

Heat Diffusion

The heat diffusion equation is

$$\frac{\partial T}{\partial t} = \frac{P}{c_{\epsilon}} + \frac{k}{\rho c_{\epsilon}} \nabla^2 T$$

where $T \equiv$ temperature, $P \equiv$ power, $c_{\epsilon} \equiv$ specific heat, $\rho \equiv$ density, $k \equiv$ thermal conductivity. The Green's function calculated for $P = \delta(t')\delta(\mathbf{r}')$ is¹

$$G(\mathbf{r}, t) = \begin{cases} 0 & t < t' \\ \frac{1}{(4\pi k/\rho c_{\epsilon}(t-t'))^{3/2}} \exp\left(-\frac{(\mathbf{r}-\mathbf{r}')^2}{4k/\rho c_{\epsilon}(t-t')}\right) & t > t' \end{cases}$$

In one-dimension the Green's function is

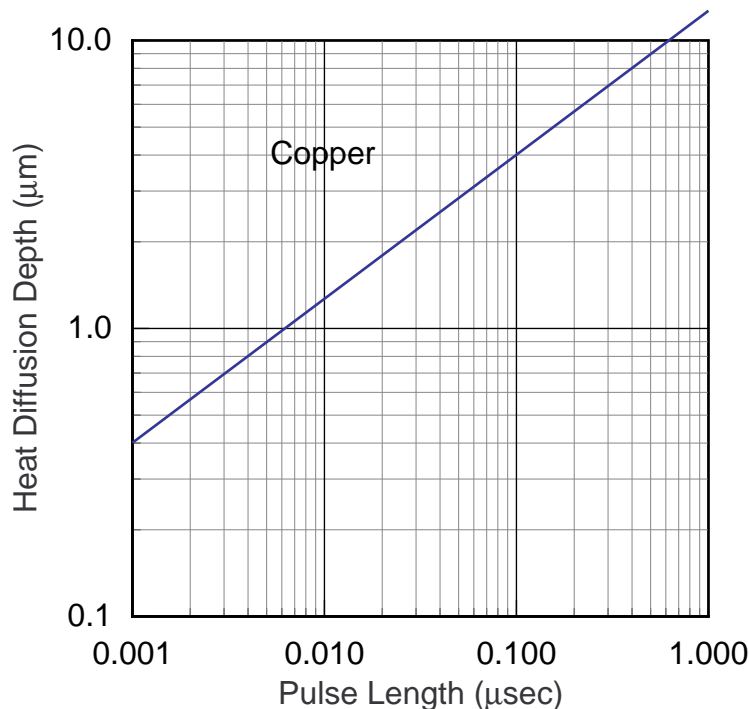
$$G(x, t) = \begin{cases} 0 & t < t' \\ \frac{1}{(4\pi k/\rho c_{\epsilon}(t-t'))^{1/2}} \exp\left(-\frac{(x-x')^2}{4k/\rho c_{\epsilon}(t-t')}\right) & t > t' \end{cases} \quad (1)$$

This is a Gaussian function with rms width

$$\sigma = \sqrt{2k/\rho c_{\epsilon}(t-t')}$$

that grows as the square root of time and is plotted in the figure.

Consider a one dimensional problem with a boundary at $x = 0$; vacuum when x is negative and metal when it is positive. The power per unit volume is



¹Morse and Feshbach, Methods Of Theoretical Physics, eq. (2.4.43)

$$\frac{dP}{dAdx'} = \frac{dP}{dA} \frac{\delta}{2} \exp(-2x'/\delta); \quad x' \geq 0 \quad (2a)$$

where dP/dA is the power per unit area and δ is the skin depth. If the surface is adiabatic (no heat flow into the vacuum), heat that diffuses in the negative x direction is reflected at the boundary. This can be accounted for with an image source or power that falls off exponentially from the surface

$$\frac{dP}{dAdx'} = \frac{dP}{dA} \frac{\delta}{2} \exp(2x'/\delta); \quad x' \leq 0 \quad (2b)$$

The Green's function in eq. (1) plus the power in equations (2) gives the temperature rise with time

$$T(x,t) = \frac{2}{\rho c_\epsilon \delta} \int_{-\infty}^t dt' \int_{-\infty}^{\infty} dx' \frac{dP(x',t')}{dAdx'} G(x-x', t-t')$$

Substituting the expressions for the power

$$T(x,t) = \frac{1}{\sqrt{\pi} \rho c_\epsilon \delta} \int_{-\infty}^t dt' \frac{1}{\sqrt{\beta}} \frac{dP(t')}{dA} \left\{ \int_{-\infty}^0 dx' \exp\left(\frac{2x'}{\delta} - \frac{(x-x')^2}{4\beta}\right) + \int_0^{\infty} dx' \exp\left(-\frac{2x'}{\delta} - \frac{(x-x')^2}{4\beta}\right) \right\}$$

where

$$\beta = \frac{k}{\rho c_\epsilon} (t-t')$$

For copper $k = 391$ W/m-K, $\rho = 8.95 \times 10^3$ kgm³, $c_\epsilon = 385$ J/kg-K and $\beta/(t-t') = 1.135 \times 10^{-4}$ m²/s. The x' integrals can be done² giving

$$T(x,t) = \frac{1}{\rho c_\epsilon \delta} \int_{-\infty}^t dt' e^{4\beta/\delta^2} \frac{dP(t')}{dA} \left\{ e^{-2x/\delta} \text{cerf}\left(\frac{2\sqrt{\beta}}{\delta} - \frac{x}{2\sqrt{\beta}}\right) + e^{2x/\delta} \text{cerf}\left(\frac{2\sqrt{\beta}}{\delta} + \frac{x}{2\sqrt{\beta}}\right) \right\} \quad (3)$$

where $\text{cerf}(z)$ is the complimentary error function. $\text{Cerf}(z)$ can be evaluated using the IMSL routines ERFC up to $z \sim 8$. This routine has underflow problems beyond that, and the exponentially scaled error function given by ERFCE should be used instead.

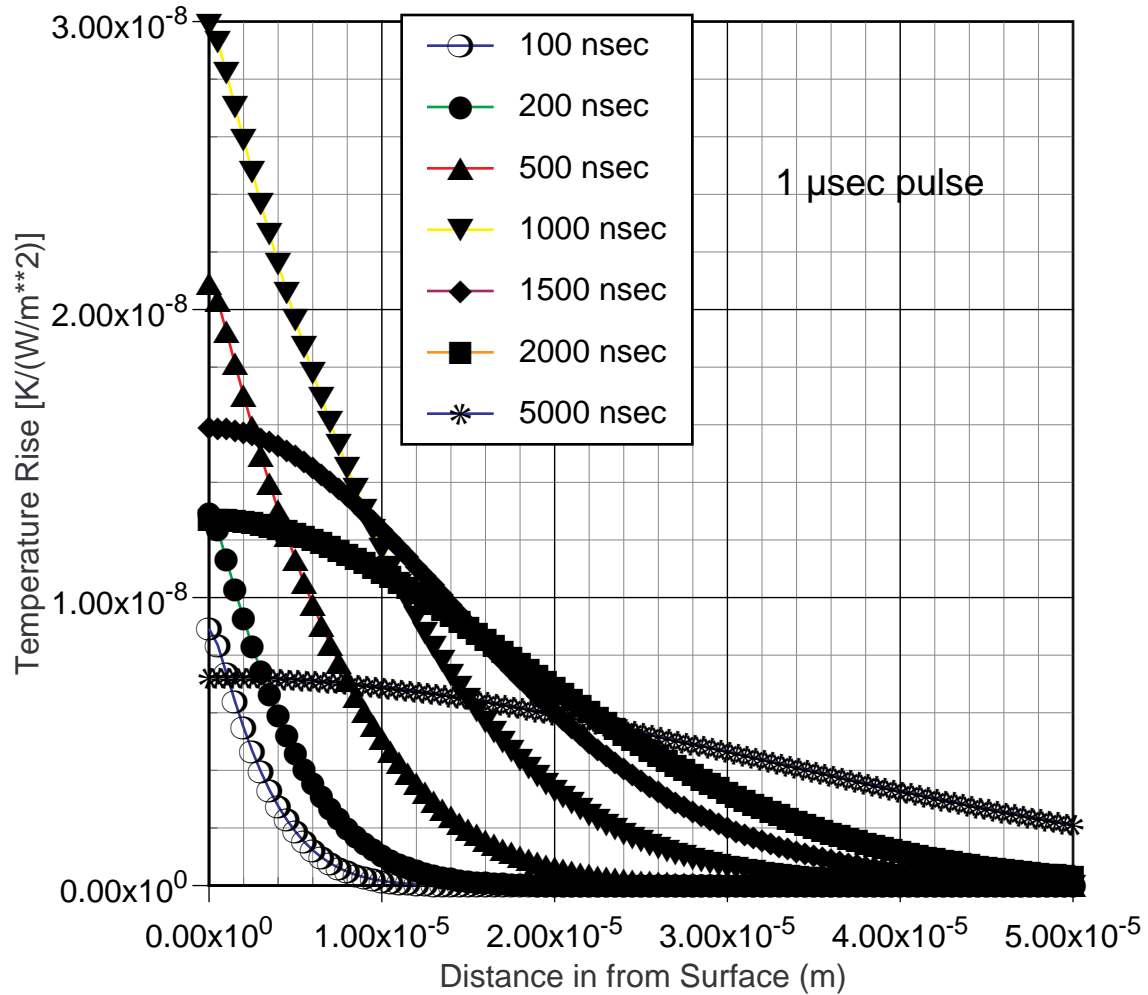
When the skin depth can be neglected, the limit of eq. (3) for $\delta \rightarrow 0$ is

$$T(x,t) = \frac{1}{\sqrt{\pi} \rho c_\epsilon} \int_{-\infty}^t \frac{dt'}{\sqrt{\beta}} e^{-x^2/4\beta} \frac{dP(t')}{dA} \quad (3')$$

² Gradshteyn and Ryzhik, eq. (3.322.2) or Abramowitz and Stegun, eq. 7.4.2)

SAMPLE CALCULATIONS

We are considering a pulsed heating experiment at X-band ($\delta = 0.66 \mu\text{m}$). Assuming a square pulse $1 \mu\text{sec}$ long the temperature evolution is given by the figure below.



The effects of pulse risetime and the appropriate coupling for maximum temperature rise in the pulsed heating experiment remains to be studied. This can be done using eq. (3).

The effect of skin depth at short pulse length is shown below for W-band ($\delta = 0.22 \mu\text{m}$) and a square RF pulse. The effects of skin resistance can be seen at 0.1 nsec when the diffusion length of $0.13 \mu\text{m}$ is less than the skin depth; the temperature rise is about one-half that without the skin effect. At 1.0 nsec, the diffusion depth is twice the skin depth and the temperature rises with and with skin effect are with 20%

