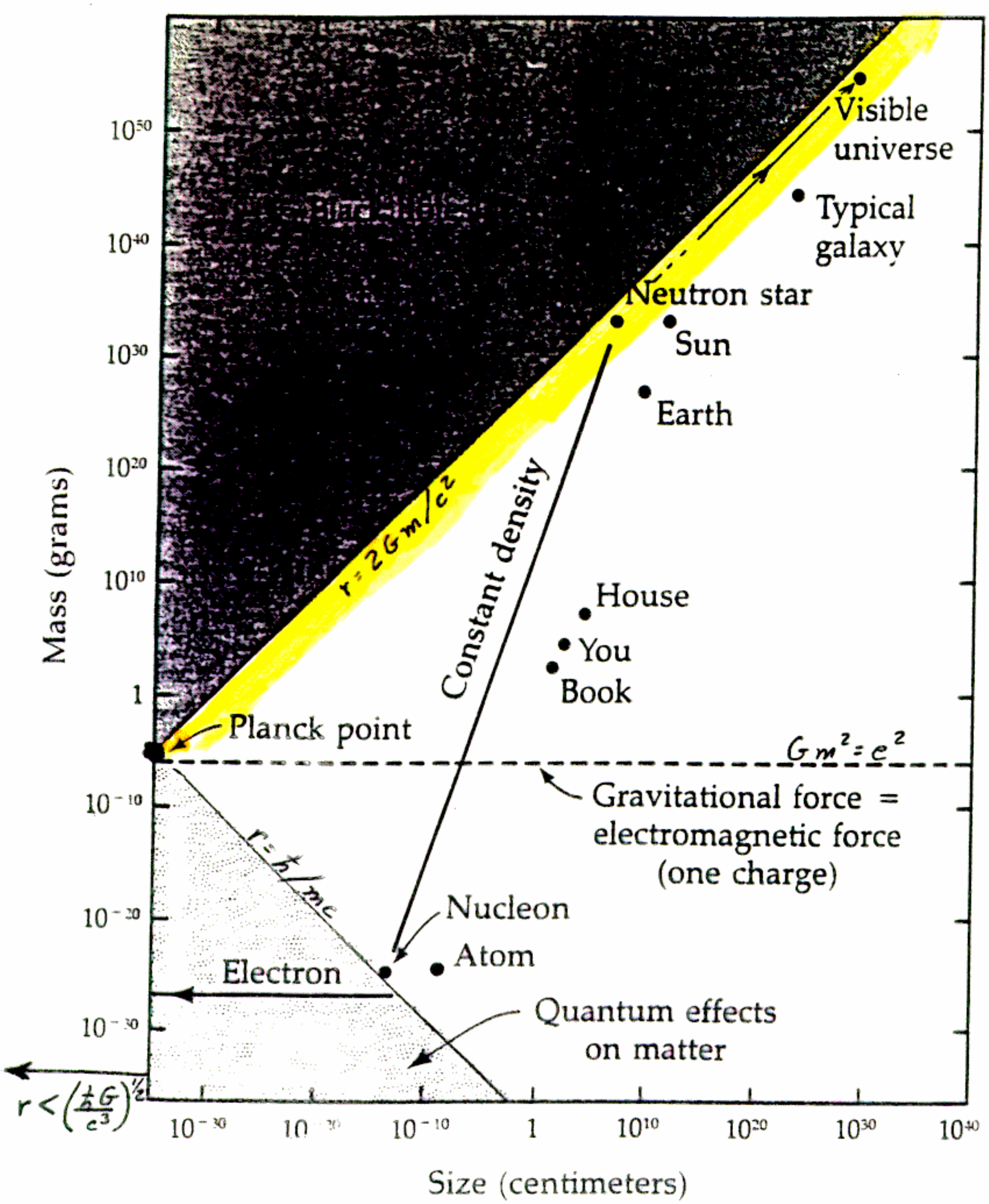
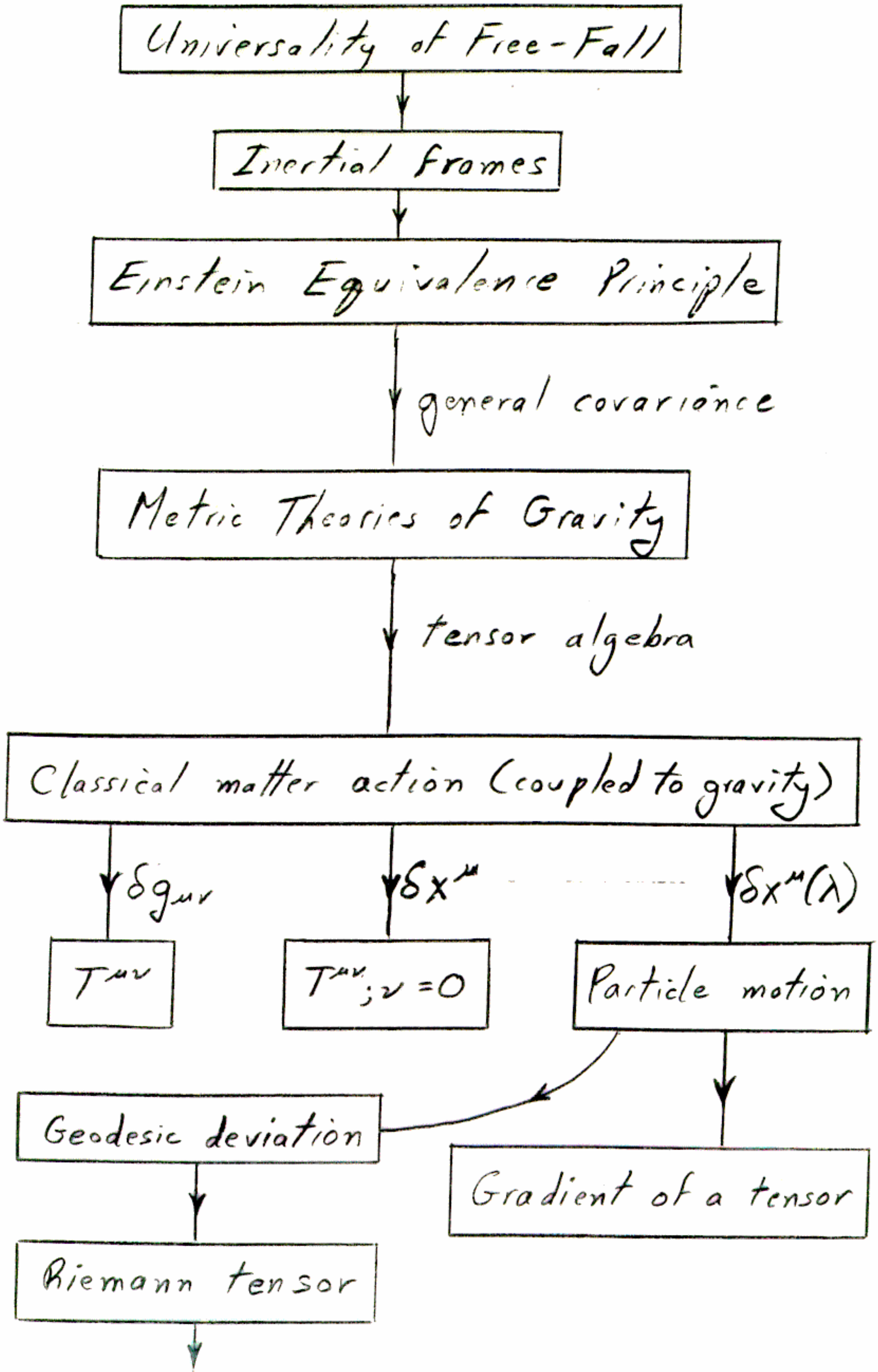
Relativistic Gravity and Some Astrophysical Applications

Robert V. Wagoner

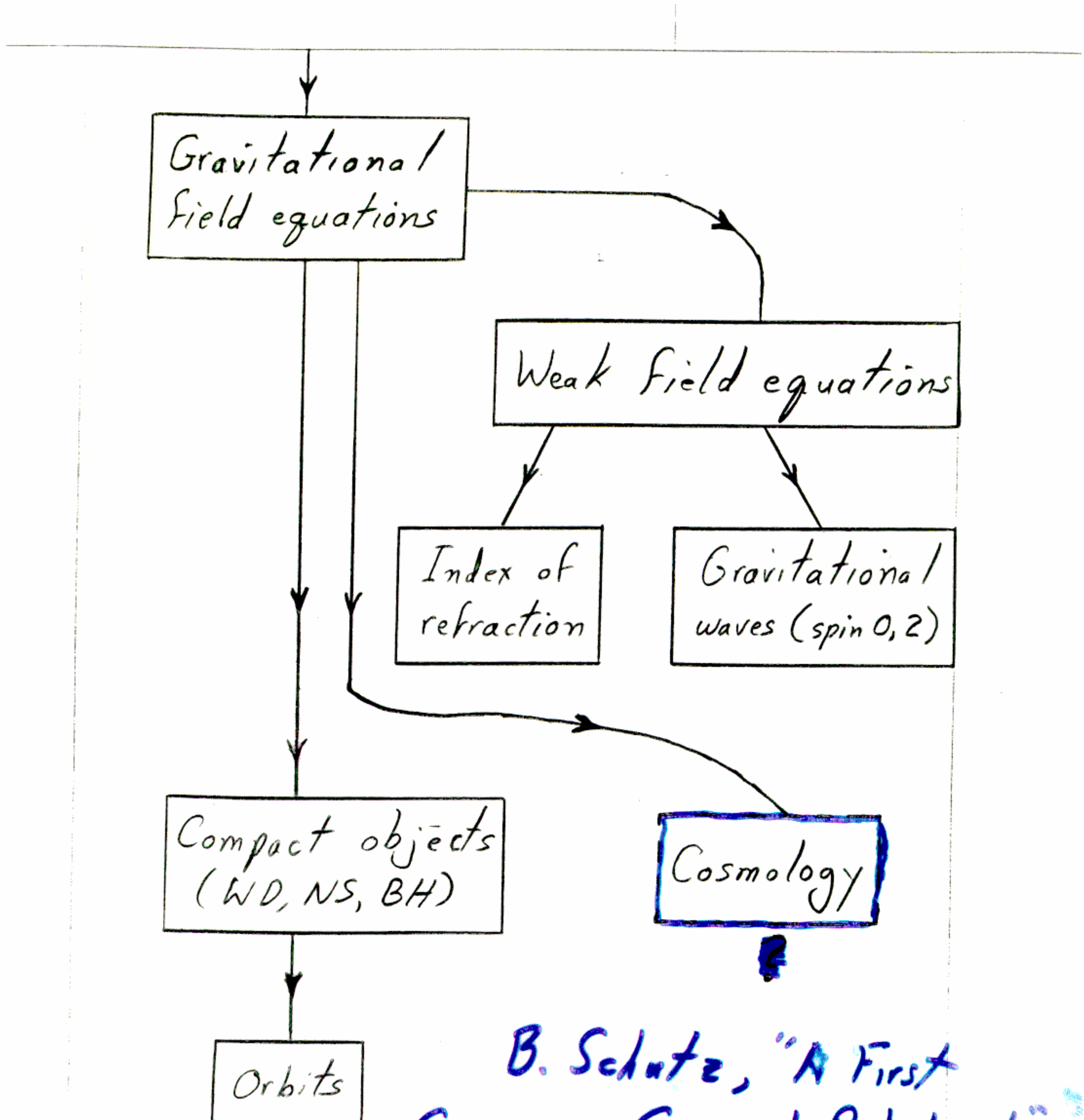
XXVI SLAC Summer Institute on Particle Physics



OUTLINE OF LECTURES



R. Wagoner, SSI 1998



B. Schutz, "A First Course in General Relativity"

Notation + conventions: MTW "Gravitation"

Relativistic Gravity and Some Astrophysical Applications

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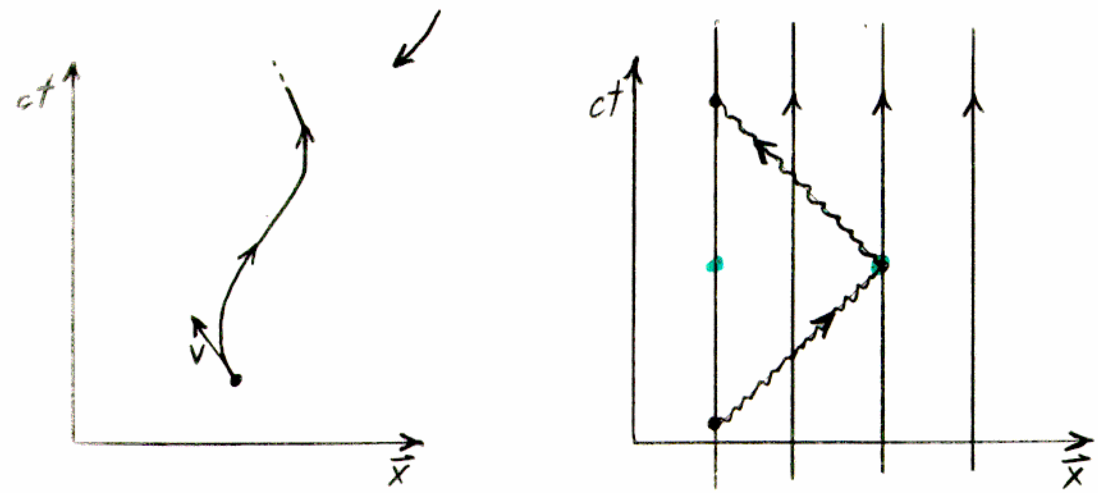
Gravitation is the most fundamental interaction, affecting all forms of mass-energy. This allows its geometrical description, at least within the classical (nonquantum) regime that we shall consider. The structure of *metric theories of gravity* is based upon a few key concepts and principles:

1) **Universality of Free–Fall (UFF)**

If a test particle is placed at an initial event in spacetime and is given an initial velocity there, its subsequent worldline will be independent of its structure (i. e., all forms of energy 'fall' at the same rate). Test particle: Charge, mass, and size reduced until experimental results are unchanged.



2) **Coordinate Frame**

A continuous set of spatially labeled 'clocks' filling spacetime. Distances best measured by radar method (round trip photon travel time).

3) **Inertial Reference Frame**

A coordinate frame in which any free test particle is unaccelerated (to a specified accuracy) within a small specified region of spacetime. It can always be constructed at *any* point (event) in spacetime (if UFF is valid). Realization: a nonrotating lab in free-fall, small enough that tidal gravitational forces are negligible.

4) **Einstein Equivalence Principle (EEP)**

In all inertial frames, the *nongravitational* laws of physics are those formulated within special relativity.

Gravity emerges from a local analysis

EEP plus covariance under general coordinate transformations will then allow us to determine how matter couples to gravity: via a metric tensor. This metric tensor has components $g_{\mu\nu}(x^\alpha)$ which can be put in the Minkowski form $\eta_{\mu\nu}$ in every inertial frame (which is then called a local Lorentz frame) $(g_{\mu\nu,\alpha} = 0)$

An equation is *generally covariant* if it preserves its form under a general coordinate transformation $x^{\mu'} = x^{\mu'}(x^\alpha)$.

- All tensor equations ($\mathbf{T} = \mathbf{0}$) are generally covariant.

- Such an equation will thus be true in all coordinate systems if it is known to be true in any one (such as a local Lorentz frame)

Tensor Algebra in Metric Spacetimes

Study of geometrical objects (scalars, curves, vectors, tensors, ...) at any fixed point $\mathcal{P}_0$. They exist independent of any coordinates, so form the proper description of physical reality.

Vector

In curved spacetime, we require a local definition. A familiar one is the tangent vector

$$\mathbf{v} = \left(\frac{\mathbf{d}\mathcal{P}}{d\lambda}\right)_{\mathcal{P}_0}$$

to some curve $\mathcal{P}(\lambda)$ at the point $\mathcal{P}_0$ where the vector exists (e. g., the four-velocity $\mathbf{U} = \mathbf{d}\mathcal{P}/d\tau$). $\mathbf{d}\mathcal{P}$ is the displacement.

Tangent Space

The vectors at any point $\mathcal{P}_0$ form this abstract four dimensional vector space. All geometrical objects at this point reside in this tangent space (not in spacetime).

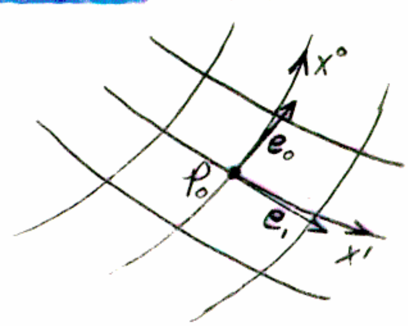
Basis

A basis is a set of four linearly independent vectors $\mathbf{e}_\alpha$ ($\alpha = 0, 1, 2, 3$) at a point $\mathcal{P}_0$. Representation of any vector $\mathbf{v}$ at $\mathcal{P}_0$ in terms of its components v^α:

$$\mathbf{v} = v^\alpha \mathbf{e}_\alpha \qquad \text{(summation convention)} .$$

Consider some coordinate system: four functions $x^\alpha(\mathcal{P})$. A (global) *coordinate basis* is then

$$\mathbf{e}_\alpha = \partial\mathcal{P}/\partial x^\alpha .$$



We shall only employ such bases.

Other aspects of tensor algebra are direct generalizations from special relativity in each tangent space:

Tensor

Can be thought of in various ways:

- Direct product: $\mathbf{T} = T^{\alpha\beta\cdots}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \cdots$
- Linear operator on vectors, giving a scalar (number):

$$\begin{aligned}\mathbf{T}(\mathbf{u}, \mathbf{v}, \ldots) &= \mathbf{T}(\mathbf{e}_\alpha, \mathbf{e}_\beta, \ldots)u^\alpha v^\beta \cdots \\ &= T_{\alpha\beta\cdots}u^\alpha v^\beta \cdots\end{aligned}$$

Metric Tensor [generalization of $\eta_{\mu\nu} = \mathrm{diag.}(-1, 1, 1, 1)$]

Produces the scalar product of vectors:

$$\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) \equiv \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} = \mathbf{g}(\mathbf{e}_\beta, \mathbf{e}_\alpha) = g_{\beta\alpha}$$

- Interval: $ds^2 = \mathbf{g}(\mathbf{d}\mathcal{P}, \mathbf{d}\mathcal{P}) = g_{\alpha\beta}dx^\alpha dx^\beta \quad (\mathbf{d}\mathcal{P} = dx^\sigma \mathbf{e}_\sigma)$
- Inverse: $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu{}_\nu$
- 'Raising and lowering indices':

$$\begin{aligned}v_\mu &\equiv \mathbf{v} \cdot \mathbf{e}_\mu = \mathbf{g}(v^\nu \mathbf{e}_\nu, \mathbf{e}_\mu) = g_{\nu\mu}v^\nu \ , \\ v^\mu &= g^{\mu\sigma}g_{\sigma\nu}v^\nu = g^{\mu\sigma}v_\sigma \ .\end{aligned}$$

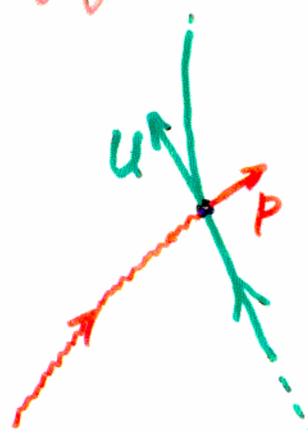
This generalizes to give

$$T_\alpha{}^\beta{}_{\gamma\cdots}{}^{\cdots} = g_{\alpha\mu}g_{\gamma\nu}T^{\mu\beta\nu\cdots} = g^{\beta\sigma}T_{\alpha\sigma\gamma\cdots}$$

and the generalized scalar product $T^\alpha{}_{\beta\gamma}V_\alpha N^{\beta\gamma}$, for instance.

All measured quantities are scalars!

Example: Energy of a photon.



$$E = -\mathbf{U}\cdot\mathbf{p}$$

$$= -U^\alpha p_\alpha$$

$$\xrightarrow[\text{LLT}]{} -U^\alpha p^\beta \eta_{\alpha\beta} \xrightarrow[v \ll c]{} -\delta^\alpha_0 \, p^\beta \eta_{\alpha\beta} = p^0$$

• *Contraction* produces a new tensor of rank two lower; for instance

$$Q_{\mu\nu} = M^{\sigma}{}_{\mu\sigma\nu} = g^{\sigma\tau} M_{\tau\mu\sigma\nu} \; ,$$

independent of choice of basis.

Change of Basis: $\mathbf{e}_{\mu'} = L^{\nu}{}_{\mu'} \mathbf{e}_{\nu}$

With the inverse transformation matrix constructed from $L^{\mu}{}_{\sigma'} L^{\sigma'}{}_{\nu} = \delta^{\mu}{}_{\nu}$, one obtains

$$T^{\alpha' \cdots}_{\beta' \cdots} = L^{\alpha'}{}_{\sigma} L^{\tau}{}_{\beta'} \cdots T^{\sigma \cdots}_{\tau \cdots} \; .$$

Between coordinate bases, $x^{\alpha'} = x^{\alpha'}(x^{\beta})$

$$L^{\alpha'}{}_{\beta} = \partial x^{\alpha'} / \partial x^{\beta} \; , \quad L^{\beta}{}_{\alpha'} = \partial x^{\beta} / \partial x^{\alpha'} \; .$$

Four-volume element (scalar)

$$dV_{(4)} = \sqrt{-g}\, dx^0 dx^1 dx^2 dx^3 \equiv \sqrt{-g}\, d^4x \; , \quad g = \det||g_{\mu\nu}|| \; .$$

Gradient of a Function $f(x^{\alpha})$

$$\mathbf{d}f = f^{,\alpha} \mathbf{e}_{\alpha} = g^{\alpha\beta} f_{,\beta} \mathbf{e}_{\alpha} = g^{\alpha\beta} \frac{\partial f}{\partial x^{\beta}} \mathbf{e}_{\alpha} \; .$$

The directional derivative of a function $f(x^{\alpha})$ along a curve $\mathcal{P}(\lambda)$ (at $\mathcal{P}_0$) is then

$$\frac{df}{d\lambda} = \frac{dx^{\alpha}}{d\lambda} \frac{\partial f}{\partial x^{\alpha}} = v^{\alpha} f_{,\alpha} \; . \quad = \mathbf{v} \cdot \mathbf{d}f$$

Extremal Action Principles

The fundamental special-relativistic laws of physics may obtained by extremizing the (scalar) action $\mathcal{I} = \int \mathcal{L} d^4x$, $\delta\mathcal{I} = 0$. Within metric theories, the effects of gravity *on* a classical material system may be obtained by the replacements

$$\mathcal{I} \to \int \mathcal{L}\sqrt{-g}\, d^4x \ , \ \mathcal{L} \to \mathcal{L}_M(\eta_{\mu\nu} \to g_{\mu\nu}; A_\mu, A_{[\mu,\nu]}, \text{'matter'}) \ .$$

Variation with respect to A_μ gives Maxwell's equations in an arbitrary metric field. ($[\alpha\beta\cdots]$ and $(\alpha\beta\cdots)$ will denote complete antisymmetrization and symmetrization of indices.) We will add the contribution of the gravitational field(s) to $\mathcal{I}$ later.

Stress-Energy Tensor

Under the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, the stress-energy tensor $T^{\mu\nu}$ is defined by

$$\delta\mathcal{I}_M = (1/2)\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x \ .$$

Energy-Momentum Conservation

Since $\mathcal{I}_M$ is a scalar, it will be unchanged under a coordinate transformation $x^\mu \to x^{\mu'} = x^\mu + \epsilon^\mu(x^\alpha)$ (here infinitesimal). It will also be unchanged by a subsequent change $x^{\mu'} \to x^\mu$ in the integration variable. This induces the net change

$$\delta g_{\mu\nu} = -(g_{\sigma\nu}\epsilon^\sigma{}_{,\mu} + g_{\mu\sigma}\epsilon^\sigma{}_{,\nu} + g_{\mu\nu,\sigma}\epsilon^\sigma) \ ,$$

the Lie derivative of $g_{\mu\nu}$. Under this variation, one obtains

$$\delta\mathcal{I}_M = \int T^\mu{}_{\sigma;\mu}\epsilon^\sigma\sqrt{-g}\, d^4x \ ,$$

$$T^\mu{}_{\sigma;\mu} \equiv (1/\sqrt{-g})[\sqrt{-g}\,T^\mu{}_\sigma]_{,\mu} - (1/2)g_{\mu\nu,\sigma}T^{\mu\nu} \ .$$

Note that the divergence $T^{\mu\sigma}{}_{;\mu}\mathbf{e}_\sigma$ must be a vector. Since $\delta\mathcal{I}_M = 0$ for arbitrary $\epsilon^\sigma(x^\alpha)$, we obtain the four components

$$T^\mu{}_{\sigma;\mu} = 0 \qquad (\text{or } T^{\mu\sigma}{}_{;\mu} = 0) ,$$

representing conservation of energy ($\sigma = 0$) and momentum ($\sigma = i = 1, 2, 3$; continuum equations of motion).

Test-Particle Equation of Motion

Our prescription gives the action

$$\mathcal{I}_M = -\int \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda$$

for a test particle, where the tangent vector to the particle's world line $x^\alpha(\lambda)$ is $\mathbf{d}\mathcal{P}/d\lambda = (dx^\mu/d\lambda)\mathbf{e}_\mu \equiv \dot{x}^\mu \mathbf{e}_\mu$. Vary the worldline, and choose λ (after the variation) so that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m^2$ (we take $c = 1$). The particle's four-momentum (component) is then $p^\mu = \dot{x}^\mu$. Then $\delta\mathcal{I}_M = -m^{-1}\int Q_\alpha \delta x^\alpha d\lambda = 0$, so

$$Q_\alpha \equiv dp_\alpha/d\lambda - (1/2)g_{\mu\nu,\alpha}p^\mu p^\nu = 0 . \tag{1}$$

Thus if the metric components $g_{\mu\nu}$ are independent of any coordinate x^α, the corresponding four-momentum component p_α *is conserved*.

We also find that the vector components

$$Q^\alpha = \frac{d^2x^\alpha}{d\lambda^2} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \equiv \frac{D}{D\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) = 0 , \tag{2}$$

p^α

where the Christoffel symbol (connection coefficient in a coordinate basis, not the components of a tensor) is

$$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ \nu\mu \end{matrix} \right\} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}) .$$

For rest masses $m \geq 0$, equations (1) and (2) govern the particle's four-momentum p^α. For $m > 0$ they also govern its four-velocity $U^\alpha = p^\alpha/m = dx^\alpha/d\tau$, where the proper time interval $d\tau = m d\lambda$.

It can be shown that in the neighborhood of a freely-falling observer, coordinates can be chosen so that $g_{\mu\nu} = \eta_{\mu\nu}$ and $g_{\mu\nu,\alpha} = 0$ along his/her worldline. It then follows from equation (2) that all test particles in that neighborhood are indeed unaccelerated, $dx^\alpha/d\tau =$ constant $\Rightarrow dx^i/dx^0 =$ constant, verifying that it is an inertial frame.

Gradient of a Tensor

Since Q^α are the components of a vector [given by equation (2)], it follows that for a vector field with components $V^\alpha(x^\mu)$,

$$\frac{DV^\alpha}{D\lambda} = \frac{dV^\alpha}{d\lambda} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} V^\mu \frac{dx^\nu}{d\lambda}$$
$$= \left(V^\alpha{}_{,\nu} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} V^\mu \right) \frac{dx^\nu}{d\lambda} \equiv V^\alpha{}_{;\nu} \frac{dx^\nu}{d\lambda} \qquad (3)$$

are also the components of a vector. It thus also follows that $V^\alpha{}_{;\nu}$ must be the components of a (rank 2) tensor: the generalization of the gradient to operate on vectors. The generalization of the directional derivative is the covariant derivative, with the above components $DV^\alpha/D\lambda$.

Denoting the directional derivative operator $D/D\lambda$ along a basis vector $\mathbf{e}_\nu$ by ∇_ν, its effect on a vector $\mathbf{V}$ can also be described as

$$\nabla_\nu(V^\sigma \mathbf{e}_\sigma) = (\nabla_\nu V^\sigma)\mathbf{e}_\sigma + V^\sigma(\nabla_\nu \mathbf{e}_\sigma) \equiv V^\alpha{}_{;\nu}\mathbf{e}_\alpha \,.$$

Comparing with equation (3), we see that the connection coefficients describe how the basis vectors vary with position:

$$\nabla_\nu \mathbf{e}_\mu = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \mathbf{e}_\alpha \, .$$

The application to tensors of any rank follows straightforwardly to give

$$T^{\alpha\cdots}{}_{\beta\cdots;\mu} = T^{\alpha\cdots}{}_{\beta\cdots,\mu} + \left\{ \begin{matrix} \alpha \\ \sigma\mu \end{matrix} \right\} T^{\sigma\cdots}{}_{\beta\cdots} + \cdots - \left\{ \begin{matrix} \sigma \\ \beta\mu \end{matrix} \right\} T^{\alpha\cdots}{}_{\sigma\cdots} - \cdots \, .$$

Note that it then follows that the gradient of the metric vanishes: $g_{\alpha\beta;\mu} = 0$.

Example: EM Field tensor

$$A_{\mu;\nu} = A_{\mu,\nu} - \left\{ \begin{matrix} \tau \\ \mu\nu \end{matrix} \right\} A_\tau$$

$$\Rightarrow F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$$

$$= 2A_{[\nu;\mu]} = 2A_{[\nu,\mu]}$$