High Order Horizontal Resonances in the Beam-Beam Interaction

T. Chen, J. Irwin, and R. H. Siemann*

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

Abstract

This paper studies the characteristics of horizontal resonances excited by the beam-beam interaction. First order perturbation theory and simulations show that high order horizontal resonances are exceptionally strong at large amplitudes. The orders of these resonances are so high that they may mistakenly be ignored, and they can result in bad lifetime.

I. Introduction

In a circular electron-positron collider, particles in one beam suffer kicks from the electromagnetic fields of the opposing beam. The beams are approximately Gaussian, and, as a result, the fields are nonlinear. Particle motion can be analyzed using first order perturbation theory that expands the beam-beam interaction potential into a resonance basis [1]. Using this method, we have found an explanation of a phenomenon observed in simulations - some high order resonances have significant effects on beam tails. In this paper we perform an analysis of horizontal resonances excited by the beam-beam interaction and set up criteria for dangerous resonances.

II. Analysis

Assuming a linear storage ring lattice and treating the beam-beam interaction as a perturbation, one can write down the Hamiltonian of a particle as

$$H(x,p_x,y,p_y,s) = H_0 + V_{BB}(x,y,s) \quad . \tag{1}$$

The variables are the horizontal and vertical coordinates, x and y, the conjugate momenta, p_x and p_y, and the position around the storage ring circumference, s. The first term, H_0, is the Hamiltonian for the unperturbed, linear betatron oscillation. It can be transformed to action-angle variables, I_x, I_x, ψ_x and ψ_y,

$$H_0(I_x,\psi_x,I_y,\psi_y) = \frac{2\pi Q_{x0}}{C} I_x + \frac{2\pi Q_{y0}}{C} I_y \tag{2}$$

where C is the accelerator circumference, and Q_{x0} and Q_{y0} are the betatron tunes in the absence of the beam-beam interaction.

The other term, V_{BB}, is the beam-beam potential that depends on the horizontal and vertical coordinates of the particle. In first order perturbation theory, these coordinates are written in terms of the action-angle variables of the linear Hamiltonian and then Fourier analyzed. Following the procedure used in Refs. 2 and 3, the beam-beam potential is

$$V_{BB} = -\frac{Nr_e}{\gamma} \sum_{p,r=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\ A_{pr}(I_x,I_y,k) \exp\left\{ i(p\psi_x + r\psi_y - ks) \right\} \tag{3}$$

where N is the number of particles in the opposing beam, γ is the particle energy in units of rest energy, and r_e is the classical electron radius.

---

* Work supported by the Department of Energy, Contract No. DE-AC03-76SF00515.

Beam-beam resonances are of the form

$$p\,Q_x + r\,Q_y + m\,Q_s = n \tag{4}$$

where p, r, m and n are integers, Q_x and Q_y are the betatron tunes including the effects of the beam-beam interaction, and Q_s is the synchrotron tune. The Fourier expansion coefficient, A_{pr}, corresponds to resonances with a horizontal order p and vertical order r. In general, A_{pr} depends on the actions, the synchrotron oscillation amplitude, the transverse beam sizes, σ_{x0} and σ_{y0}, and the bunch length.[2, 3] In this paper the beam is assumed to be flat, *i. e.,* $R = \sigma_{y0}/\sigma_{x0} << 1$, and the focus will be on horizontal resonances, $pQ_x = n$. Bunch length effects depend on vertical motion and the vertical order of a resonance. Therefore, they can be ignored in the present analysis. Putting r = 0 gives the following expression

$$A_{p0}(I_x,I_y,k) = \frac{1}{C} T_{p0}(I_x,I_y,k) \sum_{n=-\infty}^{\infty} \delta(k - 2\pi n/C) \tag{5}$$

where

$$T_{p0}(I_x,I_y) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta_x e^{-ip\theta_x} \int_0^{2\pi} d\theta_y \int_0^{\infty} \frac{dq}{\sqrt{(2\sigma_{x0}^2 + q)(2\sigma_{y0}^2 + q)}}$$
$$\exp\left\{-\left[\frac{2\beta_x^* I_x \cos^2\theta_x}{2\sigma_{x0}^2 + q} + \frac{2\beta_y^* I_y \cos^2\theta_y}{2\sigma_{y0}^2 + q}\right]\right\} \ . \tag{6}$$

The average value of the perturbation gives the tunes as a function of action

$$Q_y = Q_{y0} - \xi_x \varepsilon_x \frac{\partial T_{00}}{\partial I_y}; \quad Q_x = Q_{x0} - \xi_x \varepsilon_x \frac{\partial T_{00}}{\partial I_x} \ , \tag{7}$$

where

$$\xi_x = \frac{N r_e}{2\pi\gamma\varepsilon_x} \tag{8}$$

is the horizontal beam-beam strength parameter and ε_x is the horizontal beam emittance. For resonance $pQ_x = n$, the detuning is

$$\Lambda_{p0} = p^2 \frac{\partial^2 T_{00}}{\partial I_x^2} \ . \tag{9}$$

A nonlinear resonance provides a fixed point. If the fixed point is stable, particles trapped by this resonance circulate around it. The resonances are usually characterized by the island tune and resonance width. The island tune is the rotation speed of the trapped particle inside the island. The resonance width is essentially the boundary of the stable region surrounding the fixed point. The standard way of calculating these resonance parameters is to perform a canonical transformation from the original action-angle variables to a new pair, K_1 and ψ_1, the action-angle coordinates of the resonance. After the canonical transformation, the island tune and resonance full width are given by

$$Q_r = \xi_x \varepsilon_x \sqrt{\left|2 T_{p0} \Lambda_{p0}\right|} \tag{10}$$

and

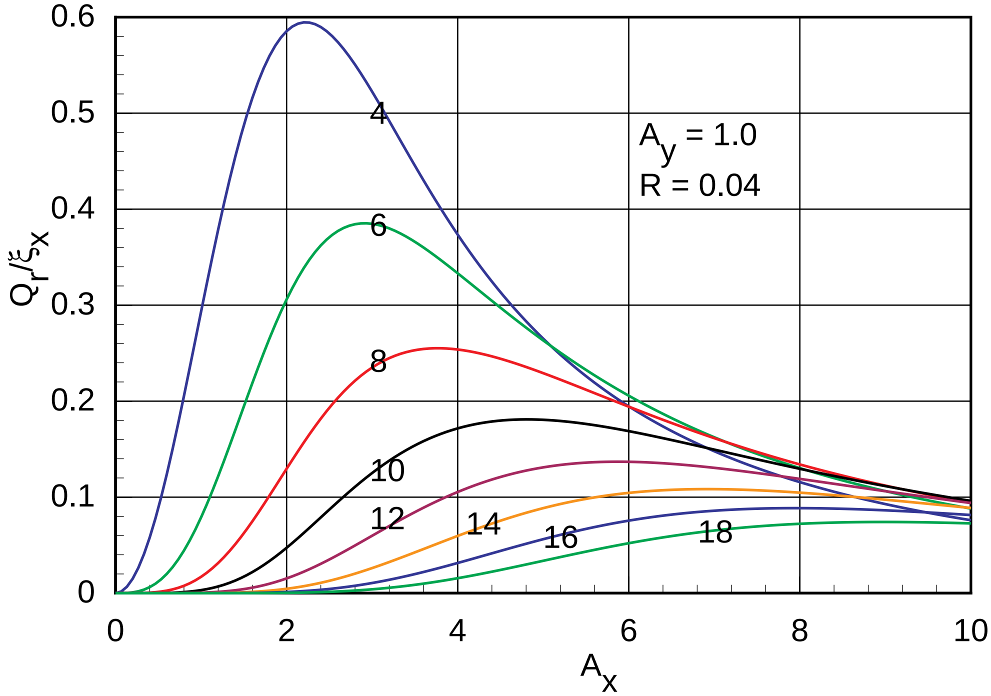

Figure 1: Island tunes normalized by horizontal tune shift for p = 4, ..., 18

$$\Delta K_1 = 4\sqrt{\left|\frac{2T_{p0}}{\Lambda_{p0}}\right|}, \tag{11}$$

respectively. In terms of normal amplitude $A_x = \sqrt{2I_x/\varepsilon_x}$, the resonance full width is[4]

$$\Delta A_x = \frac{\Delta I_x}{\varepsilon_x A_x} = \frac{p\Delta K_1}{\varepsilon_x A_x} . \tag{12}$$

III. Results

The island tunes for different resonance orders are given in the Figure 1 for $A_y = \sqrt{2I_y/\varepsilon_y} = 1$ (ε_y is the vertical emittance) and R = 0.04. As the figure shows, the island tune is large at large horizontal amplitudes even for high order resonances. For example, for A_x =6, Q_r ranges from $0.05\xi_x$ to $0.20\xi_x$. For a beam-beam strength parameter $\xi_x = 0.0455$ chosen for examples in this paper, $Q_r = 9.1\times10^{-3}$ to 2.3×10^{-3}. This should be compared with the inverse of a typical damping time, $1/\tau \sim 2\times10^{-4}$. A particle that falls inside this resonance circulates many times during a damping time; i.e. that particle will be transported from the minimum amplitude to the maximum amplitude of the resonance island in a time short compared to the damping time. If there is an aperture at an amplitude smaller than this maximum amplitude, the particle is lost. The effective aperture has been moved to the minimum amplitude of the resonance island!

The resonance width tells whether this is an important effect. If the resonance is narrow, the minimum and maximum amplitudes of the resonance are almost equal, and the effective aperture is not changed significantly. The full widths of resonances are plotted in Figure 2; they can be several units of amplitude.

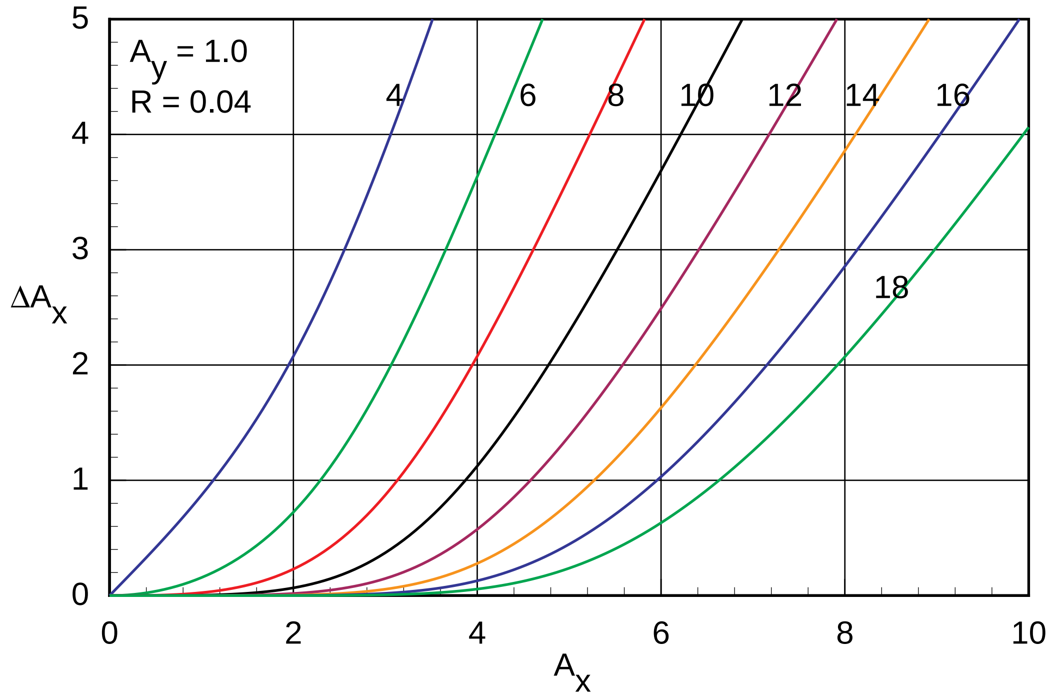

Figure 2: Full width of resonance islands for p = 4, ..., 18

IV. Simulations

Simulations have been performed to check the results of the analytical calculations. The simulation is a simple weak-strong model that consists of a linear transfer matrix and a beam-beam kick. To obtain the island tunes, a single particle is launched near the resonance center and tracked for $1024 \times p$ turns. Then, the island tune can be found by the FFT of the coordinates.

The resonances p = 8, ..., 16 are checked by simulation. For each resonance, the horizontal tune is set such that the resonance is located at $A_x \approx 7\sigma_x$. The vertical tune Q_y is chosen as 0.64, and the results are not sensitive to the choice. The horizontal beam-beam tune shift is $\xi_x = 0.0455$. The analytical calculation and simulation results are listed in Table 1 for comparison. The agreement is good for high order resonances but not good for low orders. The discrepancy may caused by the strength of the low order resonances which make first order perturbation theory a bad approximation. However, despite this discrepancy the conclusion that the island tune is much higher than radiation damping rate is still true.

Table 1: Comparison of island tunes from the analytical calculation and simulation

Resonance order	p=8	p=10	p=12	p=14	p=16
Q_{x0}	0.6236	0.5986	0.5819	0.57	0.5611
Analytical calculation	0.0073	0.0066	0.0059	0.0049	0.0041
Simulation	0.0047	0.00537	0.00545	0.00502	0.00439

The island width can be found by looking at the phase space plot and measuring the distance between the separatrices. Alternatively, the islands are evident in the beam tail distributions generated by the simulation program written to study beam-beam lifetime[5]. This program gives not only the island widths but also the effect on the beam distribution. Figure 3 gives the contour plots of distributions in amplitude space. The accelerator conditions are the same as mentioned previously, and the contour lines are lines of equal density on a logarithmic scale. One can easily see the bulge in the horizontal tail, which results from the resonance. In

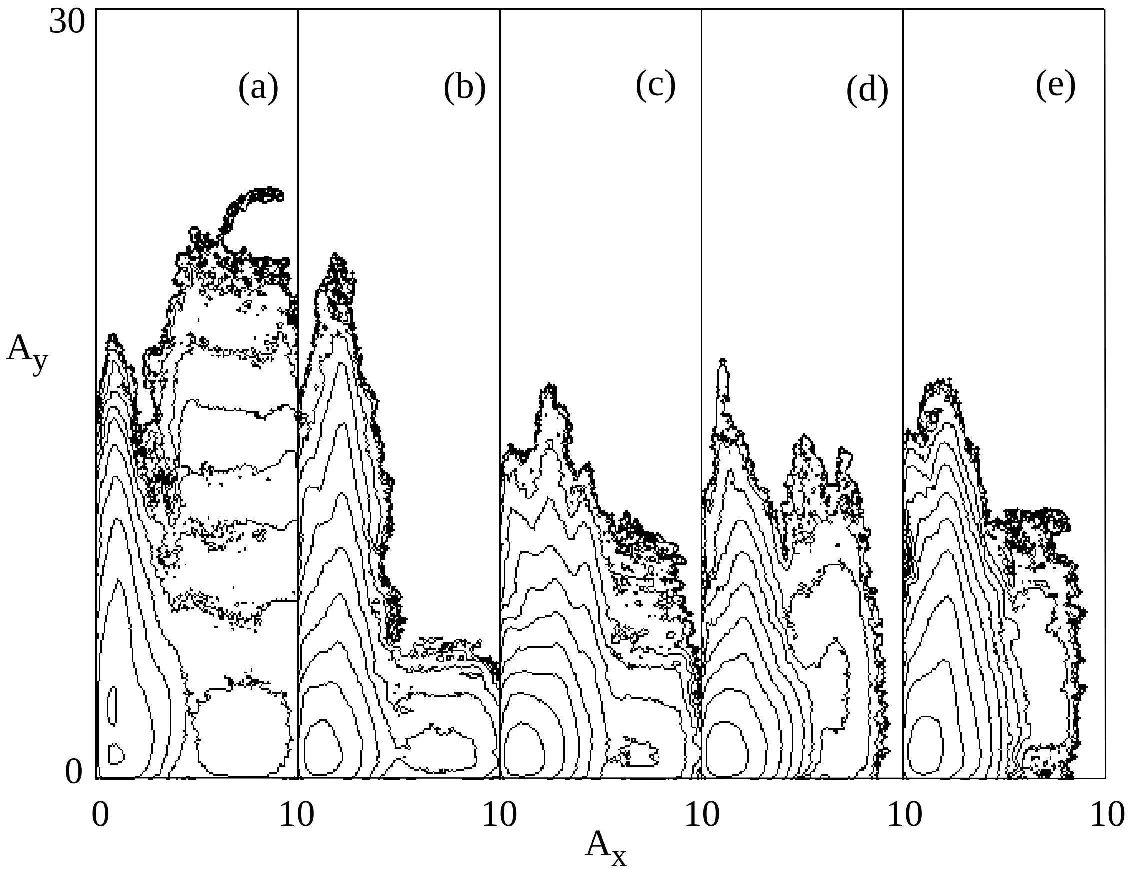

Figure 3: Beam tail distributions in transverse amplitude space: a) Q_{x0} = 0.6236 for p = 8 resonance, b) Q_{x0} =0 .5986 for p = 10 resonance, c) Q_{x0} = 0.5819 for p = 12 resonance, d) Q_{x0} = 0.57 for p = 14 resonance, and e) Q_{x0} = 0.5611 for p = 16 resonance.

Table 2, the analytical results of island full widths at $A_x = 7$ are listed for the different resonances. These widths are comparable to the width of the bulges in Figure 3. This comparison between the analytical calculation and simulation shows that the first order perturbation theory of the beam-beam interaction gives a good estimate of the island widths.

Table 2: Analytical calculation of island widths

Resonance order	p = 8	p = 10	p = 12	p = 14	p = 16
Full island width, ΔA_x	7.5	5.2	3.8	2.7	1.9

V. Discussion

These results can be interpreted as follows. Assume there is an aperture at $A_x = 10$ and that amplitudes $A_x \leq 6$ are populated sufficiently by statistical fluctuations to be important for the lifetime. Then if a resonance occurs such that

$$A_x(\text{minimum}) \leq 6 \text{ and } A_x(\text{maximum}) \geq 10 \tag{13}$$

the lifetime is reduced because a sufficient number of particles will be transported to the aperture by the resonance. The situation is illustrated using p = 12 in the Figure 4(a). If the resonance

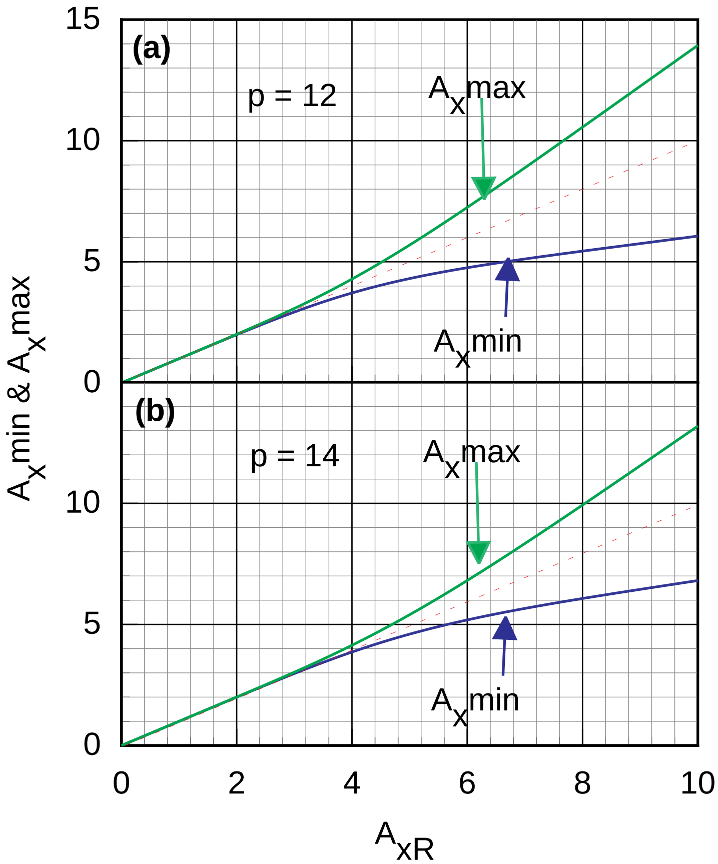

Figure 4: Minimum and maximum A_x as a function of A_{xR}, the resonance value of A_x, the value where (a) 12 $Q_x(A_{xR}) = n$, (b): 14 $Q_x(A_{xR}) = n$.

value of A_x , A_{xR}, is between $7.6 \leq A_{xR} \leq 10$, the lifetime is reduced. When $A_{xR} < 7.6$, the resonance affects the beam distribution, but particles are not transported out to the aperture at $A_x = 10$. When $A_{xR} > 10$, A_x(minimum) > 6, and there is no flux into the resonance. The tune shifts for these two values of A_x are ΔQ_x (A_{xR} = 7.6) = $0.036\xi_x$ and ΔQ_x (A_{xR} = 10) = $0.02\xi_x$. The relatively narrow tune region

$$\frac{n}{12} - 0.036\xi_x \leq Q_{x0} \leq \frac{n}{12} - 0.02\xi_x \tag{14}$$

will have unacceptably short lifetime.

The maximum and minimum horizontal amplitudes for resonance p = 14 are plotted in Figure 4(b); the lifetime is acceptable for all values of A_x(resonance). When $A_{xR} < 8$, A_x(maximum) < 10, and particles are not lost. When $A_{xR} > 8$, A_x(minimum) > 6, and there is not significant flux into the resonance. The resonances of importance are twelfth order and below. Table 3 gives the tune regions of unacceptable lifetime due to horizontal resonances.

A set of stop bands has been introduced by the beam-beam interaction. The resonance orders are high, so there are a large number of stopbands. However, they are narrow and can be avoided by an appropriate choice of horizontal tune. This is illustrated in Figure 5 for A_x(maximum) = 10. These results are based on linear lattice, and in the presence of lattice nonlinearities, the results can be significantly different. For example, the tune shift with amplitude due to lattice can change the detuning and distort the resonance bucket either increasing or decreasing resonance widths. Studying this interplay between the beam-beam

Table 3: Tune Regions of Reduced Lifetime for A_x(minimum) = 6

Resonance order	Aperture, A_x(maximum)	Unacceptable resonance center location	Unacceptable tune stopbands
p = 4	8	$4.4 \leq A_{xR} \leq \infty$	$n/4 - 0.107\xi_x \leq Q_{x0} \leq n/4$
	10	$5.4 \leq A_{xR} \leq \infty$	$n/4 - 0.070\xi_x \leq Q_{x0} \leq n/4$
	12	$6.3 \leq A_{xR} \leq \infty$	$n/4 - 0.051\xi_x \leq Q_{x0} \leq n/4$
p = 6	8	$5.1 \leq A_{xR} \leq \infty$	$n/6 - 0.079\xi_x \leq Q_{x0} \leq n/6$
	10	$6.2 \leq A_{xR} \leq \infty$	$n/6 - 0.052\xi_x \leq Q_{x0} \leq n/6$
	12	$7.2 \leq A_{xR} \leq \infty$	$n/6 - 0.040\xi_x \leq Q_{x0} \leq n/6$
p = 8	8	$5.7 \leq A_{xR} \leq \infty$	$n/8 - 0.064\xi_x \leq Q_{x0} \leq n/8$
	10	$6.7 \leq A_{xR} \leq \infty$	$n/8 - 0.047\xi_x \leq Q_{x0} \leq n/8$
	12	$7.8 \leq A_{xR} \leq \infty$	$n/8 - 0.033\xi_x \leq Q_{x0} \leq n/8$
p = 10	8	$6.1 \leq A_{xR} \leq \infty$	$n/10 - 0.055\xi_x \leq Q_{x0} \leq n/10$
	10	$7.2 \leq A_{xR} \leq \infty$	$n/10 - 0.040\xi_x \leq Q_{x0} \leq n/10$
	12	$8.4 \leq A_{xR} \leq \infty$	$n/10 - 0.030\xi_x \leq Q_{x0} \leq n/10$
p = 12	8	$6.5 \leq A_{xR} \leq 10$	$n/12 - 0.049\xi_x \leq Q_{x0} \leq n/12 - 0.020\xi_x$
	10	$7.7 \leq A_{xR} \leq 10$	$n/12 - 0.035\xi_x \leq Q_{x0} \leq n/12 - 0.020\xi_x$
	12	$8.8 \leq A_{xR} \leq 10$	$n/12 - 0.027\xi_x \leq Q_{x0} \leq n/12 - 0.020\xi_x$
p = 14	8	$6.8 \leq A_{xR} \leq 8$	$n/14 - 0.045\xi_x \leq Q_{x0} \leq n/14 - 0.032\xi_x$
	10	No limit	No stopbands
	12	No limit	No stopbands
$p \geq 16$	≥ 8	No limit	No stopbands

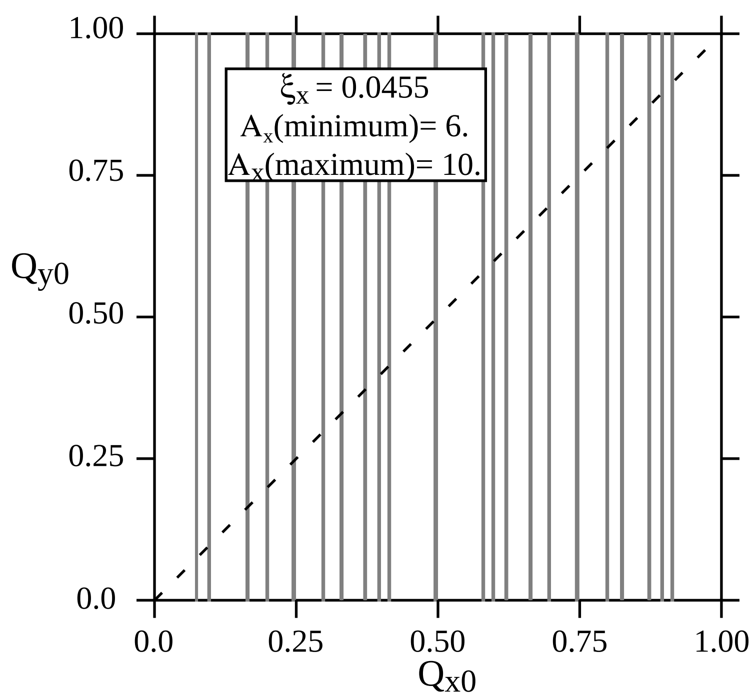

Figure 5: The shaded regions of the tune plane have reduced lifetime due to horizontal beam-beam resonances with p = 4, ..., 12.

interaction and lattice nonlinearities will be an important new direction for understanding collider luminosity limits.

VI. Acknowledgements

This work was supported by the Department of Energy, Contract No. DE-AC03-76SF00515.

VII. References

[1] F. M. Izrailev and I. B. Vasserman, Proceedings of the 7th All Union Conference on Charged Particle Accelerators, Dubna, USSR, 288 (1980).
[2] S. Krishnagopal and R. H. Siemann, Phys. Rev. D **41**, 2312 (1990).
[3] R. H. Siemann, Frontiers of Particle Beams: Factories with e^+e^- Rings (Springer-Verlag, Lecture Notes in Physics, Volume 425 edited by M. Dienes, M. Month, B. Strasser and S. Turner), 327 (1994).
[4] A. L. Gerasimov, F. M. Izrailev, and J. L. Tennyson, AIP Conf Proceedings 153, 474 (1987); appendix C.
[5] T. Chen, J. Irwin, and R. H. Siemann, Phys. Rev. E **49**, 2323 (1994).