Geodesic Deviation $\Longleftrightarrow$ Riemann Tensor

Consider two freely falling test particles with infinitesimal separation $\Delta x^\alpha \mathbf{e}_\alpha$. Subtracting their (geodesic) equations of motion (2) gives ΔQ^α:

$$\frac{d^2\Delta x^\alpha}{d\lambda^2} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}_{,\sigma} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Delta x^\sigma + 2\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{d\lambda}\frac{d\Delta x^\nu}{d\lambda} = 0\,.$$

Now, since $D\Delta x^\alpha/D\lambda$ (but not $d\Delta x^\alpha/d\lambda$) are the components of a vector, so is the result of applying the operator $D/D\lambda$ [defined by equation (3)] again. Employing the above equation, this operation produces the *equation of geodesic deviation*

$$\frac{D^2\Delta x^\alpha}{D\lambda^2} + R^\alpha{}_{\mu\sigma\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Delta x^\sigma = 0\,. \tag{4}$$

Since both terms in this equation are the components of vectors, the quantities

$$R^\alpha{}_{\mu\sigma\nu} = \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}_{,\sigma} - \begin{Bmatrix} \alpha \\ \mu\sigma \end{Bmatrix}_{,\nu} + \begin{Bmatrix} \alpha \\ \sigma\tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \mu\nu \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \tau\nu \end{Bmatrix}\begin{Bmatrix} \tau \\ \mu\sigma \end{Bmatrix}$$

must be the components of a (rank 4) tensor, called the Riemann (curvature) tensor. It plays a role similar to that of the electromagnetic field tensor $F_{\mu\nu}$ in the extension of the equation of motion (2) to charged ($m > 0$) test particles:

$$\frac{D}{D\tau}\left(\frac{dx^\alpha}{d\tau}\right) = \frac{q}{m}F^\alpha{}_\sigma\frac{dx^\sigma}{d\tau}\,,$$

the generalized Lorentz force equation. We see, however, that the Riemann tensor represents the physical field *gradients* (tidal forces), and only *relative* gravitational acceleration has physical meaning.

The symmetry properties of the Riemann tensor (analogous to $F_{\mu\nu} = F_{[\mu\nu]}$) are

$$R_{\alpha\beta\mu\nu} = R_{([\alpha\beta][\mu\nu])} , \qquad R_{[\alpha\beta\mu\nu]} = 0 ,$$

the first giving $6 \cdot 7/2$ independent components and the second giving one less, for a total of 20 independent components. Its (unique) contractions are

$$R_{\mu\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} , \qquad R = g^{\mu\nu} R_{\mu\nu} ,$$

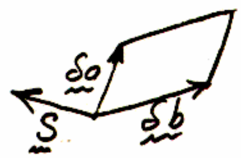
the components of the (symmetric) Ricci tensor and the Ricci scalar. It also obeys the Bianchi identities $R_{\alpha\beta[\mu\nu;\zeta]} = 0$ (analogous to the Maxwell equations $F_{[\mu\nu;\zeta]} = 0$), and their (unique) double contraction

$$G^{\mu\nu}{}_{;\nu} = 0 \qquad (G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) ,$$

which involves the components of the Einstein tensor $\mathbf{G}$.

Parallel Transport Around a Closed Curve

Consider an infinitesimal area, generated by displacement vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$.
Parallel transport a vector $\mathbf{S}$ around the boundary (clockwise), using



$$\nabla_{\mathbf{u}}\mathbf{S} = 0 \quad \left(\frac{dS^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} S^\alpha \frac{dx^\beta}{d\lambda} = 0\right), \quad \mathbf{u} = \frac{dx^\alpha}{d\lambda}\mathbf{e}_\alpha$$

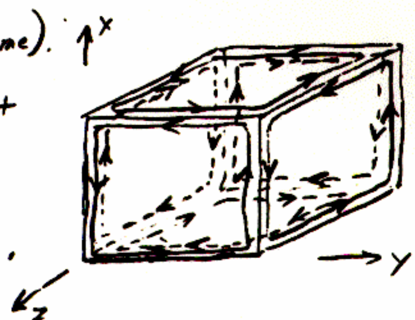
It then follows (Schutz, p. 167–169) that the change in $\mathbf{S}$ after one circuit is given by

$$\Delta S^\mu = -R^\mu{}_{\alpha\beta\gamma} S^\alpha \delta a^\beta \delta b^\gamma .$$

(Can be seen by transporting a vector on a globe.)

"The Boundary of a Boundary Vanishes" ⇒ Bianchi Identity.

Transport a vector $\mathbf{S}$ around all six faces of an infinitesimal cube (in a local Lorentz frame).



The two faces x = constant and $x + \delta x$ = constant give a change

$$-\Delta S^\mu = (\partial R^\mu{}_{\alpha yz}/\partial x) S^\alpha \delta y \delta z \delta x .$$

Since all portions of the path cancel in pairs, this will give $\Delta S^\mu = 0$. So adding the contribution from the other faces,

$$0 = (R^\mu{}_{\alpha yz;x} + R^\mu{}_{\alpha zx;y} + R^\mu{}_{\alpha xy;z}) S^\alpha \delta x \delta y \delta z$$
$$= 3 R^\mu{}_{\alpha[xy;z]} .$$

Gravitational Field Equations

We now complete the implementation of the extremal action principle by adding to the matter Lagrangian density $\mathcal{L}_M$ (specified above) a Lagrangian density $\mathcal{L}_G$ which depends solely on the gravitational field(s). Adopting the principle of simplicity that has worked so well in deriving the laws of physics, we are tempted to include nature's simplest (scalar, spin 0) field φ in addition to the metric tensor field $\mathbf{g}$. In analyzing the field equations, more insight is gained by employing the 'spin representation', in which $\hat{\mathbf{g}}$ denotes the metric tensor which corresponds to pure spin 2, while $\mathbf{g} = A^2(\varphi)\hat{\mathbf{g}}$ is the metric tensor discussed above through which gravity couples to matter.

If we require only that the field equations be of at most second differential order, the most general Lagrangian density is then

$$\mathcal{L}_G = (16\pi G)^{-1}[\hat{R} - 2\hat{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - 2\Lambda(\varphi)] ,$$

where G is the bare gravitational constant. Thus there are two free functions in this theory: the matter coupling function $A(\varphi)$ and the 'cosmological function' $\Lambda(\varphi)$. Extremizing the action with respect to variations in $\hat{g}_{\mu\nu}$ and φ then gives the field equations

$$\hat{G}_{\mu\nu} = 8\pi G\hat{T}_{\mu\nu} - \hat{g}_{\mu\nu}\Lambda(\varphi) + 2\varphi_{,\mu}\varphi_{,\nu} - \hat{g}_{\mu\nu}\hat{g}^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} \quad (5)$$

$$\hat{g}^{\mu\nu}\varphi_{,\mu;\nu} = -4\pi G\alpha(\varphi)\hat{T} + \tfrac{1}{2}d\Lambda/d\varphi , \quad (6)$$

[range = $(2\Lambda'')^{-1/2}$]

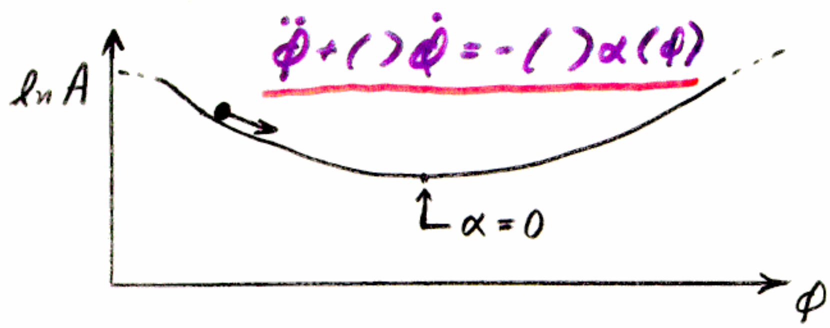
where the related coupling function

$$\alpha(\varphi) \equiv d\ln A/d\varphi = a_1 + a_2(\varphi - \varphi_0) + \cdots$$

if one expands about the present cosmological value φ_0 of the scalar field. The stress-energy tensor $\hat{\mathbf{T}}$ is defined with respect to variations of $\hat{\mathbf{g}}$, and obeys the modified conservation laws

$$\hat{T}_\mu{}^\nu{}_{;\nu} = \alpha(\varphi)\hat{T}\varphi_{,\mu} \ .$$

If the coupling function $A(\varphi)$ has a minimum, Damour & Nordtvedt (1993) and Santiago, Kalligas, & Wagoner (1998) have shown that in many cases the theory is attracted toward that minimum during the expansion of the universe, so it approaches general relativity (φ = constant, $A(\varphi)$ = constant). This is in accord with the small experimental limits $a_1^2 < 10^{-3}$. (The Brans-Dicke theory is the special case $\alpha(\varphi)$ = constant, $\Lambda(\varphi) = 0$.)



Although there is local interest in this broad class of theories, we shall concentrate on general relativity for the remainder of these lectures (except for gravitational waves).

In addition, we will employ the fact that most matter in the universe is well approximated as a perfect fluid, described by the stress-energy tensor (obtained from the EEP)

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \ , \tag{7}$$

where ρ is the mass-energy density, p is the pressure, and $\mathbf{U}$ is the four-velocity of the fluid.

Weak-Field Equations

Throughout almost the entirety of all regions much smaller than that of the observable universe, gravitational fields can be considered weak. This means that (except near black holes and neutron stars) one can choose coordinates such that the metric assumes the nearly Minkowski form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x^\alpha) , \qquad |h_{\mu\nu}| \ll 1 .$$

For instance, within the solar system, $|h_{\mu\nu}| \lesssim GM_\odot/R_\odot c^2 = 2.12 \times 10^{-6}$. We shall consider isolated sources $T_{\mu\nu}$, and can neglect the cosmological constant $\Lambda(\varphi_0)$ within such regions.

We work to first order in $h_{\mu\nu}$, and utilize our freedom of general infinitesimal coordinate transformations $x^{\alpha'}(\mathcal{P}) = x^\alpha(\mathcal{P}) + \xi^\alpha(\mathcal{P})$, which produces

$$h_{\mu'\nu'} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} .$$

This is directly analogous to the gauge transformation of $A_{\mu'}$ $= A_\mu + \chi_{,\mu}$ in electrodynamics; and leaves $R_{\alpha\beta\mu\nu}$, like $F_{\alpha\beta}$, invariant. We can then use our freedom in choosing the four functions $\xi^\alpha(\mathcal{P})$ to impose the coordinate condition

$$\bar{h}^{\mu\sigma}{}_{,\sigma} = 0 , \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h \quad (h = \eta^{\alpha\beta} h_{\alpha\beta}) ,$$

analogous to the Lorentz gauge condition in electrodynamics. $(A^\sigma{}_{,\sigma} = 0)$

The Einstein field equations (5)(with $\varphi = \varphi_0$) then produce the weak field equations

$$\eta^{\alpha\beta}\bar{h}_{\mu\nu,\alpha\beta} \equiv \Box\,\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} ,$$

identical in structure with those in electrodynamics. (With the Lorentz gauge, this is consistent with the conservation laws

$T^{\mu\nu}{}_{,\nu} = 0$, analogous to $J^{\mu}{}_{,\mu} = 0$ in electrodynamics.) Thus the solution is of the same form:

$$\bar{h}_{\mu\nu}(x^\alpha) = 4G \int T_{\mu\nu}(x^0 - |x^i - x^i_*|, x^i_*)|x^i - x^i_*|^{-1} d^3x^i_* \,. \quad (8)$$

A Newtonian system is one in which all (macroscopic and microscopic) velocities are nonrelativistic (so that the above retardation is negligible), in addition to having a weak field. In such systems the dominant component of the stress-energy tensor (7) is seen to be $T_{00} \cong \rho$. Therefore the dominant component of the (trace-reversed) metric perturbation is $\bar{h}_{00} = -4\Phi$, where Φ is the Newtonian gravitational potential. Thus the spacetime interval becomes

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \cong -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \,. \quad (9)$$

Incidentaly, the result $h_{00} = -2\Phi$ can be obtained more generally by comparing Newton's second law with the geodesic equation of motion (2) for slowly-moving test particles. In the same limit (in which $dx^\alpha/d\tau \cong \delta^\alpha{}_0$), the spatial components of the equation of geodesic deviation (4) become

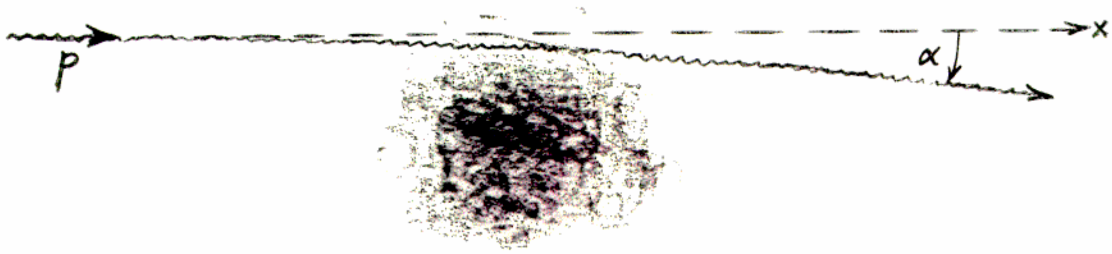
$$\frac{d^2 \Delta x^i}{d\tau^2} \cong -R^i{}_{0j0} \Delta x^j \cong 2 \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \Delta x^j \,,$$

showing how the tidal gravitational forces affect the separation of nearby particles.

Finally, the gravitational coupling constant is identified with G by examining the field equations (8) in the Newtonian limit.

Weak-Field Applications

A. Index of Refraction of Gravity



Consider a photon passing through a Newtonian gravitational potential (for which $|\partial\Phi/\partial t| \ll |\nabla\Phi|$) produced by some localized distribution of mass, a good approximation for all observed systems. In the geometrical optics limit, we follow a photon initially travelling in the direction $\mathbf{e}_x$ far from the masses, so subsequently $p^x = dx/d\lambda \cong p^t \equiv p$. It will be deflected by a very small angle $\vec{\alpha}$, with components $\alpha^N \cong p^N/p$, where the index $N = y, z$.

The equation of motion (2) then gives

$$\frac{dp^N}{d\lambda} \cong -\left[\left\{\begin{matrix} N \\ tt \end{matrix}\right\} + 2\left\{\begin{matrix} N \\ tx \end{matrix}\right\} + \left\{\begin{matrix} N \\ xx \end{matrix}\right\}\right] p^2 \cong -2\Phi_{,N}p^2$$

when the weak-field metric (9) is inserted in the Christoffel symbols. With $d\ell \equiv dx \cong p d\lambda$ and $d\alpha^N/dx \cong p^{-1}dp^N/dx$, we obtain

$$\vec{\alpha} = -2\int \nabla_\perp \Phi \, d\ell = \int \nabla_\perp n \, d\ell \, ,$$

where we have identified the *gravitational index of refraction* $n(x^i) \cong 1 - 2\Phi$. (≥ 1)

$\cong g_{tt}\,dt^2 + g_{xx}\,dx^2$

Using the fact that $ds^2 = 0$ for the photon, we obtain its coordinate velocity

$$\frac{dx}{dt} \cong \left(\frac{-g_{tt}}{g_{xx}}\right)^{1/2} \cong 1 + 2\Phi \cong 1/n \, ,$$

as expected. The time delay relative to a photon traveling (in flat space) between the same initial and final values of x, far from the masses where t is proper (clock) time τ, is then

$$\Delta t = -2 \int \Phi \, d\ell = 2 \int |\Phi| d\ell = \int (n-1) d\ell \, .$$

Thus in both respects, empty space acts as if it had this index of refraction.