B. **Gravitational Waves**

Unfortunately, all gravitational waves will be weak when they reach our detectors. The vacuum weak-field equations $\Box\bar{h}^{\mu\nu} = 0$ and Lorentz gauge condition $\bar{h}_{\mu}{}^{\sigma}{}_{,\sigma} = 0$ allow a representation in terms of plane monochromatic waves

$$\bar{h}_{\mu\nu} = A_{\mu\nu}e^{ik_\alpha x^\alpha} \; ; \quad A_{\mu\sigma}k^\sigma = 0, \; k_\sigma k^\sigma = 0 \; ,$$

with wave vector $\mathbf{k}$. However, both the field equations and Lorentz gauge condition are preserved under another infinitesimal coordinate (gauge) transformation if the generator also satisfies the wave equation $\Box\xi_\mu = 0$. One can then use the four additional degrees of freedom to set $\bar{h}_{0i} = 0$ and $\bar{h} = 0$ (so now $\bar{h}_{\mu\nu} = h_{\mu\nu}$). In summary, we have constructed the transverse-traceless (TT) gauge, in which the eight independent conditions

$$h_{\mu 0} = h_{jk,k} = h^k{}_k = 0$$

leave two independent polarization states, again in direct analogy with electrodynamics.

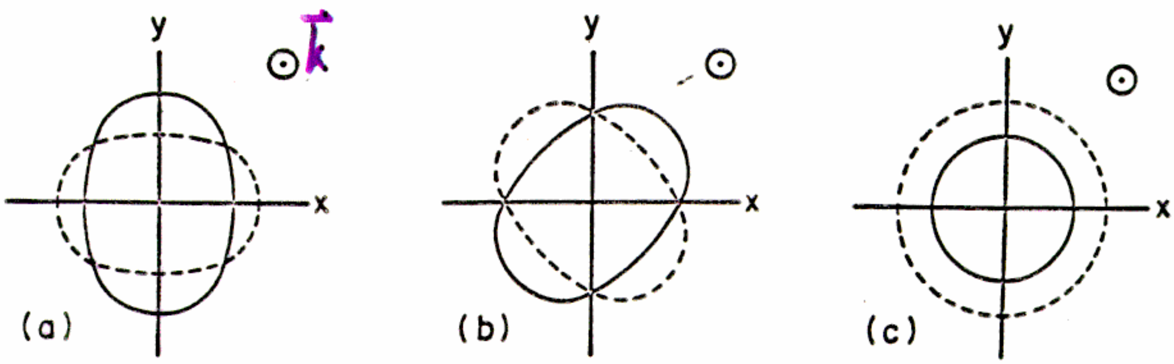
We can also include the possibility of a weak scalar wave $\varphi_1 \equiv \varphi - \varphi_0$, since the tensor field equations (5) are unaffected through first order in φ_1 (except that $h_{\mu\nu} \to \hat{h}_{\mu\nu}$ in the above equations). The scalar field equation (6) becomes $\Box\varphi_1 = 0$, giving the same plane-wave representation.

To understand the response of a gravitational-wave detector, consider slowly moving free test particles whose separation is much less than the gravitational wavelengths involved. Now employ a local Lorentz frame, in which physical (e. g., radar) and coordinate distances are equal through first order in the particle separation $\mathbf{\Delta x}$. The equation of geodesic deviation becomes $d^2\Delta x^i/d\tau^2 \cong -R^i{}_{0j0}\Delta x^j$ as before, and involves only the matter coupling metric $\mathbf{g} = A^2(\varphi)\hat{\mathbf{g}}$. In terms of our spin representation,

$$R_{i0j0} = \hat{R}_{i0j0} + a_1(\varphi_{,ij} - \delta_{ij}\varphi_{,00}) \, .$$

In the previous TT gauge, one obtains $\hat{R}^{TT}_{i0j0} = -\frac{1}{2}\hat{h}^{TT}_{ij,00}$. However, the gauge (coordinate) invariance of the weak-field Riemann tensor allows us to use this expression in the above equation of geodesic deviation, giving

$$d^2\Delta x^i/d\tau^2 \cong [\tfrac{1}{2}\hat{h}^{TT}_{ij,00} + a_1(\delta_{ij}\varphi_{,00} - \varphi_{,ij})]\Delta x^j \, .$$



In the figure above we show the resulting positions of an initially circular ring of test particles (at phases $\pi/2$ and $3\pi/2$) for each polarization state: (a) $\hat{h}^{TT}_{xx} = -\hat{h}^{TT}_{yy}$, (b) $\hat{h}^{TT}_{xy} = \hat{h}^{TT}_{yx}$, (c) φ_1. They remain in the plane transverse to the propagation vector $\vec{k}$ shown.

For separations $\vec{\Delta x} \parallel \vec{k}$, the above equation also shows that there is no response to any of the three wave components.

Strong-Field Applications

A. Compact Objects

Throughout any spherically symmetric spacetime, we can choose Schwarzschild coordinates, in which the interval assumes the form

$$ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2] ,$$

so that the proper area (measured by local observers) of any spherical surface is $4\pi r^2$. We consider here isolated bodies, so the metric potentials $\Phi, \lambda \to 0$ as $r \to \infty$. In addition, we shall consider static ($U^i = 0$, $\partial/\partial t = 0$) bodies.

Then the only non-trivial momentum conservation equation $T_{r\ ;\sigma}^{\ \sigma} = 0$ gives hydrostatic equilibrium:

$$\frac{dp}{dr} = -(\rho + p)\frac{d\Phi}{dr} ,$$

also indicating that $\Phi(r)$ is the generalized Newtonian potential. The only structural difference from the Newtonian equation is the addition of the pressure to the inertial mass-energy density.

The $\{tt\}$ component of the Einstein field equation gives, after an integration,

$$e^{2\lambda} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1} , \quad m(r) = 4\pi \int_0^r \rho(r_*)r_*^2 dr_* .$$

Let $r = R$ be the radius of the star, defined by $p(R) = 0$. Then $m(r \geq R) = M$, the total gravitational mass (defined by applying Kepler's third law to distant orbits).

The $\{rr\}$ component of the Einstein field equation gives the generalization of the Newtonian field equation:

$$\frac{d\Phi}{dr} = G\frac{(m + 4\pi r^3 p)}{r(r - 2Gm)} .$$

We see that pressure also contributes to the gravitational mass-energy density, and strong fields ($2Gm/r \to 1$) also steepen the potential (and pressure) gradient. The $\{\theta\theta\}$ and $\{\phi\phi\}$ components of the field equation are redundant.

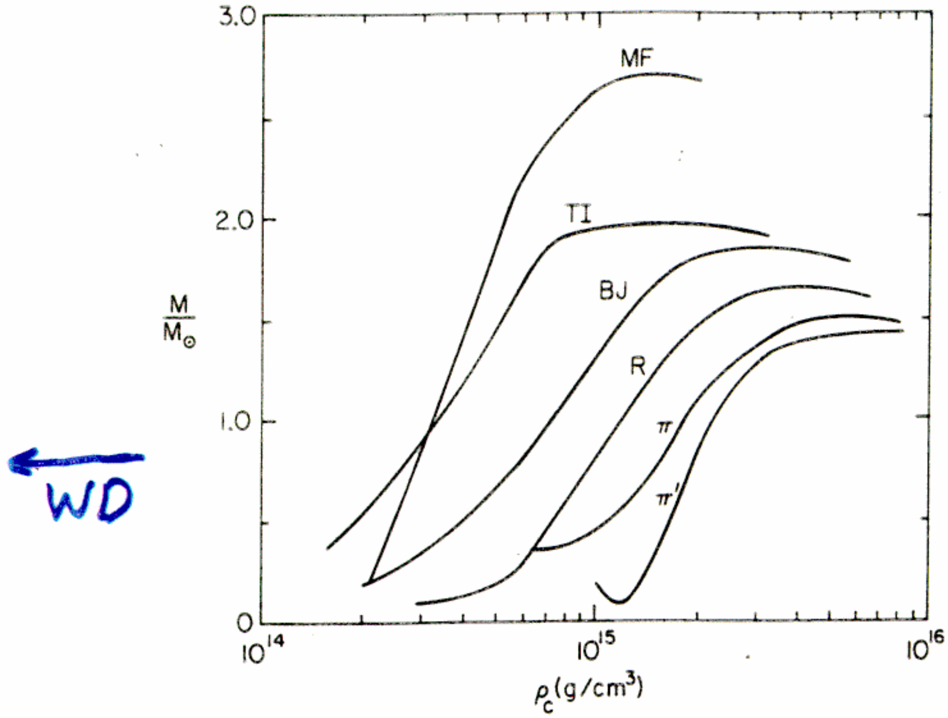
Finally, if the equation of state assumes the form

$$p = p(\rho, s(\rho)) \quad \text{(specific entropy } s \text{ determined separately)} ,$$

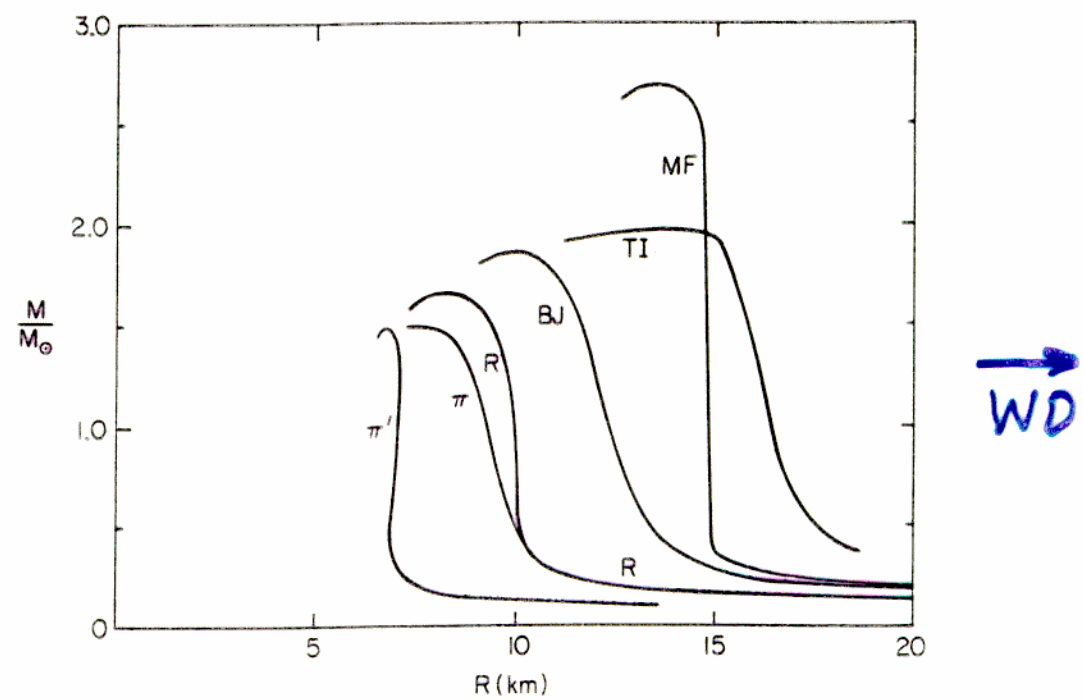
we have four equations to determine Φ, m(or λ), p, and ρ. (For the major applications, white dwarfs and neutron stars, the entropy effectively vanishes.) Continuity of $\Phi(r)$ and $\lambda(r)$ at the stellar surface allows matching to the exterior (or black hole) Schwarzschild solution

$$e^{2\Phi} = e^{-2\lambda} = 1 - \frac{2GM}{r} ,$$

also obtained from the above equations. With the additional boundary condition $m \propto r^3 (r \to 0)$, a set of stellar models is then a one-parameter $[p(0)$ or $\rho(0)]$ family.



Gravitational mass (in solar units) versus central density for a variety of equations of state. The rising portions of the curves represent stable neutron stars.



Gravitational mass (in solar units) versus radius for the same equations of state.

B. **Orbits**

One of the most powerful probes of the strong gravitational fields near neutron stars and black holes is analysis of the orbits of test particles (or those comprising gaseous accretion disks), here taken to have rest mass $m > 0$. Within the Schwarzschild geometry, we can define the orbital plane as $\theta = \pi/2$. From equation (1), with $p^\mu = m dx^\mu/d\tau$, we then see that both the energy and angular momentum (per unit mass)

$$\tilde{E} = -\frac{p_t}{m} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} , \qquad \tilde{L} = \frac{p_\phi}{m} = r^2 \frac{d\phi}{d\tau} ,$$

are conserved along each orbit. The relation $g_{\mu\nu} p^\mu p^\nu = -m^2$ then gives us the remaining (energy) equation

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(\tilde{L}, r) , \qquad \tilde{V}^2 = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) .$$

As in Newtonian theory, we can understand the orbits via plots of the effective potential $\tilde{V}$, shown below for $r > 2GM$ (black hole horizon).

Note that a particle is captured by the black hole if its specific angular momentum is low enough ($\tilde{L} < 2\sqrt{3}GM$) or its specific energy is high enough ($\tilde{E} > \tilde{V}_{max}$).

Correspondingly, stable circular orbits only exist for

$$\tilde{L} > 2\sqrt{3}GM \text{ , at } r > 6GM \text{ .}$$

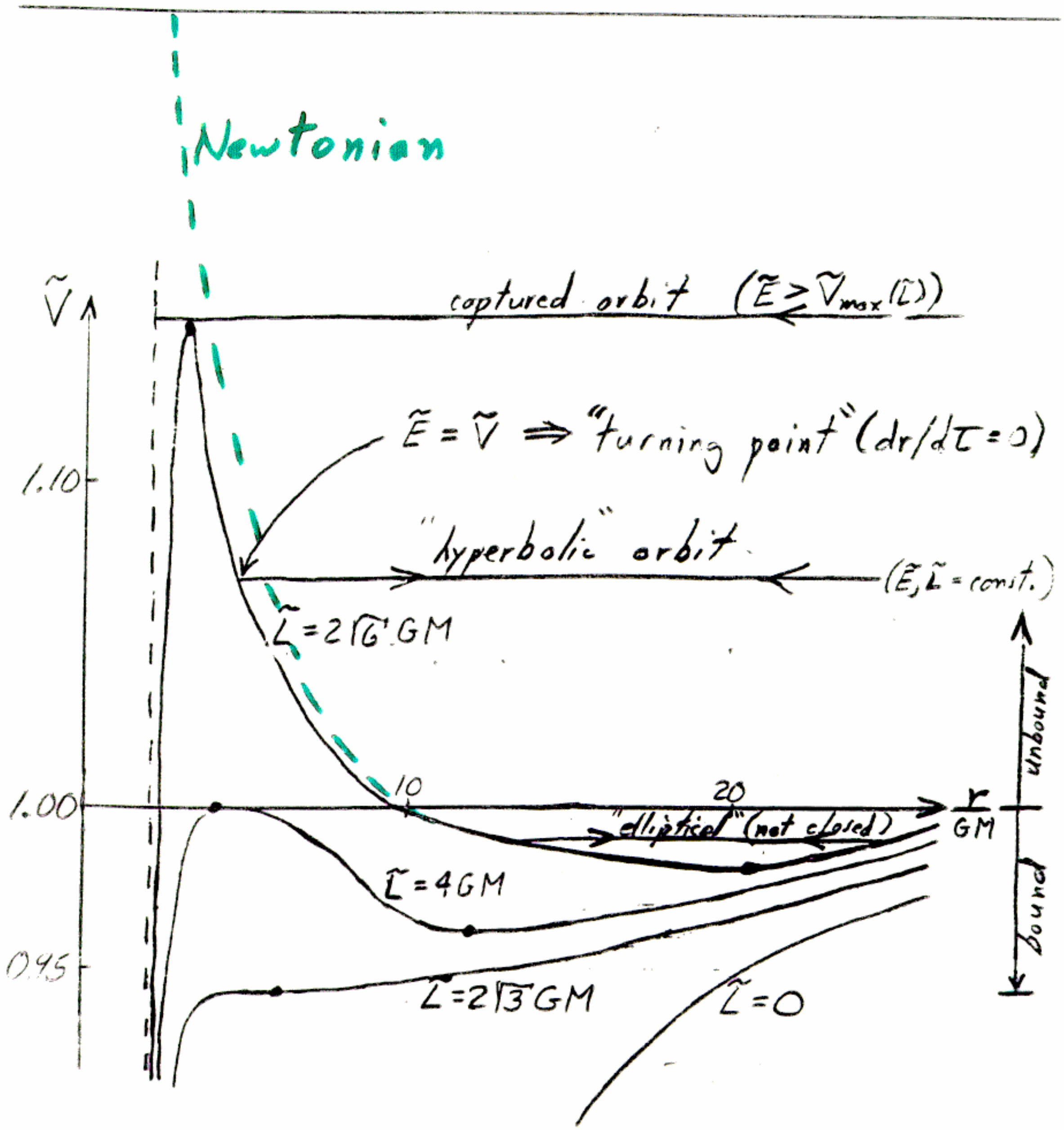


Figure 1. The effective potential $\tilde{V}$ is plotted for various choices of $\tilde{L}$. Also shown are the three classes of orbits (constant $\tilde{E}, \tilde{L}$).

For rotating (Kerr) black holes, the metric tensor is much more complicated, but effective potentials can still be obtained for orbits with angular momentum parallel (or anti-parallel) to that (**J**) of the black hole. These are shown below.

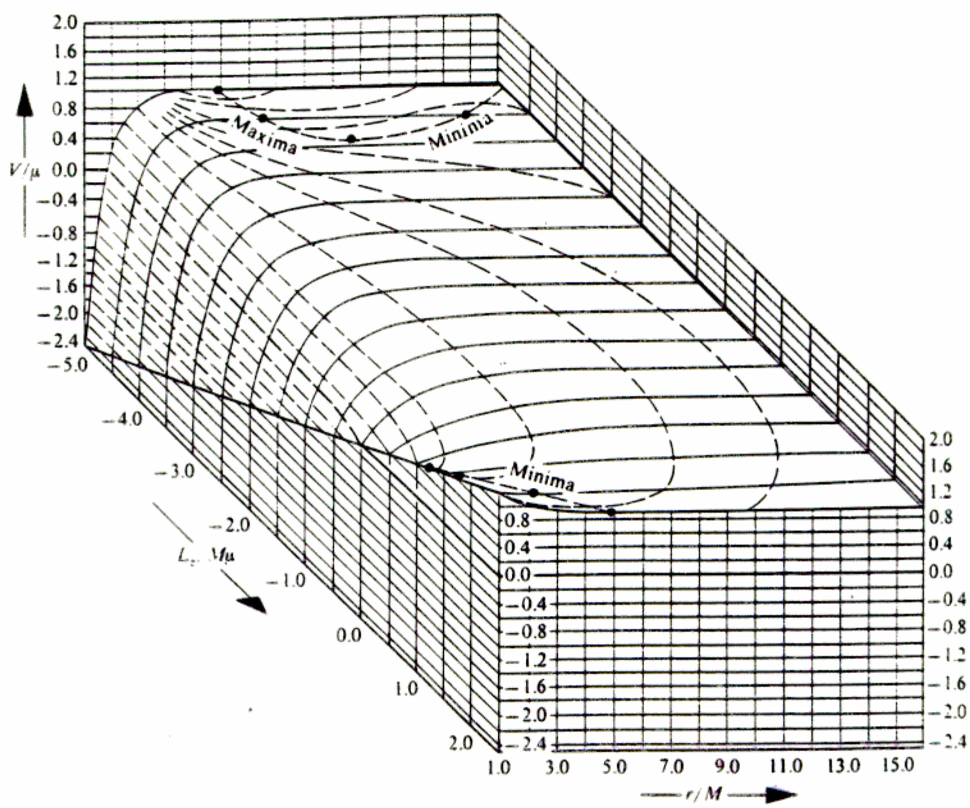
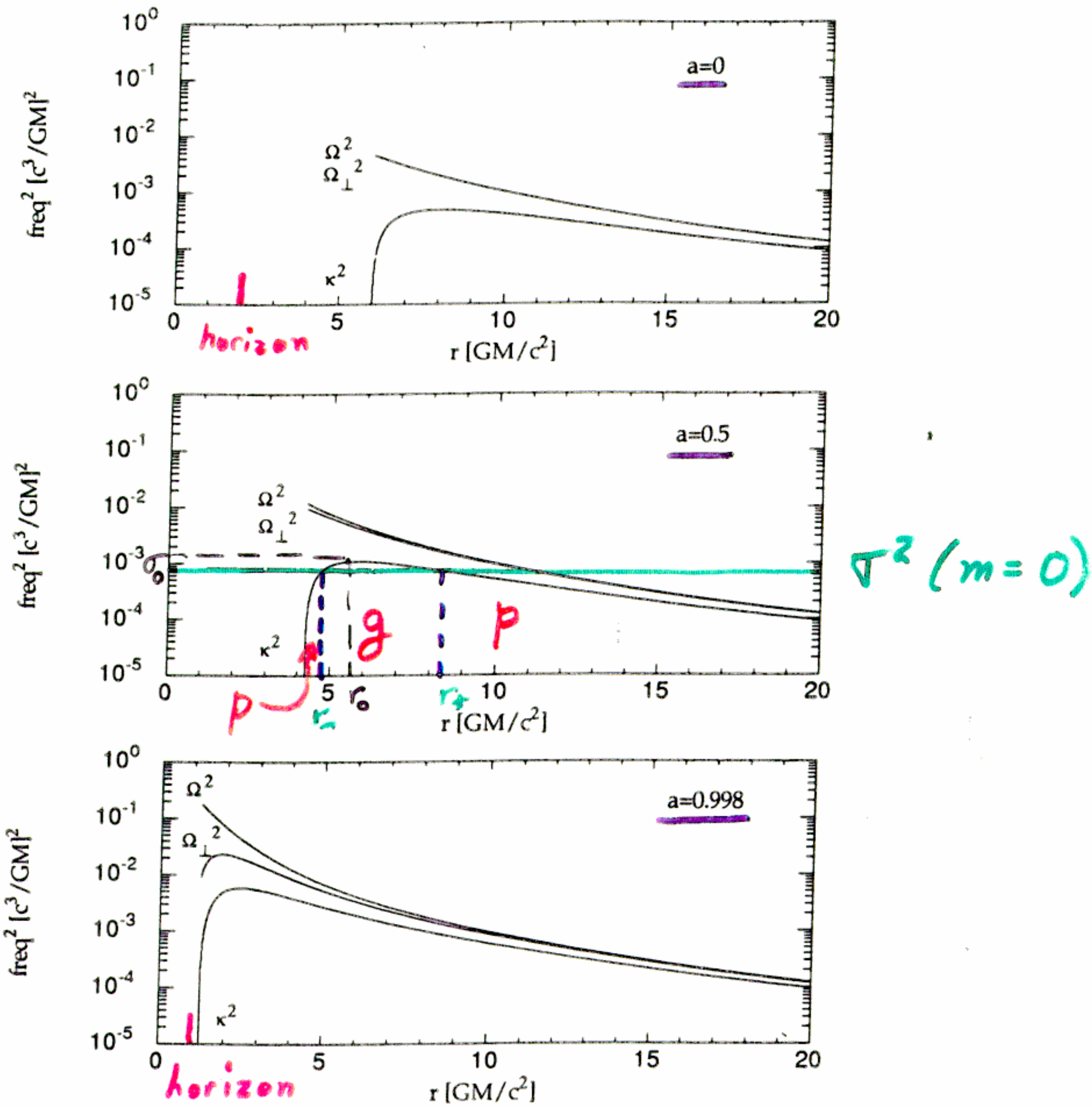


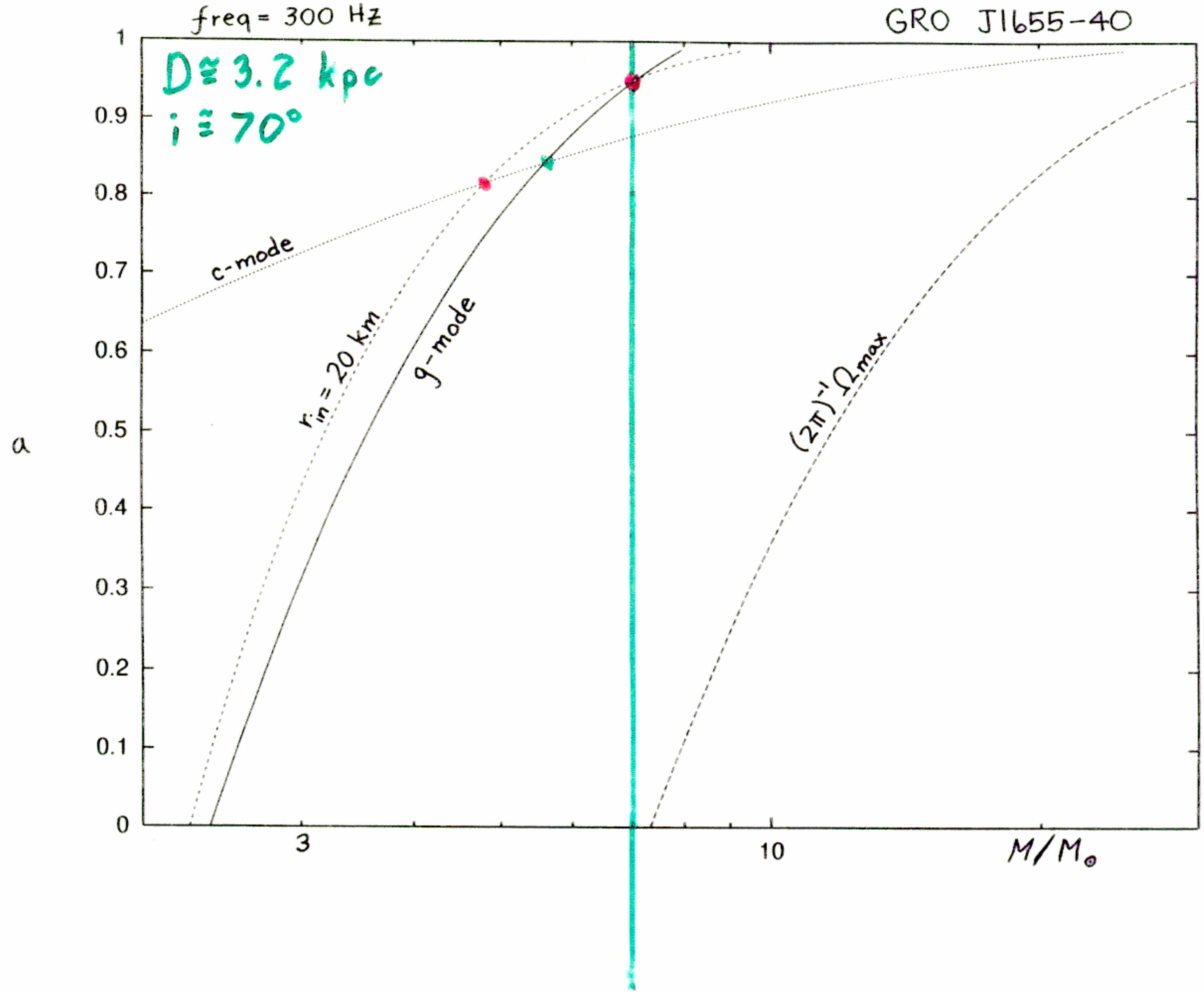
Figure 2. Plots of $\tilde{V}(r)$ for various values of $\tilde{L}/GM$, for a maximally rotating black hole ($J = GM^2$). (From MTW.)



Basis of relativistic diskoseismology

(AS, DL, MN, RW)

[redacted]—The radial dependence of the square of the key frequencies characterizing the disk: Keplerian (Ω), and radial (κ) and vertical ($\Omega_\perp$) epicyclic. Three values of the black hole angular momentum parameter $a = cJ/GM^2$ are chosen.


freq = 300 Hz
GRO J1655-40
D ≅ 3.2 kpc
i ≅ 70°
c-mode
r_{in} = 20 km
g-mode
$(2\pi)^{-1}\Omega_{max}$
a
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
3
10
$M/M_\odot$

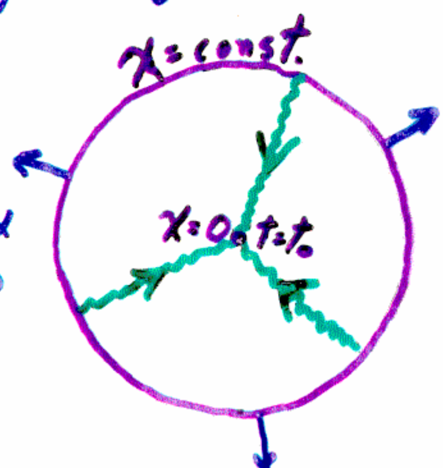
COSMOLOGY

- **Principles**

Copernican Principle + Observed isotropy

$\Rightarrow$ Universal Isotropy $\Rightarrow$ Homogeneity (Cosmological Principle)

- **Coordinates**



$\left.\begin{array}{l}\text{Comoving spatial}\\ \text{Proper time}\end{array}\right\} \Rightarrow U^\alpha = \delta^\alpha_0$

- **Metric** ($\Leftarrow$ CP)

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \Sigma^2(\chi)\, d\Omega^2\right]$$

$$\Sigma = \begin{cases} \sin\chi, & k = +1 \\ \chi, & 0 \\ \sinh\chi, & -1 \end{cases}$$

Curvature: $K = \dfrac{k}{a^2}$

- **Photons**

Physical (radar) distance: $d\ell = a(t)\, d\chi$

Redshift: $1 + z \equiv \dfrac{\lambda_0}{\lambda} = \dfrac{a(t_0)}{a(t)}$

(emitted) $\rightarrow \lambda$

- Stress-Energy Tensor ($\Leftarrow$ CP)

$$T_{\mu\nu} = (\rho_* + p_*) U_\mu U_\nu + p_* g_{\mu\nu}$$

$\uparrow$ Metric theories

- Einstein Field Equations

(a) $\frac{1}{2}\left(\frac{da}{dt}\right)^2 + U(a) = -\frac{k}{2}$ $\quad \left\{ U = -\frac{a^2}{6}(8\pi G\rho_* + \Lambda) \right\}$

(b) $\frac{d}{dt}(\rho_* a^3) + p_* \frac{d}{dt}(a^3) = 0$ $\quad \left[\begin{array}{l} p_* = w\rho_* \\ \Rightarrow \rho_* \propto a^{-3(1+w)} \end{array} \right]$

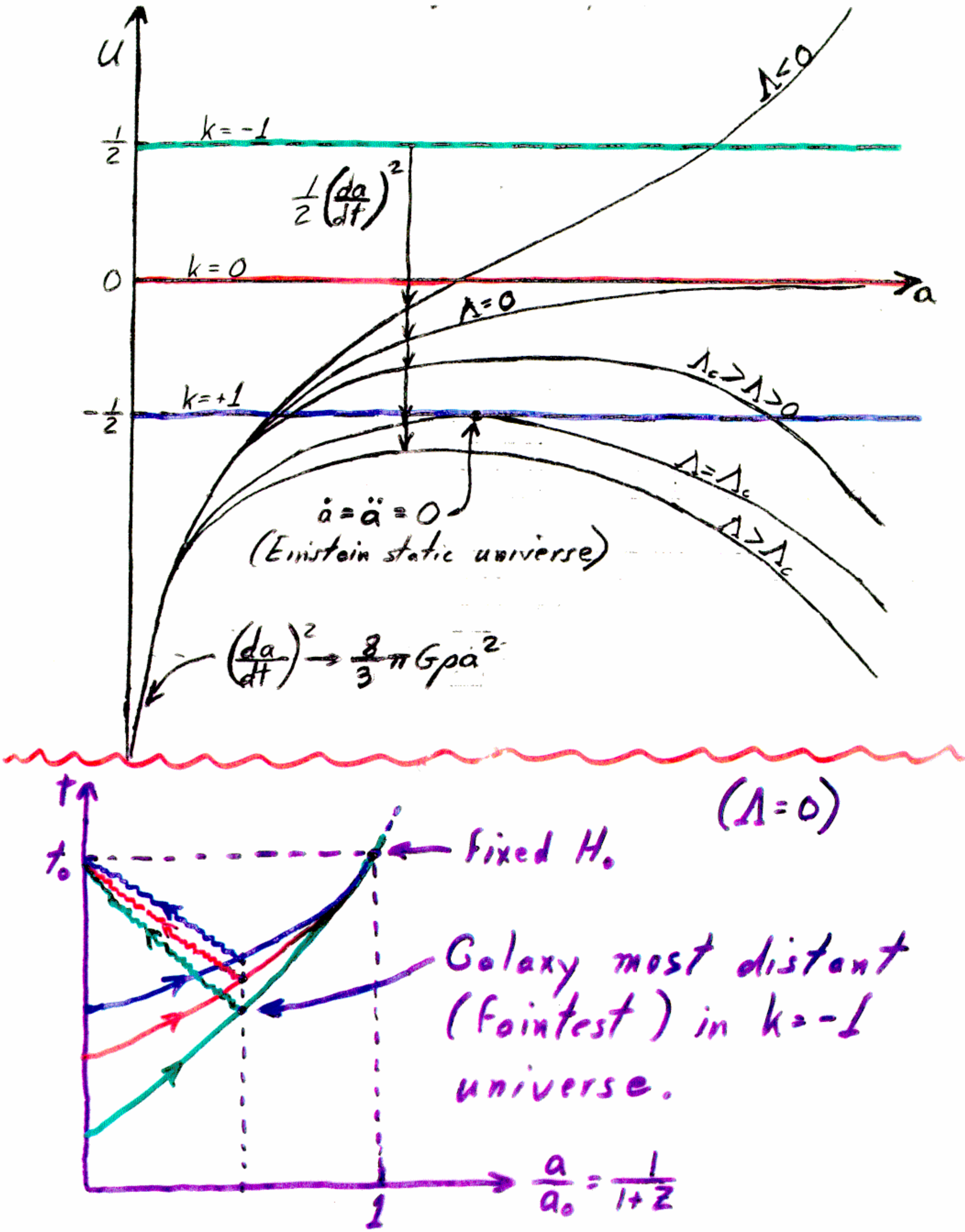
(Conservation of (a) macroscopic + (b) microscopic energy)

<u>Evaluated today</u> ($p_* \ll \rho_* = \rho_0$)

(a) $\Rightarrow$ $\underline{\Omega_0 + \Omega_k + \Omega_\Lambda = 1}$ $\quad \left[H_0 = \left(\frac{1}{a}\frac{da}{dt}\right)_0 \right]$

$$\left[\Omega_0 \equiv \frac{8\pi G\rho_0}{3H_0^2},\ \Omega_k \equiv \frac{-k}{(a_0 H_0)^2},\ \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2} \right]$$

(a)+(b) $\Rightarrow \left(\frac{1}{a}\frac{d^2a}{dt^2}\right)_0 = H_0^2\left[\Omega_\Lambda - \frac{1}{2}\Omega_0\right)$



U
$\frac{1}{2}$
$k=-1$
$\frac{1}{2}\left(\frac{da}{dt}\right)^2$
0
$k=0$
a
$\Lambda<0$
$\Lambda=0$
$\Lambda_c>\Lambda>0$
$-\frac{1}{2}$
$k=+1$
$\Lambda=\Lambda_c$
$\Lambda>\Lambda_c$
$\dot{a}=\ddot{a}=0$
(Einstein static universe)
$\left(\frac{da}{dt}\right)^2 \to \frac{8}{3}\pi G\rho a^2$
$(\Lambda=0)$
t
t_0
Fixed H_0
Galaxy most distant (faintest) in $k=-1$ universe.
$\frac{a}{a_0}=\frac{1}{1+z}$
1