collide the like sign beams (1 & 4) and (2 & 3) repel each other and the opposite charge beams attract each other. For an initial offset of $2\Delta/\sigma_r = 0.05$ used in these simulations, the beams are substantially offset when the separation is ~ 20Δ. At this point the collision is not well neutralized. Particles will start to be outside the charge distribution of one of the beams, and the expression for the field in the beginning of section I.1 is no longer valid. Therefore, the calculation is no longer valid. The figure below shows that the offsets and changes in size are substantial.

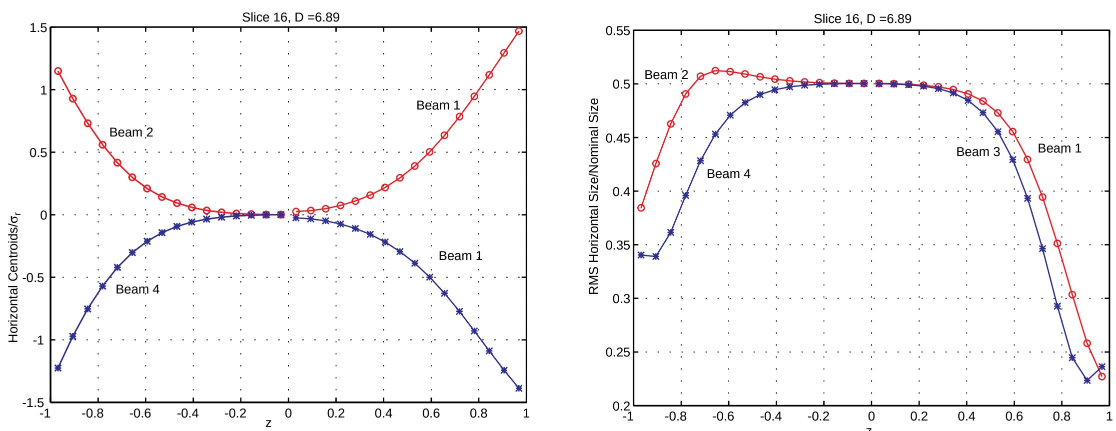

Figure 9: Horizontal offsets and sizes of slice 16 of individual beams for D = 6.9. Note that there is a factor of 2 between the RMS and nominal sizes.

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1 S. Krishnagopal and R. H. Siemann, Physical Review Letters **67**, 2461 (1991).

B. Podobedov and R. H. Siemann, Physical Review E, **52**, 3066 (1995).

2 The table below presents information about the maximum radial bin occupied during the simulation.

D	0.49	0.98	1.99	2.95	3.94	4.92	5.91	6.89
Max. Radial Bin	102	102	105	111	123	147	173	211