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Extension of Beam-Beam Calculations in SLAC-PUB-6073

I. INTRODUCTION

The calculation in section 4.2 of "The Beam-Beam Interaction in e^+e^- Storage Rings" (SLAC-PUB-6073) can be extended to solve a number of problems including multiple IR's, parasitic collisions, resonance locations, etc. This paper documents some of those extensions. Equation numbers that appear in places refer to equations in SLAC-PUB-6073. Several previous documents were combined in this one, and this leads to some repetition.

II. FORMALISM

Assume B collision points located at S_b, b = 1, ..., B. Each of these collision points has β-functions, nominal beam sizes, etc., and the beam is separated from the center of the other beam by D_{xb} and D_{yb} in the horizontal and vertical, respectively. The generalization of eq. (4.7) in SLAC-PUB-6073 is

$$V_{BB} = \frac{-Nr_e}{\gamma}\tilde{V}_{BB}$$

$$\tilde{V}_{BB} = \sqrt{\frac{2}{\pi\sigma_L^2}}\sum_{n=-\infty}^{\infty}\sum_{b=1}^{B} V_F(x,y,s)\times\exp\left\{-2(s-(nC+S_b+c\tau))^2/\sigma_L^2\right\}.$$

Index n is a sum over turns, and

$$\tau = \frac{\hat{\tau}}{2}\cos(2\pi Q_s s/C).$$

V_F is given by eq. (4.8)

$$V_F = \int_0^\infty \frac{dq}{\sqrt{(2\sigma_{xb}^2+q)(2\sigma_{yb}^2+q)}}\exp\left\{-\left[\frac{x^2}{2\sigma_{xb}^2+q}+\frac{y^2}{2\sigma_{yb}^2+q}\right]\right\}$$

Rewrite in terms of action-angle variables following eq. (4.11)

$$V_F \approx \int_0^\infty \frac{dq}{\sqrt{(2\sigma_{xb}^2+q)(2\sigma_{yb}^2+q)}}$$
$$\exp\left\{-\left[\frac{(D_x+\sqrt{2\beta_x I_x}\cos(\psi_x+\chi_x))^2}{2\sigma_{xb}^2+q}+\frac{(D_y+\sqrt{2\beta_y I_y}\cos(\psi_y+\chi_y))^2}{2\sigma_{yb}^2+q}\right]\right\}.$$

The ψ's are the phase advances in the smooth approximation, and the χ's are the (periodic) deviations from these smooth phase advances.

Fourier analyze the beam-beam potential

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$$V_{BB} = -\frac{Nr_e}{\gamma}\sum_{p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, A_{pr}(I_x, I_x, k)\exp\left\{i(p\psi_x + r\psi_y - ks)\right\}$$

where

$$A_{pr} = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\psi_x \int_0^{2\pi} d\psi_y \int_{-\infty}^{\infty} ds \exp\left\{-i(p\psi_x + r\psi_y - ks)\right\}\tilde{V}_{BB}$$

Angular integrals of the following form will appear and can be changed by making a change of variables $\theta = \psi + \chi$

$$\int_0^{2\pi} d\psi_x e^{-ip\psi_x} \exp\left\{-\left[\frac{(D_x + \sqrt{2\beta_x I_x}\cos(\psi_x + \chi_x))^2}{2\sigma_{xb}^2 + q}\right]\right\} =$$

$$e^{ip\chi_x}\int_0^{2\pi} d\theta_x e^{-ip\theta_x} \exp\left\{-\left[\frac{(D_x + \sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_{xb}^2 + q}\right]\right\}$$

If the approximation that the beam sizes vary slowly near the collision points is made, the only remaining s dependence in the transverse factors is in the χ's, and the equation for A_{pr} becomes

$$A_{pr} = \sum_{b=1}^{B} T_{pr}^{b}\frac{1}{2\pi}\int_{-\infty}^{\infty} ds \exp\left\{i(p\chi_x + r\chi_y + ks)\right\}\sqrt{\frac{2}{\pi\sigma_L^2}}\sum_{n=-\infty}^{\infty}\exp\left\{-2(s-(nC + S_b + c\tau))^2/\sigma_L^2\right\}$$

where

$$T_{pr}^{b} = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_{xb}^2 + q)(2\sigma_{yb}^2 + q)}}$$

$$\exp\left\{-\left[\frac{(D_x^b + \sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_{xb}^2 + q} + \frac{(D_y^b + \sqrt{2\beta_y I_y}\cos\theta_y)^2}{2\sigma_{yb}^2 + q}\right]\right\}$$

Note that the χ's have been taken out of the sum over turns since they are periodic in the circumference. They are given by eq (4.4)

$$\chi = \int_0^{s}\frac{d\zeta}{\beta(\zeta)} - Q_0\frac{2\pi s}{C}\ .$$

Near collision point b

$$\chi = \int_{S_b}^{s}\frac{d\zeta}{\beta(\zeta)} + \left\{\int_0^{S_b}\frac{d\zeta}{\beta(\zeta)} - Q_0\frac{2\pi s}{C}\right\} \approx \frac{s - S_b}{\beta_b} + \left\{\int_0^{S_b}\frac{d\zeta}{\beta(\zeta)} - Q_0\frac{2\pi S_b}{C}\right\} \approx \frac{s - S_b}{\beta_b} + 2\pi\Delta Q^b$$

$2\pi\Delta Q^b$ is the difference between the actual phase advance to S_b and the fraction of the total phase advance to that point. Doing the s integral

$$A_{pr} = \frac{1}{2\pi}\sum_{b=1}^{B} T_{pr}^{b} \exp\left\{2\pi i(p\Delta Q_x^b + r\Delta Q_y^b) - \left(k_{pr}^{b}\sigma_L\right)^2/8\right\}$$
$$\times \sum_{n=-\infty}^{\infty} \exp\left\{\frac{ik_{pr}^{b}\hat{\tau}c}{2}\cos(2\pi Q_s(n+S_b/C)) + ikC(n+S_b/C)\right\}$$

where

$$k_{pr}^{b} = k + p(1/\beta_{xb} - 2\pi Q_x/C) + r(1/\beta_{yb} - 2\pi Q_y/C)$$

Expanding the exponential with a Bessel function sum and using the Poisson sum rule gives

$$A_{pr} = \frac{1}{C}\sum_{b=1}^{B} T_{pr}^{b}(I_x, I_y)\exp\left\{2\pi i(p\Delta Q_x^b + r\Delta Q_y^b) - \left(k_{pr}^{b}\sigma_L\right)^2/8\right\}$$
$$\times \sum_{m,n=-\infty}^{\infty} i^m J_m(k_{pr}^{b}\hat{\tau}c/2)\exp\{i(kC + m2\pi Q_s)S_b/C\}\delta(kC - 2\pi(n - mQ_s))$$

This expression can be substituted back into the expression for the beam-beam potential, and, after doing the k integral, the beam-beam potential is

$$V_{bb} = -\frac{Nr_e}{C\gamma}\sum_{b=1}^{B}\sum_{m,n,p,r=-\infty}^{\infty} T_{pr}^{b}(I_x, I_y)\exp\left\{2\pi i(p\Delta Q_x^b + r\Delta Q_y^b) - \left(k_{pr}^{b}\sigma_L\right)^2/8\right\}$$
$$\times i^m J_m(k_{pr}^{b}\hat{\tau}c/2)\exp\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C) + i2\pi nS_b/C\}$$

where

$$k_{pr}^{b} = 2\pi(n - mQ_s)/C + p(1/\beta_{xb}^{*} - 2\pi Q_{x0}/C) + r(1/\beta_{yb}^{*} - 2\pi Q_{y0}/C)$$

and

$$T_{pr}^{b} = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_{xb}^2 + q)(2\sigma_{yb}^2 + q)}}$$
$$\exp\left\{-\left[\frac{(D_x^b + \sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_{xb}^2 + q} + \frac{(D_y^b + \sqrt{2\beta_y I_y}\cos\theta_y)^2}{2\sigma_{yb}^2 + q}\right]\right\}$$

Tune shifts and tune as a function of amplitude depend on the average value of the potential which is given by putting p = r = m = n = 0 in the equation above

$$\langle V_{BB}\rangle = -\frac{Nr_e}{C\gamma}\sum_{b=1}^{B} T_{00}^{b}(I_x, I_y)$$

$$= -\frac{Nr_e}{(2\pi)^2 C\gamma}\sum_{b=1}^{B}\int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^{\infty} \frac{dq}{\sqrt{(2\sigma_{xb}^2 + q)(2\sigma_{yb}^2 + q)}}$$

$$\exp\left\{-\left[\frac{(D_x^b + \sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_{xb}^2 + q} + \frac{(D_y^b + \sqrt{2\beta_y I_y}\cos\theta_y)^2}{2\sigma_{yb}^2 + q}\right]\right\}$$

Tune shifts are given by

$$\Delta Q_y = \frac{C}{2\pi}\frac{\partial\langle V_{BB}\rangle}{\partial I_y};\ \ \Delta Q_x = \frac{C}{2\pi}\frac{\partial\langle V_{BB}\rangle}{\partial I_x}$$

The interaction regions do not interfere when it comes to calculating total tune shift. The procedure is to calculate the tune shift for each interaction point as a function of $\{I_x, I_y\}$ and then add them to get the total tune shift at those actions. The phase advances between interaction points and the relative signs of offsets do not enter since a change of variables $\theta \to \theta + \pi$ will eliminate any sign of D. A calculation showing this for an example is in the next section. Phase shifts and signs of separations will be important for calculating resonance strengths. This is discussed in a section related to resonance strengths at LEP.

III. SINGLE AND MULTIPLE COLLISION POINTS

Start with the expression for the beam-beam potential at a single collision point.

$$\langle V_{BB}\rangle = -\frac{Nr_e}{(2\pi)^2 C\gamma}\int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)}}$$
$$\exp\left\{-\left[\frac{(D_x+\sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_x^2+q}+\frac{(D_y+\sqrt{2\beta_y I_y}\cos\theta_y)^2}{2\sigma_y^2+q}\right]\right\}$$

Taking the derivative with respect to I_y, making a change of variables

$$\eta = \frac{2\sigma_y^2}{2\sigma_y^2+q};\quad dq = \frac{-2\sigma_y^2 d\eta}{\eta^2};\quad 2\sigma_x^2+q = \frac{2\sigma_x^2}{\eta}\left(\eta + R^2(1-\eta)\right)$$

normalizing to the tune shift, and writing I_x and I_y in terms of actions normalized to emittances gives

$$\Delta Q_y = \frac{\xi_y(\sigma_x+\sigma_y)\sqrt{2/J_y}}{2(2\pi)^2\sigma_x}\int_0^{2\pi} d\theta_x \int_0^{2\pi}\cos\theta_y (D_y/\sigma_y+\sqrt{2J_y}\cos\theta_y)d\theta_y \int_0^1 \frac{d\eta}{\sqrt{(\eta+R^2(1-\eta))}}$$
$$\exp\left\{-\eta\left[\frac{(D_x/\sigma_x+\sqrt{2J_x}\cos\theta_x)^2}{2(\eta+R^2(1-\eta))}+\frac{(D_y/\sigma_y+\sqrt{2J_y}\cos\theta_y)^2}{2}\right]\right\}$$

Note that this expression does not have a divergence as $J_y \to 0$. Interchange the orders of integration to do the η integral last

$$\Delta Q_y = \frac{\xi_y(\sigma_x+\sigma_y)}{2\sigma_x}\int_0^1 \frac{d\eta}{\sqrt{(\eta+R^2(1-\eta))}} F_0(\Gamma_x, M_x)\left\{\sqrt{\frac{2}{J_y}}\frac{D_y}{\sigma_y}F_1(\Gamma_y, M_y) + 2F_2(\Gamma_y, M_y)\right\}$$

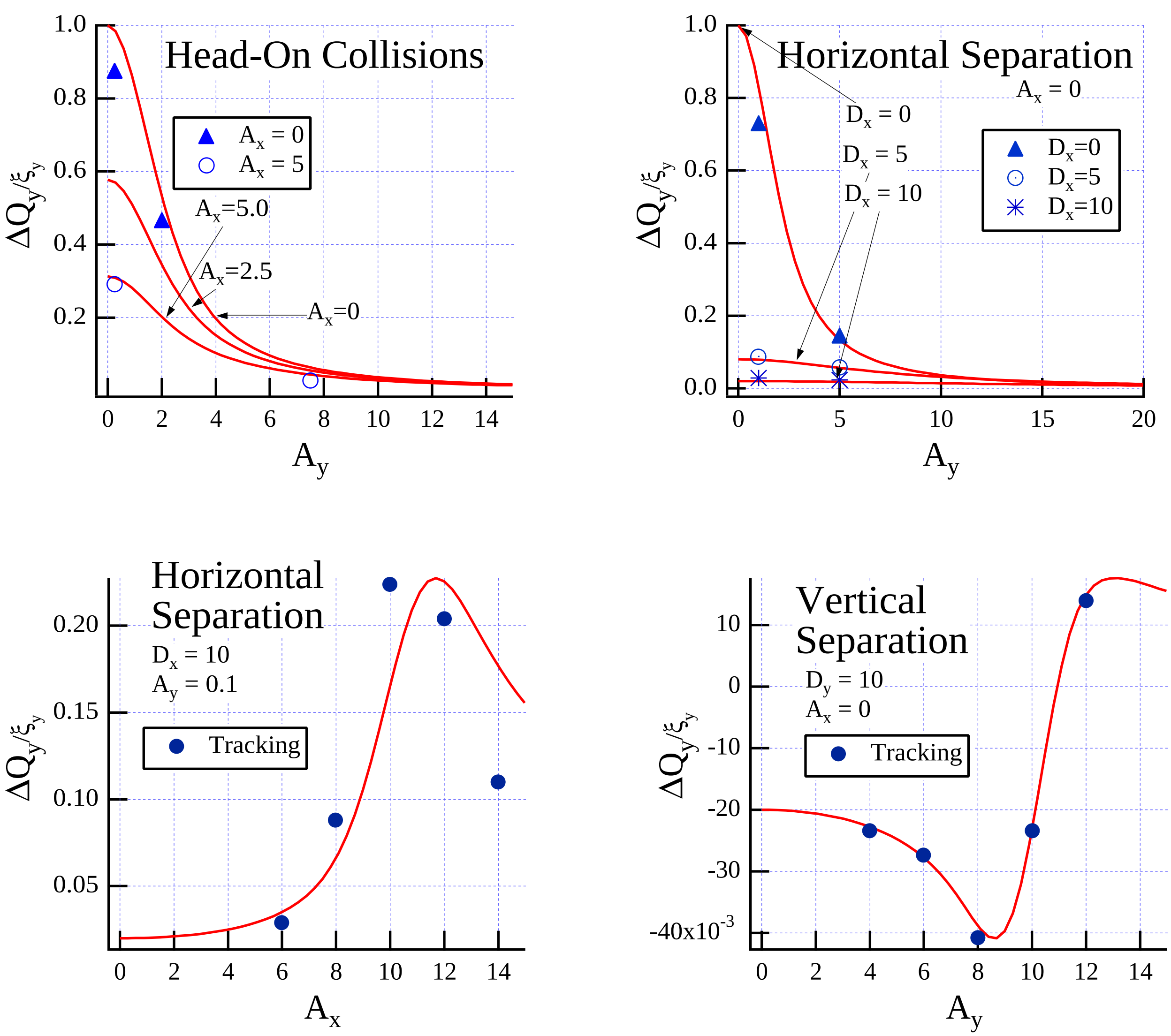
where F_0, F_1, and F_2 are given in the appendix. The arguments are ($A_{x,y} = (2J_{x,y})^{1/2}$ are the betatron amplitudes normalized to beam sizes)

$$\Gamma_x^2 = \left(\frac{D_x}{\sigma_x}\right)^2 \frac{\eta}{2(\eta+(1-\eta)R^2)};\quad M_x = \frac{\sqrt{2J_x}}{D_x/\sigma_x} = \frac{A_x}{D_x/\sigma_x}$$
$$\Gamma_y^2 = \left(\frac{D_y}{\sigma_y}\right)^2 \frac{\eta}{2};\quad M_y = \frac{\sqrt{2J_y}}{D_y/\sigma_y} = \frac{A_y}{D_y/\sigma_y}$$

There are some limits where this general expression cannot be used:

$D_y = 0$	$F_1 = 0;\quad F_2 = I_0^e(B_y/2) - I_1^e(B_y/2);\quad B_y = \eta J_y$
$D_y \neq 0,\ J_y = 0$	$\sqrt{\frac{2}{J_y}}\,\frac{D_y}{\sigma_y}F_1(\Gamma_y, M_y) = -\eta\left(\frac{D_y}{\sigma_y}\right)^2 e^{-\Gamma_y^2}$
$D_x = 0$	$F_0 = I_0^e\left(\frac{B_x}{2}\right);\quad B_x(\eta) = \eta\frac{J_x}{\eta + R^2(1-\eta)}$

This expression has been tested by tracking a single particle in the weak-strong approximation for a set of parameters that approximated a PEP-II parasitic crossing: $\sigma_x = 0.3$ mm, $\sigma y = 0.3$ mm, $\beta_x = 1.0$ m, $\beta y = 25.$ m, $\xi_x = 0.0034$, $\xi_y = 0.0857$. The plots below show the comparison.



The calculations of tune shift are available in the program **PARASITIC** which must be linked with **PARASITIC_FUNCTIONS** and **FM_FUNCTIONS** to run.

The horizontal tune shift is given by the same expressions with x and y interchanged (R becomes 1/R when x and y are interchanged.)

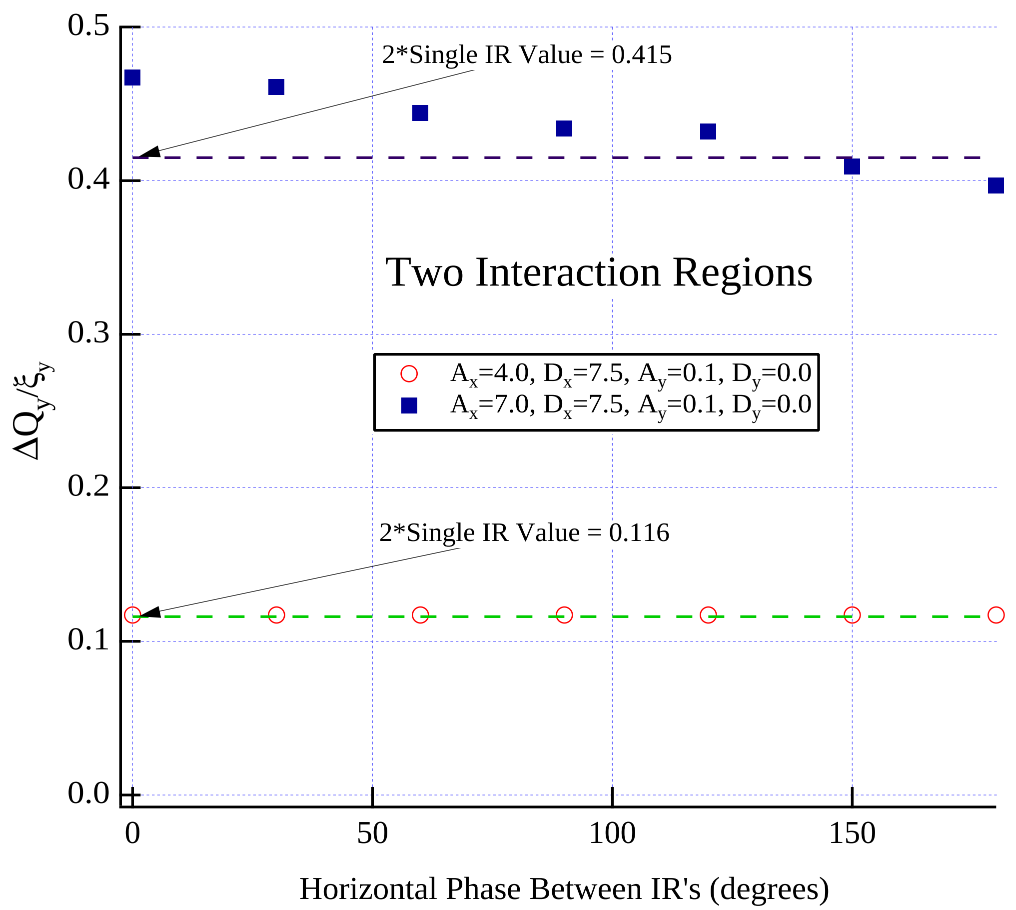
$$\Delta Q_x = \frac{\xi_x(\sigma_x+\sigma_y)}{2\sigma_y}\int_0^1 \frac{d\eta}{\sqrt{(\eta+(1-\eta)/R^2)}} F_0(\Gamma_y, M_y)\left\{\sqrt{\frac{2}{J_x}}\frac{D_x}{\sigma_x}F_1(\Gamma_x, M_x)+2F_2(\Gamma_x, M_x)\right\}$$

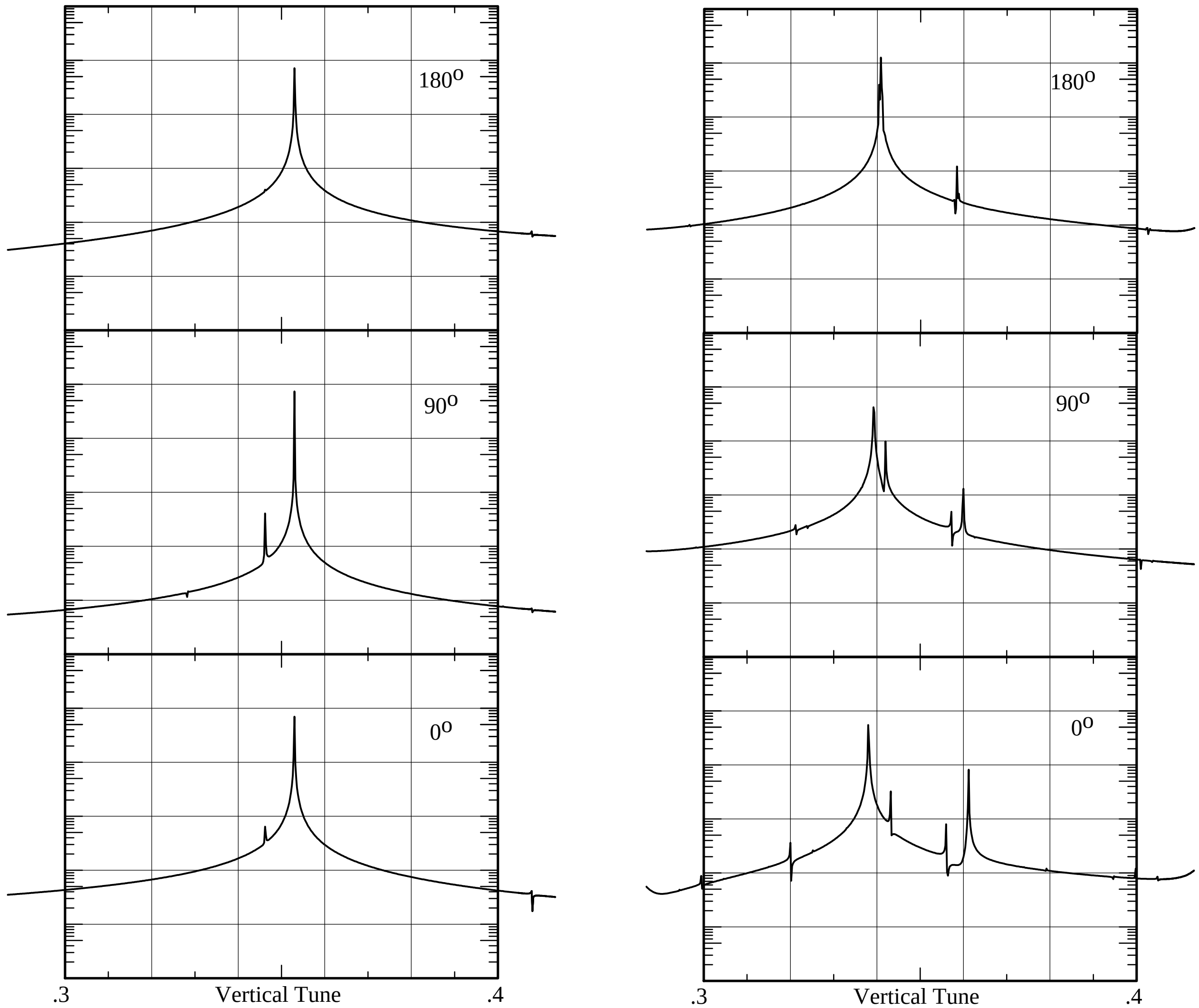
The arguments of the functions are

$$\Gamma_x^2 = \left(\frac{D_x}{\sigma_x}\right)^2 \frac{\eta}{2}; \quad M_x = \frac{\sqrt{2J_x}}{D_x/\sigma_x} = \frac{A_x}{D_x/\sigma_x}$$

$$\Gamma_y^2 = \left(\frac{D_y}{\sigma_y}\right)^2 \frac{\eta}{2(\eta+(1-\eta)/R^2)}; \quad M_y = \frac{\sqrt{2J_y}}{D_y/\sigma_y} = \frac{A_y}{D_y/\sigma_y}$$

One of the results from the first section is that the tune shifts from individual collision points can be added to give the total tune shift without accounting for the phase advance between the collision points. This was tested with two IR's for the conditions above by tracking with different phase advances between IR's. The results are below





Vertical spectra for the two cases above. The plots on the left are for A_X = 4.0 and the ones on the right are for A_X = 7.0. The nominal tune is 0.641 = 0.359when reflected about the half-integer. The beam-beam interaction shifts tunes to the left.

The figure above shows the spectra for the two cases tracked for different values of phase shift between the interaction points. The spectra have some structure in addition to the main tune peak, but that peak can be clearly distinguished and its value measured.

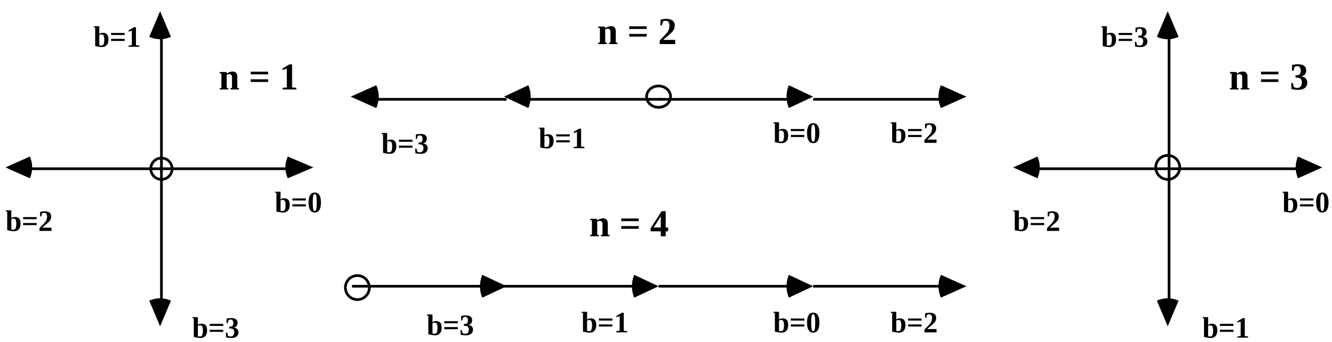
IV. MULTIPLE IR's

This summarizes the connection between single IR and multiple IR colliders and the effects of phase advance errors and differences in β-functions. A draft of this note was written earlier and used slightly different notation: i) the sum over collision points extends from 0 to B_{IR}-1; ii) the IR's are assume equally spaced around the circumference of the ring, and iii) nominally there are equal phase advances between IR's. This analysis has no offsets at the collision points. Much of this work was published at PAC95 in *The Effect Of Phase Advance Errors Between Interaction Points On Beam Halos* by T. Chen, J. Irwin, and R. Siemann.

CASE 1 - The IR's are identical and there are no phase advance errors.

$$H = H_0(I_x,I_y) - \frac{Nr_e}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty} T_{pr}(I_x,I_y)\exp\left\{-\left(k_{pr}\sigma_L\right)^2/8\right\} i^m J_m(k_{pr}\hat{\tau}c/2)$$
$$\times \exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right\}\sum_{b=0}^{B_{IR}-1}\exp\left\{2\pi ibn/B_{IR}\right\}$$

The sum over b equals zero unless n is a multiple of B_{IR}. This is illustrated for B_{IR} = 4 below.



If n is a multiple of B_{IR}, the sum equals B_{IR}.

$$H = H_0(I_x,I_y) - \frac{Nr_eB_{IR}}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty} T_{pr}(I_x,I_y)\exp\left\{-\left(k_{pr}\sigma_L\right)^2/8\right\} i^m J_m(k_{pr}\hat{\tau}c/2)$$
$$\times \exp\left\{i(p\psi_x + r\psi_y - 2\pi(nB_{IR} - mQ_s)s/C)\right\}$$

The resonance conditions are

$$\frac{d}{ds}\left(p\psi_x + r\psi_y - 2\pi(nB_{IR} - mQ_s)\frac{s}{C}\right) = 0 \qquad \textit{ALLOWED RESONANCES}$$
$$pQ_x + rQ_y + mQ_s = nB_{IR}$$

The resonances that appear in this case are those allowed by symmetry. When a single IR collider with horizontal tune Q_x / B_{IR}, etc. is modeled, these allowed resonances satisfy

$$p\left(\frac{Q_x}{B_{IR}}\right) + r\left(\frac{Q_y}{B_{IR}}\right) + m\left(\frac{Q_s}{B_{IR}}\right) = n \qquad \textit{ALLOWED RESONANCES}$$

The indices p and r are even because of the symmetry of T^b_{pr}.

CASE 2 - The average value of Hamiltonian
In this case n = m = p = r = 0. Then independent of phase advance errors

$$H = H_0(I_x, I_y) - \frac{Nr_e}{C\gamma}\sum_{b=0}^{B_{IR}-1} T^b_{00}(I_x, I_y)$$

The average Hamiltonian is equal to the unperturbed Hamiltonian with the perturbation from each IR added. If the IR's are identical

$$H = H_0(I_x, I_y) - \frac{Nr_e B_{IR}}{C\gamma} T_{00}(I_x, I_y)$$

and the tune shifts with amplitude are B_{IR} times the tune shift from a single IR.

CASE 3 - Small Errors
Consider small phase advance and β errors.

$$\begin{aligned}
H = {} & H_0(I_x, I_y) \\
& - \frac{Nr_e B_{IR}}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty} T_{pr}(I_x, I_y)\exp\left\{-\left(k_{pr}\sigma_L\right)^2/8\right\} i^m J_m(k_{pr}\hat{\tau}c/2) \\
& \times \exp\left\{i(p\psi_x + r\psi_y - 2\pi(nB_{IR} - mQ_s)s/C)\right\} \\
& - \frac{Nr_e}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty} T_{pr}(I_x, I_y)\exp\left\{-\left(k_{pr}\sigma_L\right)^2/8\right\} i^m J_m(k_{pr}\hat{\tau}c/2) \\
& \times \exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right\}\sum_{b=0}^{B_{IR}-1}\exp\left\{2\pi i(bn/B_{IR} + p\Delta Q^b_x + r\Delta Q^b_y)\right\} \\
& - \frac{Nr_e}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty}\exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C + m\pi/2)\right\} \\
& \sum_{b=0}^{B_{IR}-1}\exp(2\pi ibn/B_{IR})\Delta\vec{\beta}_b \bullet \vec{\nabla}_\beta\left(T^b_{pr}(I_x, I_y)\exp\left\{-\left(k^b_{pr}\sigma_L\right)^2/8\right\} J_m(k^b_{pr}\hat{\tau}c/2)\right)
\end{aligned}$$

The first term gives the allowed resonances while the second and third terms give the resonances from unequal phase advances and unequal β's respectively. These resonances arise from an imperfect cancellation of the phasors in the figure above. When there are phase advance errors, the phasors don't point in exactly the right directions, and when there are β-function errors, the phasors are not all the same length. In all cases indices p and r remain even because of the symmetry of T^b_{pr} and its derivatives with respect to the β functions. The resonance condition for these error driven resonances is

$$\frac{d}{ds}\left(p\psi_x + r\psi_y - 2\pi(n - mQ_s)\frac{s}{C}\right) = 0$$
$$pQ_x + rQ_y + mQ_s = n$$

ERROR DRIVEN RESONANCES

In terms of the single IR model for a multiple IR collider, these resonances are

$$p\left(\frac{Q_x}{B_{IR}}\right) + r\left(\frac{Q_y}{B_{IR}}\right) + m\left(\frac{Q_s}{B_{IR}}\right) = \frac{n}{B_{IR}}$$

ERROR DRIVEN RESONANCES

The resonance order is $|p| + |r| + |m|$.

V. RESB

RESB is a Beam-Beam resonance program with this formalism applied to PEP-II. It is a second generation following on the program **RESA**. The main difference is that **RESA** used FFT routines to do the angle integrals. These integrals depended on the number of bins in the FFT with the consequence that resonance locations were dependent on numerical integration parameters.

Definitions

Symbol	*Definition*
$\xi_y = \frac{r_e}{2\pi} \frac{N\beta_y^*}{\gamma\sigma_{y0}(\sigma_{x0}+\sigma_{y0})}$ $\xi_x = \frac{r_e}{2\pi} \frac{N\beta_x^*}{\gamma\sigma_{x0}(\sigma_{x0}+\sigma_{y0})}$	Beam-Beam Tune Shifts
ε_x, ε_y	Weak Beam Emittances
I_x, I_y	Weak Beam Actions in Units of meters
$J = I/\varepsilon$	Weak Beam Actions Normalized to Emittance
$A = \sqrt{2I/\varepsilon} = \sqrt{2J}$	Weak Beam Betatron Amplitude Normalized to Nominal Weak Beam Size
HEAD ON COLLISIONS	
σ_{x0}, σ_{y0}, $R = \sigma_{y0}/\sigma_{x0}$	Strong Beam Sizes and Size Ratio
β_x^*, β_y^*	Weak Beam β's at Collision Point
σ_{xw}, σ_{yw}	Weak Beam Nominal Sizes $\sigma_{xw} = \sqrt{\varepsilon_x\beta_x^*}$
PARASITIC COLLISIONS	
$\pm S$	Distance from Collision Point to Parasitic Collisions
D	Horizontal Separation at Parasitic Collisions
σ_{xp}, σ_{yp}, $R_p = \sigma_{yp}/\sigma_{xp}$	Strong Beam Sizes and Size Ratio and Parasitic Crossing Point
β_x^p, β_y^p	Weak Beam β's at Parasitic Crossing Point
σ_{xwp}, σ_{ywp}	Weak Beam Nominal Sizes $\sigma_{xwp} = \sqrt{\varepsilon_x\beta_x^p}$
ξ_{yp}, ξ_{xp}	Parasitic Tune Shifts - Equations Above with Parasitic Crossing Parameters
DETUNING & RESONANCE WIDTHS	
Λ_{pr}	Proportional To Rate Of Change Of Tune With Action, eq. (4.32)
T_{pr}, F_{prm}	Fourier expansion coefficients for the resonance $pQ_x + rQ_y + mQ_s = n$, eqs. (4.15), (4.28)

Results

Equation	**Description**
1	Vertical Tune Shift from Head-On Collisions
2	Horizontal Tune Shift from Head-On Collisions
3	Vertical Tune Shift from Parasitic Collisions
4	Horizontal Tune Shift from Parasitic Collisions
5, 6	Tune Shifts from Lattice Nonlinearities
7	Resonance detuning
8	Resonance width
9	Resonance Tune

Head-on Beam-Beam Tune Shifts (Subroutine YTSHFT, XTSHFT, Functions GY2, GX2)

There is a single IR with $D_x = D_y = 0$. Take the derivative with respect to I_y

$$\Delta Q_y\Big|_{HO} = \frac{2Nr_e\beta_y^*}{\gamma(2\pi)^3}\int_0^{2\pi} d\theta_y \cos^2\theta_y \int_0^{2\pi} d\theta_x \int_0^{\infty} \frac{dq}{\sqrt{(2\sigma_{x0}^2+q)(2\sigma_{y0}^2+q)^3}}$$
$$\exp\left\{-\left[\frac{2I_y\beta_y^*\cos^2\theta_y}{(2\sigma_{y0}^2+q)} + \frac{2I_x\beta_x^*\cos^2\theta_x}{(2\sigma_{x0}^2+q)}\right]\right\}$$

Make a change of variables

$$\eta = \frac{2\sigma_{y0}^2}{2\sigma_{y0}^2+q}; \quad dq = \frac{-2\sigma_{y0}^2 d\eta}{\eta^2}; \quad 2\sigma_{x0}^2 + q = \frac{2\sigma_{x0}^2}{\eta}\left(\eta + R^2(1-\eta)\right),$$

normalize to ξ_y, and write I_x and I_y in terms of the actions normalized to emittances to get

$$\frac{\Delta Q_y\Big|_{HO}}{\xi_y} = \frac{\sigma_{x0}+\sigma_{y0}}{(2\pi)^2\sigma_{x0}}\int_0^{2\pi} d\theta_y \cos^2\theta_y \int_0^{2\pi} d\theta_x \int_0^{1} \frac{d\eta}{\sqrt{\eta + R^2(1-\eta)}}$$
$$\exp\left\{-\eta\left[J_y\cos^2\theta_y \frac{\sigma_{yw}^2}{\sigma_{y0}^2} + J_x\cos^2\theta_x \frac{\sigma_{xw}^2}{\sigma_{x0}^2(\eta + R^2(1-\eta))}\right]\right\}$$

Interchange the order of integration and use the results in the appendix for the θ_x and θ_y integrals

$$\frac{\Delta Q_y\Big|_{HO}}{\xi_y} = \frac{\sigma_{x0}+\sigma_{y0}}{2\sigma_{x0}}\int_0^{1} \frac{d\eta}{\sqrt{\eta + R^2(1-\eta)}} I_0^e\left(\frac{B_x}{2}\right)\left\{I_0^e\left(\frac{B_y}{2}\right) - I_1^e\left(\frac{B_y}{2}\right)\right\}$$
$$B_x(\eta) = \eta\frac{J_x\sigma_{xw}^2/\sigma_{x0}^2}{\eta + R^2(1-\eta)}; \; B_y(\eta) = \eta J_y \sigma_{yw}^2/\sigma_{y0}^2 \qquad (1)$$

The η integral is done numerically. The tune shift is calculated in subroutine **YTSHFT**, and the argument of the numerical integral is function **GY2**. The only difference between this result and that in an earlier section is the square of the beam size ratios appearing in $B_{x,y}$.

For the horizontal tune shift the result is the same as above with x and y interchanged everywhere. Note that in doing this R becomes 1/R.

$$\frac{\Delta Q_x\Big|_{HO}}{\xi_x} = \frac{\sigma_{x0}+\sigma_{y0}}{2\sigma_{y0}}\int_0^{1} \frac{d\eta}{\sqrt{\eta + (1-\eta)/R^2}} I_0^e\left(\frac{B_y}{2}\right)\left\{I_0^e\left(\frac{B_x}{2}\right) - I_1^e\left(\frac{B_x}{2}\right)\right\}$$
$$B_y(\eta) = \eta\frac{J_y\,\sigma_{yw}^2/\sigma_y^2}{\eta + (1-\eta)/R^2}; \; B_x(\eta) = \eta J_x\, \sigma_{xw}^2/\sigma_x^2 \qquad (2)$$

The η integral is done numerically. The tune shift is calculated in subroutine **XTSHFT**, and the argument of the numerical integral is function **GX2**.

PARASITIC COLLISIONS

There are two parasitic collision points with horizontal separation. These points are at ±S from the collision point, and the separation of the beams is D_x = +D at one point and D_x = - D at the other. As argued earlier, the relative sign does not matter, and the total parasitic tune shift is equal to twice that from a single parasitic crossing point. There is no vertical separation, $D_y = 0$. Bunch length is assumed short compared to β's at the parasitic crossing point.

Following results derived earlier and taking the beam size ratio into account, the total parasitic tune shift, normalized to the head-on tune shift, is

$$\frac{\Delta Q_y|_P}{\xi_y} = \frac{2\xi_{yp}}{\xi_y}\frac{(\sigma_{xp}+\sigma_{yp})}{2\sigma_{xp}}\int_0^1 \frac{d\eta}{\sqrt{(\eta + R_p^2(1-\eta))}} F_0(\Gamma_x, M_x)\left\{I_0^e(B_y/2) - I_1^e(B_y/2)\right\} \tag{3}$$

where F_0 is given in the appendix. The arguments are

$$\Gamma_x^2 = \left(\frac{D}{\sigma_{xp}}\right)^2 \frac{\eta}{2(\eta + (1-\eta)R_p^2)}; \quad M_x = \frac{\sqrt{2J_x}\,\sigma_{xpw}/\sigma_{xp}}{D/\sigma_{xp}}$$

$$B_y = \eta J_y \frac{\sigma_{ypw}^2}{\sigma_{yp}^2}$$

The integral is performed numerically. **GY2P** is the function giving the integrand.

The horizontal tune shift from parasitic crossings also follows from previous results. It is

$$\frac{\Delta Q_x|_P}{\xi_x} = \frac{2\xi_{xp}}{\xi_x}\frac{(\sigma_{xp}+\sigma_{yp})}{2\sigma_{yp}}\int_0^1 \frac{d\eta}{\sqrt{(\eta + (1-\eta)/R_p^2)}} I_0^e(B_y\ \ 2)\left\{\sqrt{\frac{2}{J_x}}\frac{D/\sigma_{xp}}{\sigma_{xpw}/\sigma_{xp}} F_1(\Gamma_x, M_x) + 2F_2(\Gamma_x, M_x)\right\} \tag{4}$$

where

$$\Gamma_x^2 = \left(\frac{D}{\sigma_{xp}}\right)^2 \frac{\eta}{2}; \quad M_x = \frac{\sqrt{2J_x}\,\sigma_{xpw}/\sigma_{xp}}{D/\sigma_{xp}}$$

$$B_y = \frac{\eta J_y}{(\eta + (1-\eta)/R_p^2)}\frac{\sigma_{ypw}^2}{\sigma_{yp}^2}$$

The integral is performed numerically. **GX2P** is the function giving the integrand.

LATTICE NONLINEARITIES

The tune is expressed as a Taylor series

$$Q_y = Q_{y0} + I_y \frac{\partial Q_y}{\partial I_y} + I_x \frac{\partial Q_y}{\partial Ix} + I_y I_x \frac{\partial^2 Q_y}{\partial I_y \partial I_x}$$

Normalizing to ξ_y and writing in terms of normalized action

$$\frac{\Delta Q_y\big|_{NL}}{\xi_y} = \frac{1}{\xi_y}\left\{J_y \varepsilon_y \frac{\partial Q_y}{\partial I_y} + J_x \varepsilon_x \frac{\partial Q_y}{\partial I_x} + J_y J_x \varepsilon_y \varepsilon_x \frac{\partial^2 Q_y}{\partial I_y \partial I_x}\right\} \tag{5}$$

Following the same argument as for the vertical

$$\frac{\Delta Q_x\big|_{NL}}{\xi_x} = \frac{1}{\xi_x}\left\{J_x \varepsilon_x \frac{\partial Q_x}{\partial I_x} + J_y \varepsilon_y \frac{\partial Q_x}{\partial I_y} + J_x J_y \varepsilon_x \varepsilon_y \frac{\partial^2 Q_x}{\partial I_y \partial I_x}\right\} \tag{6}$$

DETUNING & RESONANCE WIDTHS
Subroutines DTJX2, DTJY2, DTJXJY

Tune shift with action for resonance $pQ_x + rQ_y + mQ_s = n$ is proportional to (eq. (4.32))

$$\Lambda_{pr} = p^2 \frac{\partial^2 T_{00}}{\partial I_x^2} + r^2 \frac{\partial^2 T_{00}}{\partial I_y^2} + 2pr \frac{\partial^2 T_{00}}{\partial I_x \partial I_y}\Bigg|_{I_{xR}, I_{yR}}$$

Changing to normalized actions

$$\Lambda_{pr} = \frac{p^2}{\varepsilon_x^2}\frac{\partial^2 T_{00}}{\partial J_x^2} + \frac{r^2}{\varepsilon_y^2}\frac{\partial^2 T_{00}}{\partial J_y^2} + 2\frac{pr}{\varepsilon_x \varepsilon_y}\frac{\partial^2 T_{00}}{\partial J_x \partial J_y}\Bigg|_{J_{xR}, J_{yR}} \tag{7a}$$

Begin by rewiting the general expression for T_{pr} with the change of variables used above

$$T_{pr} = \frac{R}{(2\pi)^2}\int_0^{2\pi} e^{ip\theta_x} d\theta_x \int_0^{2\pi} e^{ir\theta_y} d\theta_y \int_0^1 \frac{d\eta}{\eta\sqrt{(\eta + R^2(1-\eta))}}$$
$$\exp\left\{-\eta\left[\frac{(D_x + \sqrt{2J_x}\sigma_{xw}\cos\theta_x)^2}{2\sigma_x^2(\eta + R^2(1-\eta))} + \frac{(D_y + \sqrt{2J_y}\sigma_{yw}\cos\theta_y)^2}{2\sigma_y^2}\right]\right\}$$

For the head-on collisions

$$T_{00} = \frac{R}{(2\pi)^2}\int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^1 \frac{d\eta}{\eta\sqrt{(\eta + R^2(1-\eta))}} \exp\left\{-\eta\left[\frac{J_x\sigma_{xw}^2\cos^2\theta_x}{\sigma_{x0}^2(\eta + R^2(1-\eta))} + \frac{J_y\sigma_{yw}^2\cos^2\theta_y}{\sigma_{y0}^2}\right]\right\}$$

and

$$\left.\frac{\partial^2 T_{00}}{\partial J_y^2}\right|_{HO} = \frac{R}{(2\pi)^2}\frac{\sigma_{yw}^4}{\sigma_{y0}^4}\int_0^1 \frac{\eta d\eta}{\sqrt{(\eta+R^2(1-\eta))}}\int_0^{2\pi} d\theta_x \int_0^{2\pi}\cos^4\theta_y d\theta_y$$
$$\exp\left\{-\eta\left[\frac{J_x\sigma_{xw}^2\cos^2\theta_x}{\sigma_{x0}^2(\eta+R^2(1-\eta))}+\frac{J_y\sigma_{yw}^2\cos^2\theta_y}{\sigma_{y0}^2}\right]\right\}$$
$$= \frac{R}{8}\frac{\sigma_{yw}^4}{\sigma_{y0}^4}\int_0^1 \frac{\eta d\eta}{\sqrt{(\eta+R^2(1-\eta))}} I_0^e(B_x/2)\left(3I_0^e(B_y/2)-4I_1^e(B_y/2)+I_2^e(B_y/2)\right) \qquad (7b)$$

$$\left.\frac{\partial^2 T_{00}}{\partial J_x^2}\right|_{HO} = \frac{R}{(2\pi)^2}\frac{\sigma_{xw}^4}{\sigma_{x0}^4}\int_0^1 \frac{\eta d\eta}{\sqrt{(\eta+R^2(1-\eta))^5}}\int_0^{2\pi} d\theta_y \int_0^{2\pi}\cos^4\theta_x d\theta_x$$
$$\exp\left\{-\eta\left[\frac{J_x\sigma_{xw}^2\cos^2\theta_x}{\sigma_{x0}^2(\eta+R^2(1-\eta))}+\frac{J_y\sigma_{yw}^2\cos^2\theta_y}{\sigma_{y0}^2}\right]\right\} \qquad (7c)$$
$$= \frac{R}{8}\frac{\sigma_{xw}^4}{\sigma_{x0}^4}\int_0^1 \frac{\eta d\eta}{\sqrt{(\eta+R^2(1-\eta))^5}} I_0^e(B_y/2)\left(3I_0^e(B_x/2)-4I_1^e(B_x/2)+I_2^e(B_x/2)\right)$$

$$\left.\frac{\partial^2 T_{00}}{\partial J_x \partial J_x}\right|_{HO} = \frac{R}{(2\pi)^2}\frac{\sigma_{xw}^2}{\sigma_{x0}^2}\frac{\sigma_{yw}^2}{\sigma_{y0}^2}\int_0^1 \frac{\eta d\eta}{\sqrt{(\eta+R^2(1-\eta))^3}}\int_0^{2\pi}\cos^2\theta_y d\theta_y \int_0^{2\pi}\cos^2\theta_x d\theta_x$$
$$\exp\left\{-\eta\left[\frac{J_x\sigma_{xw}^2\cos^2\theta_x}{\sigma_{x0}^2(\eta+R^2(1-\eta))}+\frac{J_y\sigma_{yw}^2\cos^2\theta_y}{\sigma_{y0}^2}\right]\right\} \qquad (7d)$$
$$= \frac{R}{4}\frac{\sigma_{xw}^2}{\sigma_{x0}^2}\frac{\sigma_{yw}^2}{\sigma_{y0}^2}\int_0^1 \frac{\eta d\eta}{\sqrt{(\eta+R^2(1-\eta))^3}}\left(I_0^e(B_y/2)-I_1^e(B_y/2)\right)\left(I_0^e(B_x/2)-I_1^e(B_x/2)\right)$$

where

$$B_x(\eta) = \eta\frac{J_x\,\sigma_{xw}^2/\sigma_{x0}^2}{\eta+R^2(1-\eta)}; \quad B_y(\eta) = \eta J_y\,\sigma_{yw}^2/\sigma_{y0}^2$$

The integrands for these three partial derivatives are in the functions *DTJY2*, *DTJX2*, and *DTJXJY*.

The resonance width in units of action for resonance $pQ_x + rQ_y + mQ_s = n$ is given by equation (4.31).

$$\Delta K_1 = 4\sqrt{\left|\frac{2F_{prm}}{\Lambda_{pr}}\right|} = 4\sqrt{\frac{2T_{pr}}{\Lambda_{pr}}\exp\left\{-\frac{1}{2}\left[\frac{r\sigma_L}{2\beta_y^*}\right]^2\right\}J_m\left(\frac{rc\hat{\tau}}{2\beta_y^*}\right)} \qquad (8a)$$

The term T_{pr} for head-on collisions is

$$T_{pr} = \frac{R}{(2\pi)^2}\int_0^{2\pi} e^{ip\theta_x} d\theta_x \int_0^{2\pi} e^{ir\theta_y} d\theta_y \int_0^1 \frac{d\eta}{\eta\sqrt{(\eta + R^2(1-\eta))}}$$
$$\exp\left\{-\eta\left[\frac{J_x\sigma_{xw}^2\cos^2\theta_x}{\sigma_x^2(\eta + R^2(1-\eta))} + \frac{J_y\sigma_{yw}^2\cos^2\theta_y}{\sigma_y^2}\right]\right\} \tag{8b}$$
$$= (-1)^{p+r} R\int_0^1 \frac{d\eta}{\eta\sqrt{(\eta + R^2(1-\eta))}} I_{p/2}^e(B_x/2) I_{r/2}^e(B_y/2)$$

B_x and B_y are given by the equation above. This integrand is in the subroutine *GPR.*

The resonance, or island, tune can be calculated from Λ_{pr} and F_{prm} also. It is given by (eq 4.30)

$$Q_r = \frac{C}{2\pi}\frac{d\psi_1}{ds} = \frac{Nr_e}{2\pi\gamma}\sqrt{\left|2F_{prm}^R\Lambda_{pr}\right|} \tag{9}$$

Flat Beams and High Order Horizontal Resonances

This section is a study of horizontal resonances, $pQ_x = n$, when the beam is flat, $R = \sigma_y/\sigma_x << 1$. Bunch length effects depend on the vertical motion, and the vertical order of the resonance, so they are ignored.

Formalism

Assume one collision point per turn. Most of the formulas needed have been developed in earlier sections. The Fourier expansion coefficient for the resonance $pQ_x + rQ_y = n$ is

$$T_{pr} = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^1 \frac{d\eta}{\eta\sqrt{1-\eta(1-R^2)}} \exp\left\{-\eta\left[J_x\cos^2\theta_x + \frac{R^2 J_y\cos^2\theta_y}{1-\eta(1-R^2)}\right]\right\} .$$

The focus of this analysis is on r = 0, so set r = 0 and perform the θ_y integral

$$T_{p0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^1 \frac{d\eta}{\eta\sqrt{1-\eta(1-R^2)}}\, I_0^e(B_{y2}) \exp\left\{-\eta J_x\cos^2\theta_x\right\}$$

where

$$B_{y2} = \frac{1}{2}\frac{\eta R^2 J_y}{1-\eta(1-R^2)} .$$

To make some of the amplitude dependence more transparent, make another change of variables

$$\zeta = \frac{\eta J_x(1-R^2)}{2} .$$

In terms of ζ,

$$T_{p0} \approx \frac{J_x^{1/2}}{2\pi}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_{\zeta=0}^{J_x(1-R^2)/2} \frac{d\zeta}{\zeta\sqrt{J_x-2\zeta}}\, I_0^e\left(B_{y2}\right)\exp\left\{-2\zeta\cos^2\theta_x\right\}$$

$$= J_x^{1/2}(-1)^{p/2}\int_{\zeta=0}^{J_x(1-R^2)/2} \frac{d\zeta}{\zeta\sqrt{J_x-2\zeta}}\, I_0^e\left(B_{y2}\right) I_{p/2}^e(\zeta)$$

$$= \frac{A_x}{\sqrt{2}}(-1)^{p/2} f_p(J_x = A_x^2/2, J_y, R)\ ;\ B_{y2} = \zeta R^2 J_y/(J_x - 2\zeta) .$$

where the last equation is the definition of f_p.

The detuning is

$$\Lambda_{p0} = p^2\frac{\partial^2 T_{00}}{\partial I_x^2} = \frac{p^2}{\varepsilon_x^2}\frac{\partial^2 T_{00}}{\partial J_x^2}$$

where

$$\frac{\partial^2 T_{00}}{\partial J_x^2} = \frac{1}{8}\int_0^1 \frac{\eta d\eta}{\sqrt{1-\eta(1-R^2)}} I_0^e(B_{y2})\left(3I_0^e(\eta J_x/2) - 4I_1^e(\eta J_x/2) + I_2^e(\eta J_x/2)\right) \ .$$

Making the same change of variables $\eta \to \zeta$ gives

$$\frac{\partial^2 T_{00}}{\partial J_x^2} = \frac{\sqrt{2}}{A_x^3}\int_0^{J_x(1-R^2)/2} \frac{\zeta d\zeta}{\sqrt{J_x - 2\zeta}} I_0^e(B_{y2})\left(3I_0^e(\zeta) - 4I_1^e(\zeta) + I_2^e(\zeta)\right)$$

$$= \frac{\sqrt{2}}{A_x^3} f_0(J_x = A_x^2 \;\; 2, J_y, R) \ .$$

The last equation defines f_0.

Results

The horizontal tune shift is

$$\frac{\Delta Q_x}{\xi_x} = \frac{1+R}{2}\int_0^1 \frac{d\eta}{\sqrt{1-\eta(1-R^2)}} I_0^e\left(\frac{B_y}{2}\right)\left\{I_0^e\left(\frac{B_x}{2}\right) - I_1^e\left(\frac{B_x}{2}\right)\right\}$$

$$B_y(\eta) = \eta\frac{J_y R^2}{1-\eta(1-R^2)};\ B_x(\eta) = \eta J_x \ .$$

It is plotted below for different values of R and J_y. As expected, the results are not sensitive to these two parameters, and, as a result, the detuning will be roughly independent of them also.

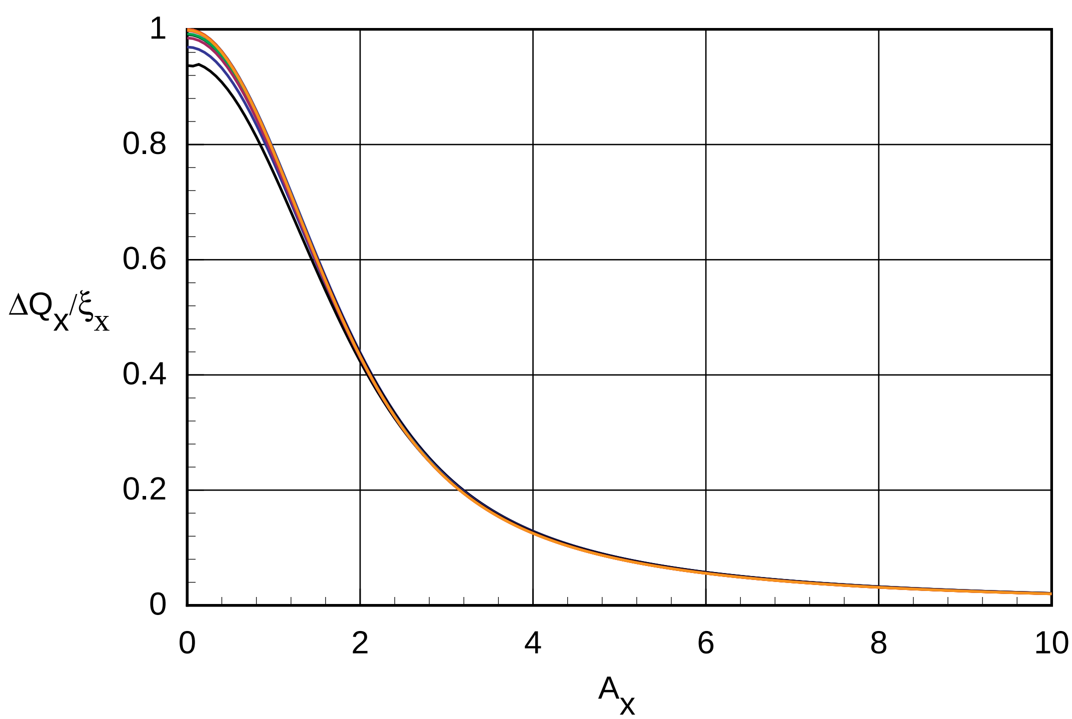


Figure: Horizontal tune shift for the twelve combinations of J_y = 0, 0.5, 2.0, 4.5 and R = 0.04, 0.01, 0.001

The island tune and resonance full width are

$$Q_r = \frac{Nr_e}{2\pi\gamma}\sqrt{\left|2T_{p0}\Lambda_{p0}\right|} = \frac{Nr_e p}{2\pi\gamma\varepsilon_x}\sqrt{\frac{2}{A_x^2}f_0 f_p} \approx \frac{\xi_x p\sqrt{2}}{A_x}\sqrt{f_p f_0} \ ,$$

and

$$\Delta K_1 = 4\sqrt{\left|\frac{2T_{p0}}{\Lambda_{p0}}\right|} = 4\frac{\varepsilon_x A_x^2}{p}\sqrt{\frac{f_p}{f_0}} \ ,$$

respectively. In terms of A_x, the resonance full width is (A. L. Gerasimov *et al*, AIP Conf Proceedings 153, 474 (1987); appendix C)

$$\Delta A_x = \frac{\Delta I_x}{\varepsilon_x A_x} = \frac{p\Delta K_1}{\varepsilon_x A_x} = 4A_x\sqrt{\frac{f_p}{f_0}} \ .$$

The island tunes for different resonance orders are given in the figure below for $J_y = 0.5$ ($A_y = 1$) and R = 0.04 which is the value for PEP-II. Interest will be focused on large amplitude particles that lead to lifetime limitations. The region $A_x \sim 5$ - 6 is populated by quantum fluctuations. As the figure shows, the island tune is large in this region even for high order resonances. For example, for $A_x = 6$, Q_r ranges from $0.05\xi_x$. to $0.20\xi_x$. For the PEP-II LER with $\xi_x = 0.03$, $Q_r = 6\times10^{-3}$ to 1.5×10^{-3}. This is to be compared with the inverse of the damping time, $1/\tau = 2\times10^{-4}$. A particle that falls inside this resonance circulates many times during a damping time; i.e. that particle will be transported from the minimum amplitude of the resonance island to the maximum amplitude of the island in a time short compared to the damping time. If there is an aperture at an amplitude smaller than this maximum amplitude, the particle is lost. The effective aperture has been moved to the minimum amplitude of the resonance island!

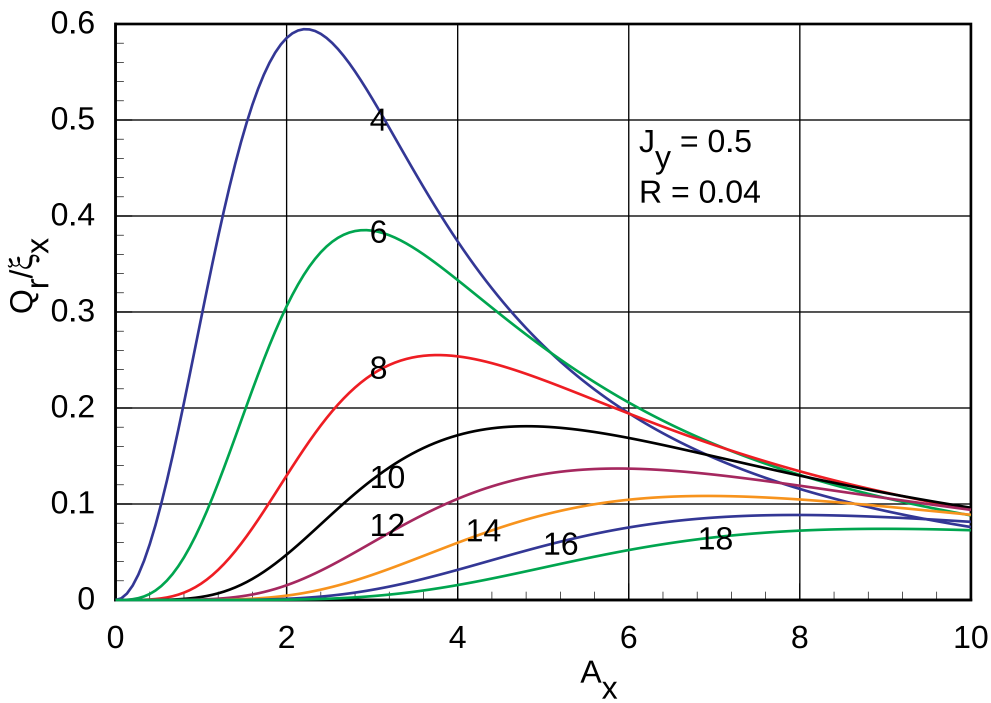


Figure: Island tunes normalized by horizontal tune shift for p = 4, ..., 18

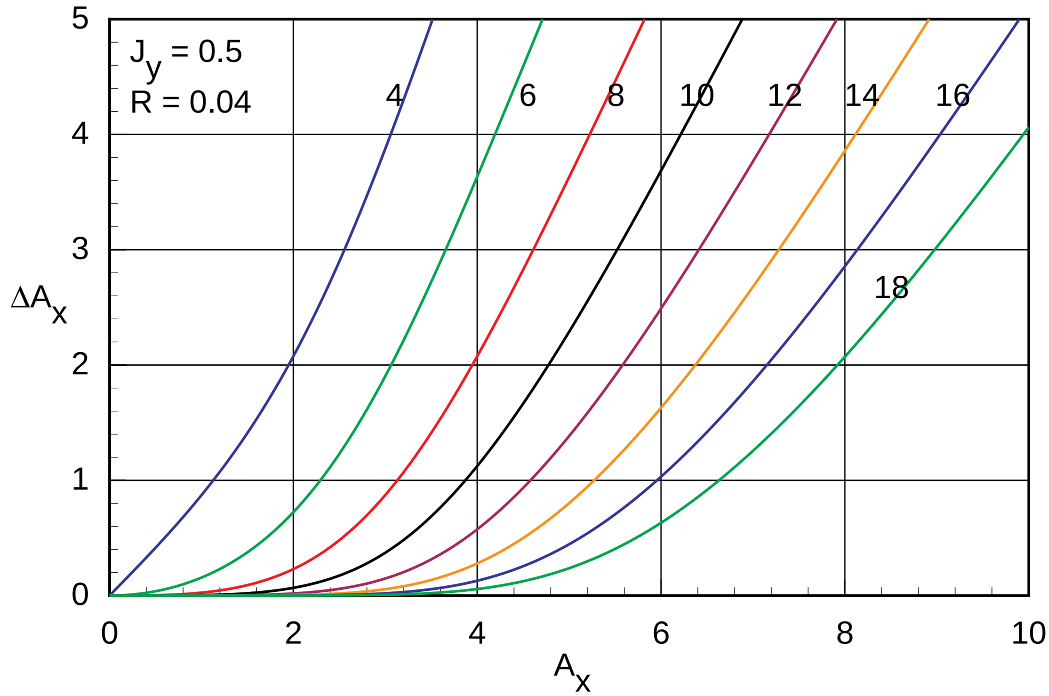


Figure: Full width of resonance islands for p = 4, ..., 18

The resonance width tells whether this is an important effect. If the resonance is narrow, the minimum and maximum amplitudes of the resonance are almost equal, and the effective aperture is not changed significantly. The full widths of resonances are plotted above, and these widths can be large.

These results can be interpreted as follows. Assume there is an aperture at $A_x = 10$ and that amplitudes $A_x \leq 6$ are populated sufficiently by statistical fluctuations to be important for the lifetime. Then if a resonance occurs such that

$$A_x(\text{minimum}) \leq 6;\ A_x(\text{maximum}) \geq 10$$

the lifetime will be reduced. A sufficient number of particles will be transported by the resonance to lower the lifetime. The situation is illustrated using p = 12 in the plot below. If the resonance value of A_x is between $7.6 \leq A_x(\text{resonance}) \leq 10$, the lifetime will be too short. When

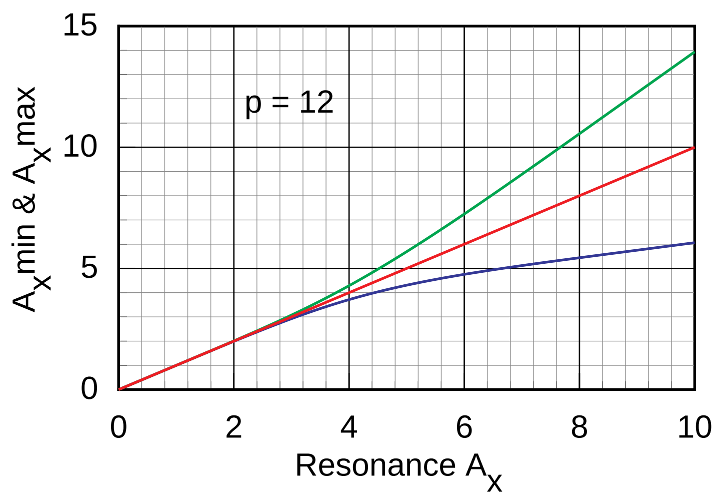


Figure: Minimum and maximum A_x as a function of the resonance value of A_x, the value where 12 $Q_x(A_x)$ = n.

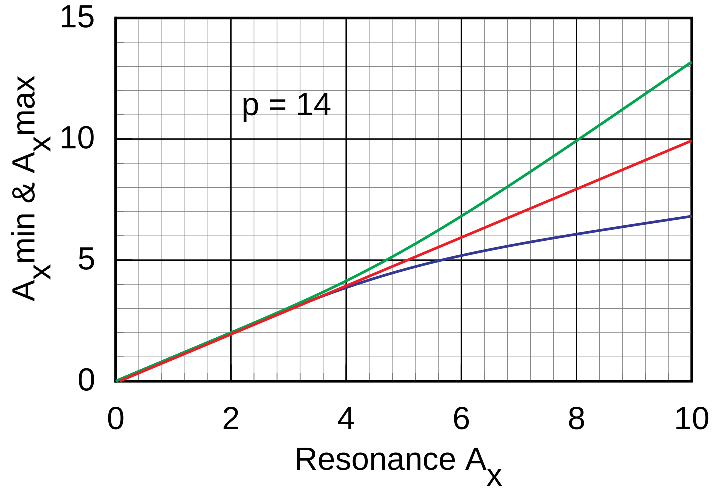


Figure: Minimum and maximum A_x as a function of the resonance value of A_x, the value where 14 $Q_x(A_x) = n$.

A_x(resonance) < 7.6, the resonance affects the beam distribution, but particles are not transported out to the aperture at $A_x = 10$. When $A_x > 10$, the A_x(minimum) > 6, and there is no flux into the resonance. The tune shifts for these two values of A_x are $\Delta Q_x(A_x = 7.6) = 0.036\xi_x = 0.0011$ for $\xi_x = 0.003$ and $\Delta Q_x(A_x = 10) = 0.02\xi_x = 0.0006$ for $\xi_x = 0.003$. The relatively narrow tune region

$$\frac{n}{12} - 0.036\xi_x \le Q_x \le \frac{n}{12} - 0.02\xi_x$$

will have unacceptably short lifetime.

The figure above for p = 14, shows that there is no region of unacceptable lifetime. When A_x(resonance) < 8, A_x(maximum) < 10, and particles are not lost. When A_x(resonance) > 8, A_x(minimum) > 6, and there is not significant flux into the resonance. The resonances of importance are twelve order and below. The table that follows gives the tune regions of unacceptable lifetime.

Tune regions of Unacceptable Lifetime for an Aperture at $A_x = 10$

p = 4	$5.4 \le A_x$(resonance) $\le \infty$	$\frac{n}{4} - 0.070\xi_x \le Q_x \le \frac{n}{4}$
p = 6	$6.3 \le A_x$(resonance) $\le \infty$	$\frac{n}{6} - 0.052\xi_x \le Q_x \le \frac{n}{6}$
p = 8	$6.8 \le A_x$(resonance) $\le \infty$	$\frac{n}{8} - 0.045\xi_x \le Q_x \le \frac{n}{8}$
p = 10	$7.2 \le A_x$(resonance) $\le \infty$	$\frac{n}{10} - 0.040\xi_x \le Q_x \le \frac{n}{10}$
p = 12	$7.6 \le A_x$(resonance) ≤ 10	$\frac{n}{12} - 0.036\xi_x \le Q_x \le \frac{n}{12} - 0.02\xi_x$
$p \ge 14$	No limit	

Bob Siemann
July 3, 1996

Crossing Angle Calculations

Formalism

The beam-beam potential is

$$V_{BB} = -\frac{Nr_e}{\gamma}\sqrt{\frac{2}{\pi\sigma_L^2}}\sum_{n=-\infty}^{\infty} V_F(x,y,s)\times\exp\left\{-2(s-(nC+c\tau))^2/\sigma_L^2\right\} = -\frac{Nr_e}{\gamma}\tilde{V}_{BB}$$

where V_F is given by eq. (4.8) of Reference 1

$$V_F = \int_0^\infty \frac{dq}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)}}\exp\left\{-\left[\frac{x^2}{2\sigma_x^2+q}+\frac{y^2}{2\sigma_y^2+q}\right]\right\}.$$

The quantities in this equation are defined in References 1 or 2. Fourier analyze V_{BB}

$$V_{BB} = -\frac{Nr_e}{\gamma}\sum_{p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, A_{pr}(I_x,I_x,k)\exp\left\{i(p\psi_x + r\psi_y - ks)\right\}$$

where the Fourier expansion coefficient is given by

$$A_{pr} = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\psi_x \int_0^{2\pi} d\psi_y \int_{-\infty}^{\infty} ds \exp\left\{-i(p\psi_x + r\psi_y - ks)\right\}\tilde{V}_{BB}\,.$$

The problem studied in this note is the case of a crossing angle in x and head-on collisions in y. The x-coordinate contains a term for the crossing angle, 2φ, plus the x betatron oscillation

$$x = s\sin 2\varphi + \sqrt{2\beta_x I_x}\cos\theta_x\,.$$

The expression for V_F becomes

$$V_F = \int_0^\infty \frac{dq}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)}}\exp\left\{-\left[\frac{(s\sin 2\varphi + \sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_x^2+q}+\frac{y^2}{2\sigma_y^2+q}\right]\right\}.$$

Write the first term in terms of its Fourier transform (derived below)

$$V_F = \frac{\sqrt{\pi}}{2\pi}\frac{1}{\sin 2\varphi}\int_0^\infty \frac{dq}{\sqrt{(2\sigma_y^2+q)}}\exp\left\{-\left[\frac{y^2}{2\sigma_y^2+q}\right]\right\}\int_{-\infty}^{\infty} e^{i\omega s}d\omega\exp\left\{-\frac{\omega^2(2\sigma_x^2+q)}{4\sin^2 2\varphi}+\frac{i\omega\sqrt{2\beta_x I_x}\cos\theta_x}{\sin 2\varphi}\right\}.$$

Substituting this expression into that for A_{pr} gives

$$A_{pr} = \frac{1}{(2\pi)^4}\frac{\sqrt{\pi}}{\sin 2\varphi}\int_0^{2\pi} d\psi_x \int_0^{2\pi} d\psi_y \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} ds \exp\left\{-i(p\psi_x + r\psi_y - ks - \omega s)\right\}\tilde{V}_F$$

where

$$\tilde{V}_F = \int_0^\infty \frac{dq}{\sqrt{(2\sigma_y^2+q)}}\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2+q}\right\}\exp\left\{-\frac{\omega^2(2\sigma_x^2+q)}{4\sin^2 2\varphi}+\frac{i\omega\sqrt{2\beta_x I_x}\cos\theta_x}{\sin 2\varphi}\right\}.$$

Following the steps in Reference 2, the s integral is performed. The only difference in the result is that k_{pr} is replaced by $k_{pr} + \omega$. The result is

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1 R. Siemann, SLAC-PUB-6073

2 R. Siemann, "Extension of Beam-Beam Calculations in SLAC-PUB-6073".

$$V_{BB} = -\frac{Nr_e}{C\gamma}\frac{1}{\sin 2\varphi}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega U_{pr}(I_x, I_y, \omega)\exp\left\{-\left((k_{pr}+\omega)\sigma_L\right)^2/8\right\}$$

$$\times i^m J_m((k_{pr}+\omega)\hat{\tau}c/2)\exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right\}$$

where

$$k_{pr} = 2\pi(n - mQ_s)/C + p(1/\beta_x^* - 2\pi Q_{x0}/C) + r(1/\beta_y^* - 2\pi Q_{y0}/C)$$

and

$$U_{pr} = \frac{\sqrt{\pi}}{(2\pi)^3}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2+q)}}$$

$$\times\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2+q}\right\}\exp\left\{-\frac{\omega^2(2\sigma_x^2+q)}{4\sin^2 2\varphi}+\frac{i\omega\sqrt{2\beta_x I_x}\cos\theta_x}{\sin 2\varphi}\right\}.$$

Make a change of variables $\zeta = \omega/\sin 2\varphi$, and these expressions become

$$V_{BB} = -\frac{Nr_e}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} d\zeta U_{pr}(I_x, I_y, \zeta)\exp\left\{-\left((k_{pr}+\zeta\sin 2\varphi)\sigma_L\right)^2/8\right\}$$

$$\times i^m J_m((k_{pr}+\zeta\sin 2\varphi)\hat{\tau}c/2)\exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right\}$$

and

$$U_{pr} = \frac{\sqrt{\pi}}{(2\pi)^3}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2+q)}}$$

$$\times\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2+q}\right\}\exp\left\{-\frac{\zeta^2(2\sigma_x^2+q)}{4}+i\zeta\sqrt{2\beta_x I_x}\cos\theta_x\right\}.$$

The crossing angle does not appear in U_{pr}, but it does appear in the factors out front related to bunch length and synchrobetatron resonance strength.

Perform a Taylor expansion in $\sin 2\varphi$

$$V_{BB} = V_{BB}\Big|_{\sin 2\varphi=0} + \sin 2\varphi\frac{\partial V_{BB}}{\partial \sin 2\varphi}\Big|_{\sin 2\varphi=0} + \ldots.$$

The first term in this expansion is

$$V_{BB}\Big|_{\sin 2\varphi=0} = -\frac{Nr_e}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} d\zeta U_{pr}(I_x, I_y, \zeta)\exp\left\{-\left(k_{pr}\sigma_L\right)^2/8\right\}$$

$$\times i^m J_m(k_{pr}\hat{\tau}c/2)\exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right\}.$$

The ζ integral of U_{pr} can be performed. Doing this

$$\int_{-\infty}^{\infty} d\zeta U_{pr}(I_x, I_y, \zeta) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2+q)(2\sigma_x^2+q)}}$$

$$\times\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2+q}-\frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2+q}\right\}$$

$$= T_{pr}(I_x, I_y).$$

T_{pr} is the resonance coefficient that appears for head-on collisions,[2] and this term in the Taylor series of the beam-beam potential is the same as for head-on collisions.

The second term is

$$\left.\frac{\partial V_{BB}}{\partial \sin 2\varphi}\right|_{\sin 2\varphi=0} = -\frac{Nr_e}{C\gamma}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta d\zeta U_{pr}(I_x,I_y,\zeta)\exp\left\{-\left(k_{pr}\sigma_L\right)^2/8\right\}$$
$$\times i^m\left[\frac{\hat{\tau}c}{2}J_{m-1}(k_{pr}\hat{\tau}c/2)-\left(\frac{k_{pr}\sigma_L^2}{4}+\frac{m}{k_{pr}}\right)J_m(k_{pr}\hat{\tau}c/2)\right]$$
$$\times\exp\left\{i(p\psi_x+r\psi_y-2\pi(n-mQ_s)s/C)\right\}.$$

Performing the ζ integral

$$\int_{-\infty}^{\infty}\zeta d\zeta U_{pr}(I_x,I_y,\zeta)=\frac{-2i}{(2\pi)^2}\int_0^{2\pi}d\theta_x e^{-ip\theta_x}\int_0^{2\pi}d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2+q)(2\sigma_x^2+q)^3}}$$
$$\times\sqrt{2I_x\beta_x}\cos\theta_x\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2+q}-\frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2+q}\right\}.$$

This second term in the Taylor series produces odd order horizontal resonances because the argument of the θ_x integral is an odd function of θ_x. It does not make contributions to even order horizontal resonances, and it can not contribute to the tune shifts. This latter point will be shown in more detail in the next section.

Tune Shifts

The average beam-beam potential is

$$\langle V_{BB}\rangle=\left\langle V_{BB}\big|_{\sin 2\varphi=0}\right\rangle+\sin 2\varphi\left\langle\left.\frac{\partial V_{BB}}{\partial\sin 2\varphi}\right|_{\sin 2\varphi=0}\right\rangle+\ldots.$$

Tune shifts are given by

$$\Delta Q_y=\frac{C}{2\pi}\frac{\partial\langle V_{BB}\rangle}{\partial I_y};\quad \Delta Q_x=\frac{C}{2\pi}\frac{\partial\langle V_{BB}\rangle}{\partial I_x}\ .$$

The tune shifts from the first term have been calculated. They are

$$\frac{\Delta Q_y}{\xi_y}=\frac{\sigma_x+\sigma_y}{2\sigma_x}\int_0^1\frac{d\eta}{\sqrt{\eta+R^2(1-\eta)}}I_0^e\left(\frac{B_x}{2}\right)\left\{I_0^e\left(\frac{B_y}{2}\right)-I_1^e\left(\frac{B_y}{2}\right)\right\}$$

$$B_x(\eta)=\eta\frac{J_x}{\eta+R^2(1-\eta)};\,B_y(\eta)=\eta J_y$$

and

$$\frac{\Delta Q_x}{\xi_x}=\frac{\sigma_x+\sigma_y}{2\sigma_y}\int_0^1\frac{d\eta}{\sqrt{\eta+(1-\eta)/R^2}}I_0^e\left(\frac{B_y}{2}\right)\left\{I_0^e\left(\frac{B_x}{2}\right)-I_1^e\left(\frac{B_x}{2}\right)\right\}$$

$$B_y(\eta)=\eta\frac{J_y}{\eta+(1-\eta)/R^2};\,B_x(\eta)=\eta J_x\ .$$

Concentrate on the contributions from the second term in the Taylor series. Putting p = r = m = n = 0 and $\hat{\tau} = 0$ gives

$$\left\langle \left. \frac{\partial V_{BB}}{\partial \sin 2\varphi} \right|_{\sin 2\varphi = 0} \right\rangle = -\frac{N r_e}{C\gamma} \int_{-\infty}^{\infty} \zeta d\zeta U_{00}(I_x, I_y, \zeta)$$

$$= \frac{N r_e}{C\gamma} \frac{2i}{(2\pi)^2} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^{\infty} \frac{dq}{\sqrt{(2\sigma_y^2 + q)(2\sigma_x^2 + q)^3}}$$

$$\times \sqrt{2 I_x \beta_x} \cos\theta_x \exp\left\{ -\frac{2 I_y \beta_y \cos^2\theta_y}{2\sigma_y^2 + q} - \frac{2 I_x \beta_x \cos^2\theta_x}{2\sigma_x^2 + q} \right\}$$

Taking the derivative with respect to I_y

$$\left\langle \left. \frac{\partial^2 V_{BB}}{\partial I_y \partial \sin 2\varphi} \right|_{\sin 2\varphi = 0} \right\rangle = -\frac{N r_e}{C\gamma} \frac{2i}{(2\pi)^2} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^{\infty} \frac{dq}{\sqrt{(2\sigma_y^2 + q)(2\sigma_x^2 + q)^3}}$$

$$\times \sqrt{2 I_x \beta_x} \cos\theta_x \frac{2\beta_y \cos^2\theta_y}{2\sigma_y^2 + q} \exp\left\{ -\frac{2 I_y \beta_y \cos^2\theta_y}{2\sigma_y^2 + q} - \frac{2 I_x \beta_x \cos^2\theta_x}{2\sigma_x^2 + q} \right\}$$

$$= 0$$

The last line follows from performing the θ_x integral.

Taking the derivative with respect to I_x

$$\left\langle \left. \frac{\partial^2 V_{BB}}{\partial I_x \partial \sin 2\varphi} \right|_{\sin 2\varphi = 0} \right\rangle = -\frac{N r_e}{C\gamma} \frac{2i}{(2\pi)^2} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^{\infty} \frac{dq}{\sqrt{(2\sigma_y^2 + q)(2\sigma_x^2 + q)^3}}$$

$$\times \left[\sqrt{\frac{\beta_x}{2 I_x}} \cos\theta_x - \frac{\sqrt{2 I_x \beta_x^3}}{2\sigma_x^2 + q} \cos^3\theta_x \right] \exp\left\{ -\frac{2 I_y \beta_y \cos^2\theta_y}{2\sigma_y^2 + q} - \frac{2 I_x \beta_x \cos^2\theta_x}{2\sigma_x^2 + q} \right\}$$

$$= 0$$

The last line follows from performing the θ_x integral also.

To first order in sin2φ the tune shifts are unchanged by a crossing angle, and the dominant effect is introduction of odd order beam-beam resonances.

Math Notes

$$\int_{-\infty}^{\infty} \exp(-p^2 x^2 \pm qx) dx = \exp\left(q^2/4p^2\right) \int_{-\infty}^{\infty} \exp(-(px \pm q/2p)^2) dx$$

$$= \frac{1}{p} \exp\left(q^2/4p^2\right) \int_{-\infty}^{\infty} \exp(-x^2) dx = \frac{\sqrt{\pi}}{p} \exp\left(q^2/4p^2\right)$$

$$\int_{-\infty}^{\infty} \exp(-p^2x^2 \pm qx)x dx = \exp\left(q^2 / 4p^2\right) \int_{-\infty}^{\infty} \exp(-(px \pm q/2p)^2)x dx$$

$$= \exp\left(q^2 / 4p^2\right) \int_{-\infty}^{\infty} \exp(-x^2)\left(\frac{x}{p^2} \mp \frac{q}{2p^3}\right) dx = \mp \frac{\sqrt{\pi} q}{2p^3} \exp\left(q^2 \quad 4p^2\right)$$

An expression of the following form appears in the beam-beam interaction

$$\exp\left(-\frac{(s \sin 2\varphi + A \cos\theta)^2}{2\sigma^2}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega B(\omega) e^{i\omega s}$$

where

$$B(\omega) = \int_{-\infty}^{\infty} ds \exp\left(-\frac{(s \sin 2\varphi + A \cos\theta)^2}{2\sigma^2}\right) e^{-i\omega s}$$

$$= \exp\left(-\frac{A^2 \cos^2\theta}{2\sigma^2}\right) \int_{-\infty}^{\infty} ds \exp\left(-\frac{s^2 \sin^2 2\varphi + 2As \cos\theta \sin 2\varphi}{2\sigma^2} - i\omega s\right)$$

$$= \exp\left(-\frac{A^2 \cos^2\theta}{2\sigma^2}\right) \frac{\sqrt{2\pi}\sigma}{\sin 2\varphi} \exp\left(\frac{(i\omega\sigma^2 + A \cos\theta \sin 2\varphi)^2}{2\sigma^2 \sin^2 2\varphi}\right)$$

$$= \frac{\sqrt{2\pi}\sigma}{\sin 2\varphi} \exp\left(-\frac{\omega^2\sigma^2}{2\sin^2 2\varphi} + \frac{i\omega A \cos\theta}{\sin 2\varphi}\right)$$

This result can be checked by taking the transform of $B(\omega)$.

APPENDIX: INTEGRALS

Notation

$$I_n^e(x) = e^{-x} I_n(x)$$

Basic Results

$$\int_0^{2\pi} d\theta\, e^{im\theta} e^{-B\cos^2\theta} = \begin{cases} 2\pi\,(-1)^{m/2}\, I_{m/2}^e(B/2) & \text{(m even)} \\ 0 & \text{(m odd)} \end{cases}$$

$$\int_0^{2\pi} d\theta\, \cos(2n\theta) e^{-B\cos^2\theta} = 2\pi\,(-1)^n\, I_n^e(B/2)$$

$$\int_0^{2\pi} d\theta\, e^{im\theta} e^{-(\Gamma+\Lambda\cos\theta)^2} = 2\pi(-1)^m \exp(-\Gamma^2 + 2\Gamma\Lambda) \sum_{k=-\infty}^{\infty} (-1)^k I_{2k+m}^e(2\Gamma\Lambda) I_k^e(\Lambda^2/2)$$

Secondary Results

$$\int_0^{2\pi} d\theta\, e^{-B\cos^2\theta} = 2\pi I_0^e(B/2)$$

$$\int_0^{2\pi} d\theta\, \cos^2\theta \exp(-B\cos^2\theta) = \pi\left(I_0^e(B/2) - I_1^e(B/2)\right)$$

$$\int_0^{2\pi} d\theta\, \cos^4\theta \exp(-B\cos^2\theta) = \frac{\pi}{4}\left(3I_0^e(B/2) - 4I_1^e(B/2) + I_2^e(B/2)\right)$$

$$\int_0^{2\pi} d\theta\, \cos^m\theta\, e^{-(\Gamma+\Lambda\cos\theta)^2} = 2\pi F_m(\Gamma, M = \Lambda/\Gamma)$$

(This is a definition of numerical integrals used in the text above)

$$\int_0^{2\pi} d\theta\, e^{im\theta} e^{-(\Gamma+\Lambda\cos\theta)^2} = 2\pi(-1)^m e^{-\Gamma^2} \sum_{k=-\infty}^{\infty} (-1)^k I_{2k+m}(2\Gamma\Lambda) I_k^e(\Lambda^2/2)$$

The Bessel function sum rules in Gradshteyn & Ryzhik 8.511.4 are relations used throughout.

$$e^{iz\cos\theta} = \sum_{k=-\infty}^{\infty} i^k J_k(z) e^{ik\theta} = 1 + 2\sum_{k=1}^{\infty} i^k J_k(z)\cos(k\theta)$$

Using the relation between Bessel functions and modified Bessel functions for real positive x (Abramowitz and Stegun 9.6.3) $J_k(ix) = (i)^k I_k(x)$ a modified expression can be written

$$e^{-x\cos\theta} = \sum_{k=-\infty}^{\infty} i^k J_k(ix) e^{ik\theta} = \sum_{k=-\infty}^{\infty} (-1)^k I_k(x) e^{ik\theta} = I_0(x) + 2\sum_{k=1}^{\infty} (-1)^k I_k(x)\cos(k\theta)$$

$$\int_0^{2\pi} d\theta\, e^{im\theta} e^{-B\cos^2\theta}.$$

Rewrite $\cos^2\theta = 1/2\ (\cos 2\theta + 1)$ and use the Bessel function sum rule:

$$\begin{aligned}
\int_0^{2\pi} d\theta\, e^{im\theta} e^{-B\cos^2\theta} &= e^{-B/2} \int_0^{2\pi} d\theta\, e^{im\theta} e^{-B/2\cos 2\theta} \\
&= e^{-B/2} \int_0^{2\pi} d\theta\, e^{im\theta} \sum_{k=-\infty}^{\infty} (-1)^k I_k(B/2) e^{i2k\theta} \\
&= 2\pi(-1)^k \delta_{-m,2k} e^{-B/2} I_k(B/2) \\
&= 2\pi(-1)^{-m/2} e^{-B/2} I_{-m/2}(B/2) \\
&= 2\pi(-1)^{m/2} e^{-B/2} I_{m/2}(B/2) \\
&= 2\pi(-1)^{m/2} I^e_{m/2}(B/2)
\end{aligned}$$

where m must be even.

$$\int_0^{2\pi} d\theta \cos(2n\theta)\, e^{-B\cos^2\theta}.$$

Rewrite $\cos^2\theta = 1/2\ (\cos 2\theta + 1)$ and use the Bessel function sum rule:

$$\int_0^{2\pi} d\theta \cos(2n\theta) e^{-B\cos^2\theta} = e^{-B/2}\int_0^{2\pi} d\theta \cos(2n\theta) e^{-B/2\cos 2\theta}$$

$$= 2e^{-B/2}\int_0^{2\pi} d\theta \cos(2n\theta)\sum_{k=1}^{\infty}(-1)^k I_k(B/2)\cos(2k\theta)$$

$$= 2\pi(-1)^k \delta_{2n,2k} e^{-B/2} I_k(B/2)$$

$$= 2\pi(-1)^n I_n^e(B/2)$$

Use $\cos^2\theta = 1/2\ (\cos 2\theta + 1)$

$$\int_0^{2\pi} d\theta \cos^2\theta \exp(-B\cos^2\theta) = \frac{1}{2}\int_0^{2\pi} d\theta (1+\cos 2\theta)\exp(-B\cos^2\theta)$$

$$= \pi\left(I_0^e(B/2) - I_1^e(B/2)\right)$$

Use $\cos^4\theta = 1/8\ (\cos 4\theta + 4\cos 2\theta + 3)$

$$\int_0^{2\pi} d\theta \cos^4\theta \exp(-B\cos^2\theta) = \frac{1}{8}\int_0^{2\pi} d\theta (3+4\cos 2\theta + \cos 4\theta)\exp(-B\cos^2\theta)$$

$$= \frac{\pi}{4}\left(3I_0^e(B/2) - 4I_1^e(B/2) + I_2^e(B/2)\right)$$

This technique can be extended to integrals of the form below, but is of limited value because the series does not converge well.

$$\int_0^{2\pi} d\theta\, e^{im\theta} e^{-(\Gamma + \Lambda\cos\theta)^2}$$

Squaring the exponent and writing out the expression for $\cos^2\theta$ gives

$$\int_0^{2\pi} d\theta\, e^{im\theta} e^{-(\Gamma + \Lambda\cos\theta)^2} = \exp(-\Gamma^2 - \Lambda^2/2)\int_0^{2\pi} d\theta\, e^{im\theta} e^{-2\Gamma\Lambda\cos\theta - \Lambda^2/2\cos 2\theta}$$

$$= \exp(-\Gamma^2 - \Lambda^2/2)\sum_{n,k=-\infty}^{\infty}(-1)^{n+k} I_n(2\Gamma\Lambda) I_k(\Lambda^2/2)\int_0^{2\pi} d\theta\, e^{i(m+n+2k)\theta}$$

$$= 2\pi\exp(-\Gamma^2 - \Lambda^2/2)\sum_{n,k=-\infty}^{\infty}(-1)^{n+k} I_n(2\Gamma\Lambda) I_k(\Lambda^2/2)\delta(m+n+2k)$$

$$= 2\pi(-1)^m e^{-\Gamma^2}\sum_{k=-\infty}^{\infty}(-1)^k I_{2k+m}(2\Gamma\Lambda) I_k^e(\Lambda^2/2)$$

This expression can be used in principle, but this series does not converge well. The exponent in the integrand is negative for all values of θ, and, therefore, the exponential is always less than 1. When the exponent is expanded there is a term -2ΓΛcosθ in the exponent that can be either positive or negative, and the exponential can be much greater than 1. This leads to large cancellations in the sum, represented by the $(-1)^k$ factor. Rather than using this expression numerical integrals are performed as discussed below.

$$\int_0^{2\pi} d\theta \cos^m \theta e^{-(\Gamma+\Lambda\cos\theta)^2}$$

This is defined as

$$\int_0^{2\pi} d\theta \cos^m \theta e^{-(\Gamma+\Lambda\cos\theta)^2} = 2\pi\left\{\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^m \theta e^{-\Gamma^2(1+M\cos\theta)^2}\right\}$$
$$= 2\pi F_m(\Gamma, M = \Lambda/\Gamma)$$

where the integral is done numerically.

There are two regions, one with M < 1 and one with M > 1. The integrand approaches an exponential as $M \to 0$ in the first case, and it approaches the modified Bessel function expressions in the second case as $M \to \infty$. Two tables are calculated for interpolation by the program **BEAMBEAM_FM**. The first is with M < 1. The indices of that table correspond to Γ and M. The second table is calculated when M > 1 and the indices in that table correspond to Λ and 1/M.

There are some limits that enter when Γ > 0 and M -> 0 that arise which offsets and zero oscillation amplitude. Those limits are

$$F_m(\Gamma, M = \Lambda/\Gamma) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^m \theta e^{-\Gamma^2(1+M\cos\theta)^2} \approx \frac{e^{-\Gamma^2}}{2\pi}\int_0^{2\pi} d\theta \cos^m \theta (1 + 2\Gamma\Lambda\cos\theta)$$

The first term matters if m is even and the second term if m is odd. Doing the integrals

$$\lim_{\Lambda\to 0} F_m = e^{-\Gamma^2}\begin{cases} 1 & m = 0 \\ \Gamma\Lambda & m = 1 \\ 1/2 & m = 2 \\ 3\Gamma\Lambda/4 & m = 3 \\ 3/8 & m = 4 \end{cases}$$

This an aside on a relation between modified Bessel functions and the confluent hypergeometric function for certain values of the arguments. Consider an integral of the form (which has been done above)

$$\int_0^{2\pi} d\theta \cos^2 \theta \exp(-\Gamma \cos^2 \theta) .$$

Change the range of integration to 0 to $\pi/2$, make a change of variables $z = \cos^2\theta$ and use $dz = -2\cos\theta\sin\theta d\theta = -2\sqrt{z}\sqrt{1-z}d\theta$.

$$\int_0^{2\pi} d\theta \cos^2\theta \exp(-\Gamma\cos^2\theta) = 4\int_0^{\pi/2} d\theta\cos^2\theta\exp(-\Gamma\cos^2\theta) = 2\int_0^1 \frac{\sqrt{z}dz}{\sqrt{1-z}}\exp(-\Gamma z)$$

This integral is in Gradshteyn & Ryzhik 3.383.1 with u =1, μ = 1/2 and ν = 3/2. Writing this expression and using a results above

$$\int_0^{2\pi} d\theta\cos^2\theta\exp(-\Gamma\cos^2\theta) = 2B(3/2,1/2)\,{}_1F_1[3/2,2,-\Gamma] = \pi\,{}_1F_1[3/2,2,-\Gamma] = \pi\left(I_0^e(\Gamma/2) - I_1^e(\Gamma/2)\right)$$

The resulting relationship is

$${}_1F_1[3/2,2,-\Gamma] = I_0^e(\Gamma/2) - I_1^e(\Gamma/2)$$