

Gravitational Waves: Sources & Signatures

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Introduction

- ◆ Goal: bring together ◆ Today:
 - experiment and theory
 - Sources for the experimentalist
 - Experiment for the theoretician
- ◆ Tomorrow:
 - Characterizing radiation, detectors & signal strength
 - Burst radiation sources
 - Periodic radiation sources
 - Stochastic signals

The Big Picture

- ◆ Gravitational waves and “fundamental” physics
 - Dynamics of a fundamental force
 - Testing general relativity
- ◆ Gravitational waves as scientific probe
 - grav. waves arise from strong field regions that are obscured or don’t radiate in other channels
 - new vista on universe

Conventions

- ◆ Geometric Units: $G = c = 1$

- Mass, time measured in units of length

$$\gg G/c^2 = 7.4 \times 10^{-29} \text{ cm/g}$$

◆ $1 M_\odot = 1.5 \times 10^5 \text{ cm}$

- Power is dimensionless

$$\gg c^5/G = 3.6 \times 10^{59} \text{ ergs/s}$$

- ◆ Indices

- Einstein summation: sum over repeated indices
 - greek indices run (t, x, y, z) , latin run (x, y, z)

Characterizing the waves

- ◆ Waveform
- ◆ Radiated power
- ◆ Energy/Power Spectrum

Waveform

- ◆ Linearized gravity
 - Metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
- ◆ Field equations:
 - Equations: $\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)h_{ij} = 16\pi T_{ij}$
 - TT Gauge: in observer rest frame . . .
 - » Purely spatial: $h_{ik} = 0$
 - » Transverse to propagation direction: $h_{ij} n_j = 0$
 - » Trace-free: $\delta_{ij} h_{ij} = 0$
 - Stress-tensor source: T_{ij}

Slow-motion expansion

- ◆ “Electric” and “magnetic” multipoles
 - Electric: charge \rightarrow mass
 - Magnetic: current \rightarrow momentum
- ◆ Quadrupole formula
 - Reduced mass quadrupole:
$$\gg Q_{ij} = \int d^3x \rho \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right)$$
 - TT gauge radiation:
$$\gg h_{ij}^{TT} = \left(2/r \right) \ddot{Q}_{ij}^{TT}$$

Radiation carries energy

- ◆ Power radiated: $L = \frac{1}{5} \frac{G}{c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle$

- ◆ Dimensional analysis:

$$L = \frac{G}{c^5} \left(\frac{MR^2}{T^3} \right)^2$$

$$= \frac{MR^2/T^3}{c^5/G} \left(\frac{MR^2}{T^3} \right)$$

- asymmetric “kinetic power” reduced by ratio of kinetic power to fundamental power (c^5/G)

Example: A rotating dumb-bell

- ◆ Quadrupole Moment:

$$\mathbf{Q} = 2MR^2 \begin{pmatrix} \cos^2 \omega t & \frac{1}{2} \sin 2\omega t & 0 \\ - & \sin^2 \omega t & 0 \\ - & - & 0 \end{pmatrix}$$

- ◆ Third time derivative:

$$\mathbf{Q} = 8MR^2 \begin{pmatrix} \sin 2\omega t & \sin 2\omega t & 0 \\ - & - \sin 2\omega t & 0 \\ - & - & 0 \end{pmatrix}$$

- ◆ Radiated power:

$$L = \frac{128}{5} \frac{G}{c^5} (MR^2 \omega^3)^2$$

$$= 1.1 \times 10^{48} \left(\frac{M}{1.4 M_\odot} \right)^2 \left(\frac{R}{200 \text{ Km}} \right)^4 \left(\frac{f_{GW}}{50 \text{ Hz}} \right)^6 \frac{\text{erg}}{\text{s}}$$

Total rotational energy
 ~~$\frac{10^{51} \text{ erg}}{10^{30} \text{ Hz}}$~~

Energy/Power Spectrum

- ◆ Characterize by energy (power) spectrum
 - $H(f) df$ = signal energy (power) in band $(f, f+df)$
- ◆ Often possible even if waveform unknown
 - Chaotic system: *e.g.*, large scale turbulence
 - Relevant (astro)physics unknown: *e.g.*, black hole or neutron star coalescence
 - Intrinsically stochastic: *e.g.*, primordial background

Radiation characterized by

- ◆ Waveform
 - when known
- ◆ Energy/Power spectrum
 - especially when waveform unknown
- ◆ Total power radiated

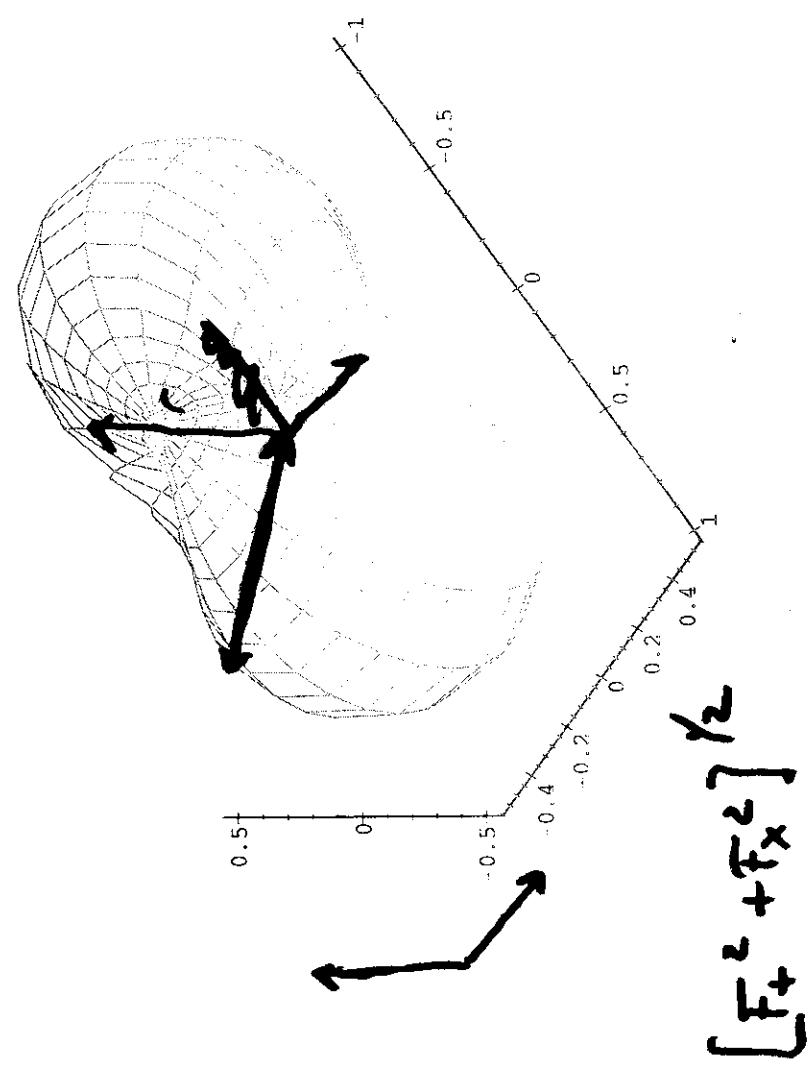
Characterizing the detector

- ◆ Response
- ◆ Noise
- ◆ Bandwidth

Response

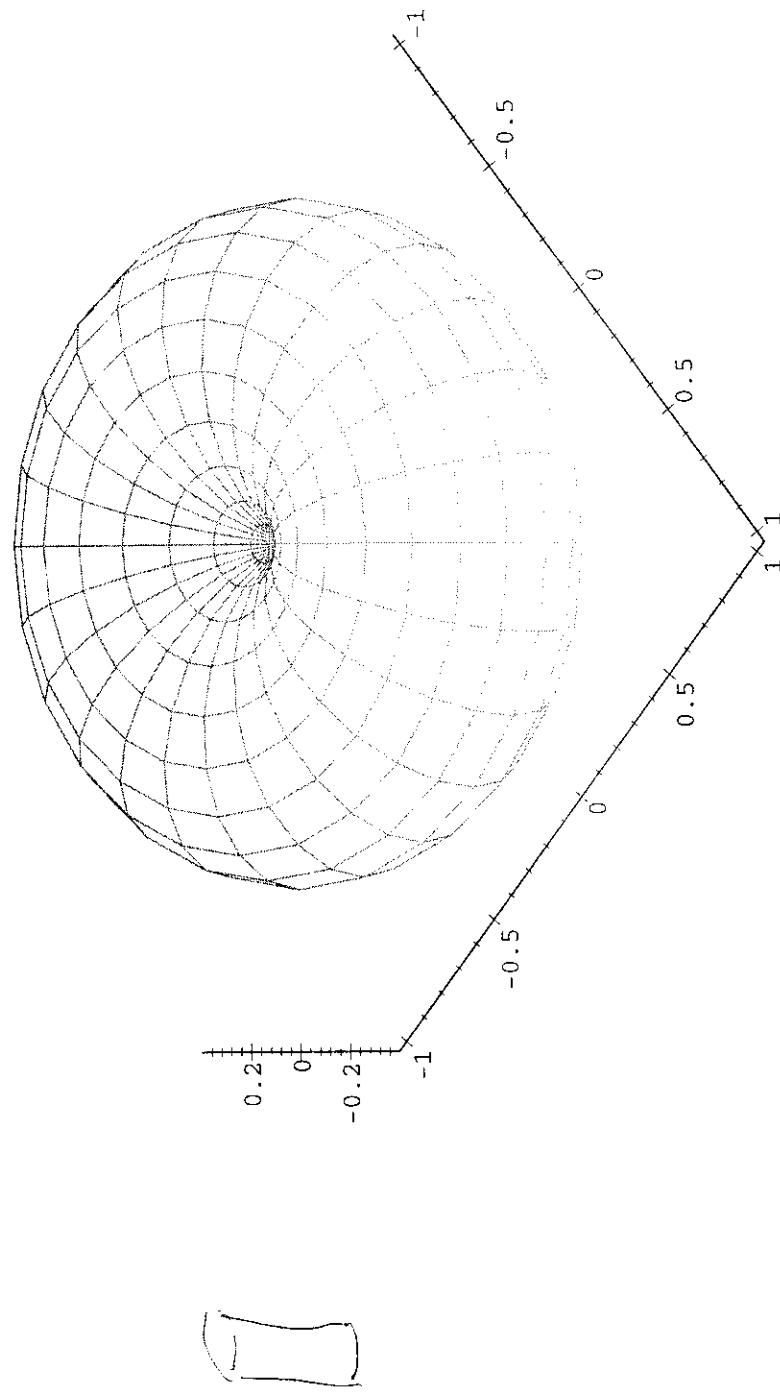
- ◆ Gravitational waves excite detector
 - differential sensitivity to radiation incident from different directions
 - differential sensitivity to +, ×
- ◆ *Antenna pattern* describes differential sensitivity
 - Response $m(t) = F_+(\theta, \phi) h_+(t) + F_\times(\theta, \phi) h_\times(t)$

Interferometer



Cylindrical acoustic detector

$$\left[F_x^2 + F_y^2 \right]^{1/2}$$



Noise

- ◆ Unpredictable excitations in output stream
 - Fundamental sources
 - » Thermal, “quantum” (i.e., counting statistics)
 - Instrumental sources
 - » Noisy amplifiers, bad contacts, creep & strain release, etc.
 - Environmental sources
 - » seismic, electromagnetic, atmospheric, etc.
- ◆ Described statistically

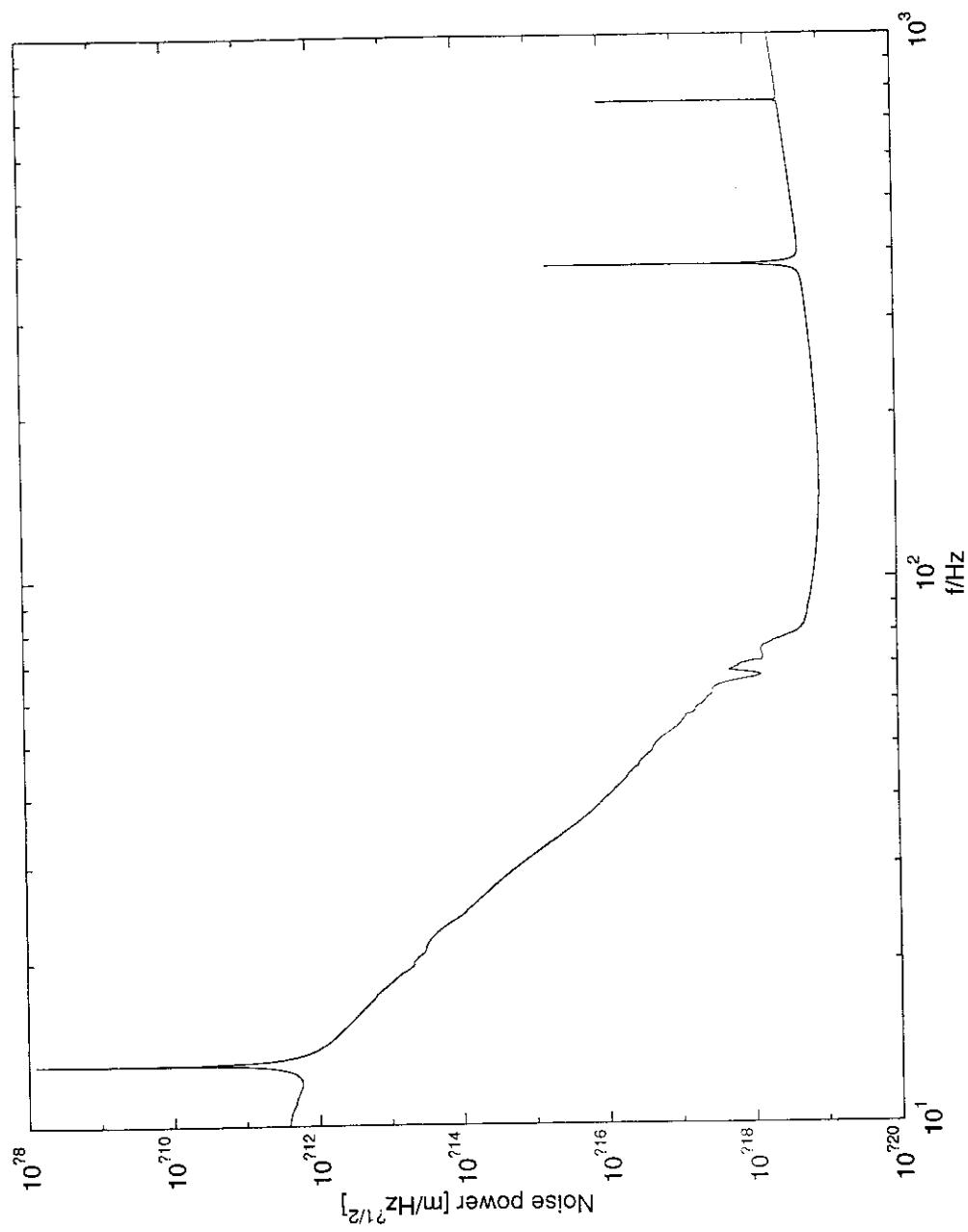
Statistical description of noise

- ◆ General auto-correlations
 - $C_N(t_1, \dots, t_N) = \overline{n(t_1) \cdots n(t_N)}$
- ◆ Stationary noise
 - Independent of absolute time
 - » $C_N(t_2 - t_1, \dots, t_N - t_1) = \overline{n(t_1) \cdots n(t_N)}$
 - Tractable noise: ensemble averages are time averages
- ◆ Correlation function: $C(\tau) = \overline{n(t)n(t + \tau)}$
 - fully characterizes *Gaussian* noise

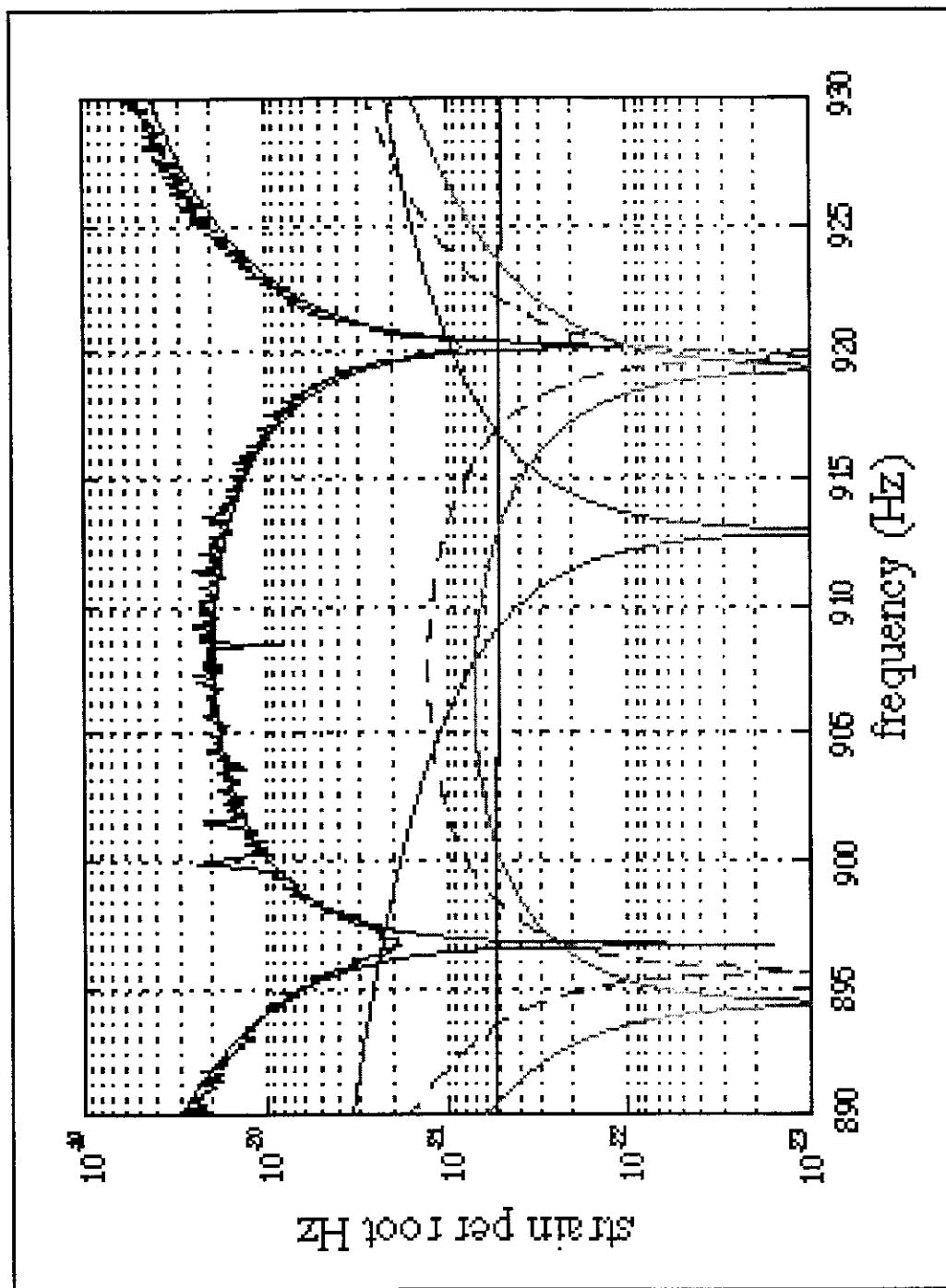
Power Spectral Density

- ◆ Consider noise $n(t)$ in interval $[0, T]$
 - Fourier series: $n(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t / T}$
 $\gg |c_k|^2$ like power at $f_k = k/T$
 - Amplitude $|c_k|$ a random variable
 - » Actual power in any measurement greater or less
- ◆ PSD $S(f_k)$ variance (mean square) of $|c_k|$
 - $S(f) df$ is noise power in bandwidth $(f, f+df)$
 - W-K Theorem: $S(f) = 4 \int_0^{\infty} C(\tau) \cos 2\pi f \tau d\tau$
 - PSD fully characterizes Gaussian noise

LIGO I Noise PSD



ALLEGRO Noise PSD

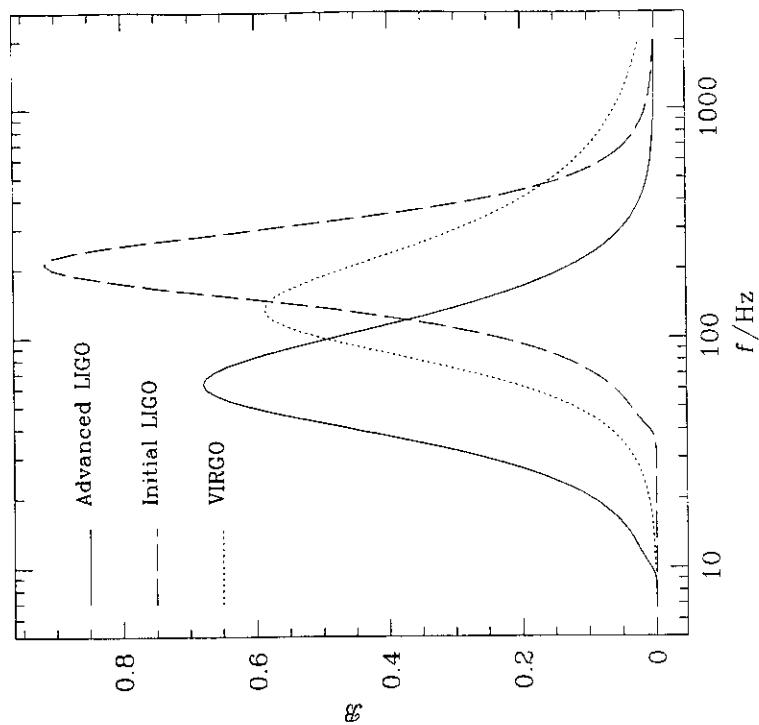


Bandwidth

- ◆ Interval Δf where noise power minimized
- ◆ Relevant bandwidth for source . . .
 - Interval Δf where ratio of signal energy to noise power maximized
 - Varies by source, detector

Detector Bandwidth: binary inspiral

- ◆ Contribution to signal “detectability”
 - Same source, different detectors



Detectors characterized by

- ◆ Antenna pattern
 - Differential sensitivity to polarization & source position
- ◆ Noise statistics
 - especially power spectrum
- ◆ Bandwidth
 - especially bandwidth relevant to source

Two approaches to detection

- ◆ Find “best” procedure for deciding between hypotheses detected, not detected
 - don’t care if right in any particular application of rule: focus on long-run batting average
 - “Frequentist” statistics
- ◆ Find *degree of belief* in alternative hypotheses
 - how sure am I *this time*?
 - “Bayesian” statistics

Signal-to-noise

- ◆ Ratio of observed to expected power:

$$-\rho^2 = 2 \int df \frac{|\tilde{g}(f)|^2}{S(f)}$$

- ◆ Looking for signal m in observation g
- ◆ Expectation value when signal m present

$$-\rho^2 = 2 \int df \frac{\tilde{m}(f) \tilde{g}^*(f)}{S(f)}$$

$$-\overline{\rho^2} = 1 + 2 \int df \frac{|\tilde{m}(f)|^2}{S(f)}$$

$$-\overline{\rho^2} = 2 \int df \frac{|\tilde{m}(f)|^2}{S(f)}$$

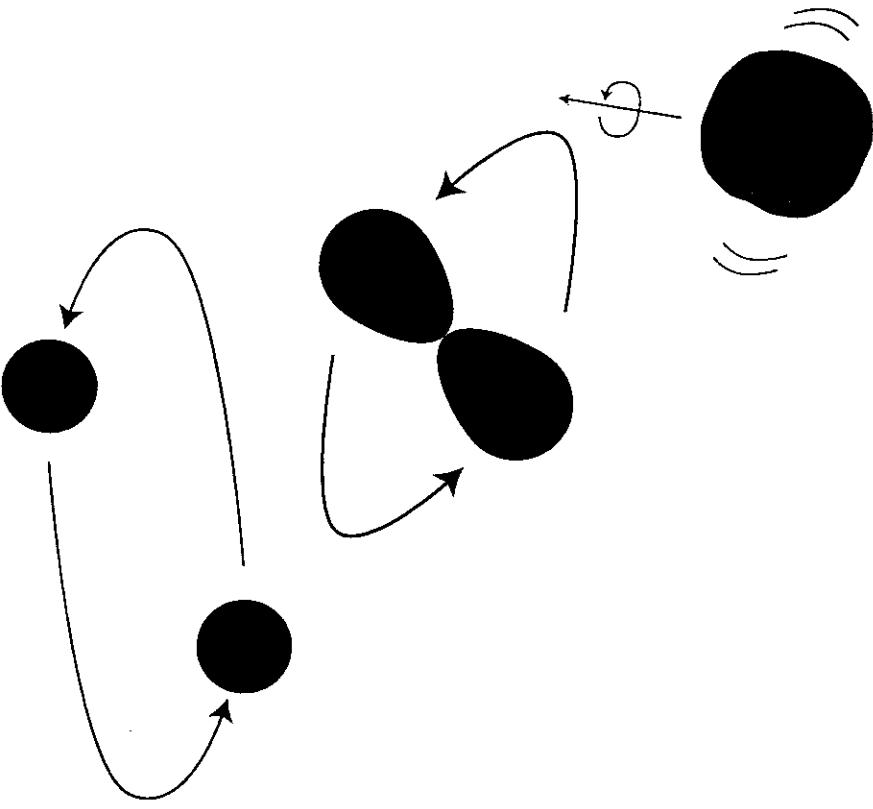
Summary

- ◆ Detection is statistical
- ◆ Signal-to-noise an important measure of signal strength

Burst Sources

- ◆ Compact binary inspiral
- ◆ Binary coalescence, supernovae, black hole formation and things that go bump in the night

Compact Binary Inspiral

- ◆ Inspiral
 - rad. reaction drives
adiabatic orbital decay
 - ◆ Coalescence
 - strong, highly
dynamical fields;
hydrodynamics
 - ◆ Ringdown
 - small perturbations
radiated away
- 

The inspiral waveform

- ◆ Source properties ◆ Detector response
- Geometry
 - » θ, φ : sky position
 - » ι : orbital inclination
- Distance
 - » d_L : Luminosity distance
 - » z : redshift
- Intrinsic properties
 - » m_1, m_2 : masses
 - T : “coalescence moment”
- quad. form. including radiation reaction

$$h = \frac{\mathcal{M}}{d_L} \Theta(\pi f \mathcal{N})^{2/3} \cos \Phi(t)$$
- $$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} (1 + z)$$
- $$\Theta(\theta, \varphi, \iota)^2 = 4 \left[F_+^2 (1 + \cos^2 \iota)^2 + 4 F_\times^2 \cos^2 \iota \right]$$
- $$f(t) = \frac{1}{\pi \mathcal{M}} \left(\frac{5}{256} \frac{\mathcal{M}}{T - t} \right)^{3/8}$$
- $$\Phi(t) = \int' 2\pi f(t') dt'$$

Observables

- ◆ Signal strength

$$\rho^2 = 8 \left(\frac{7}{32} \right)^{1/3} \frac{r_0}{d_L} \left(\frac{\mathcal{M}}{1.2 M_\odot} \right)^{5/6}$$
- ◆ Geometry & distance
 - Available if observed in several separated detectors
- ◆ Rate

$$\frac{r_0^2}{M_\odot^2} = \frac{5}{192\pi} \left(\frac{243}{7 \times 10^5} \right)^{1/3} \int \frac{df}{f^{7/3} S(f)} \quad$$
 - $r_0 \approx 20$ Mpc for LIGO I
- ◆ Chirp mass \mathcal{M}
 - high precision
 - m_1, m_2 at higher order
- ◆ $\dot{N} \approx 4/3 \pi r_0^3 \dot{n}$
- ◆ $\dot{n} \approx 10^{-7 \pm 2} \text{ Mpc}^3 \text{ yr}^{-1}$

Detecting binary inspiral

- ◆ Detection needs $\Phi(t)$

$$-\rho^2 = 2 \int df \frac{\tilde{m}(f)\tilde{g}^*(f)}{S(f)}$$

- $\arg(m) = \arg(g) (= \Phi(t))$
for positive integrand

- ◆ Quadrupole formula

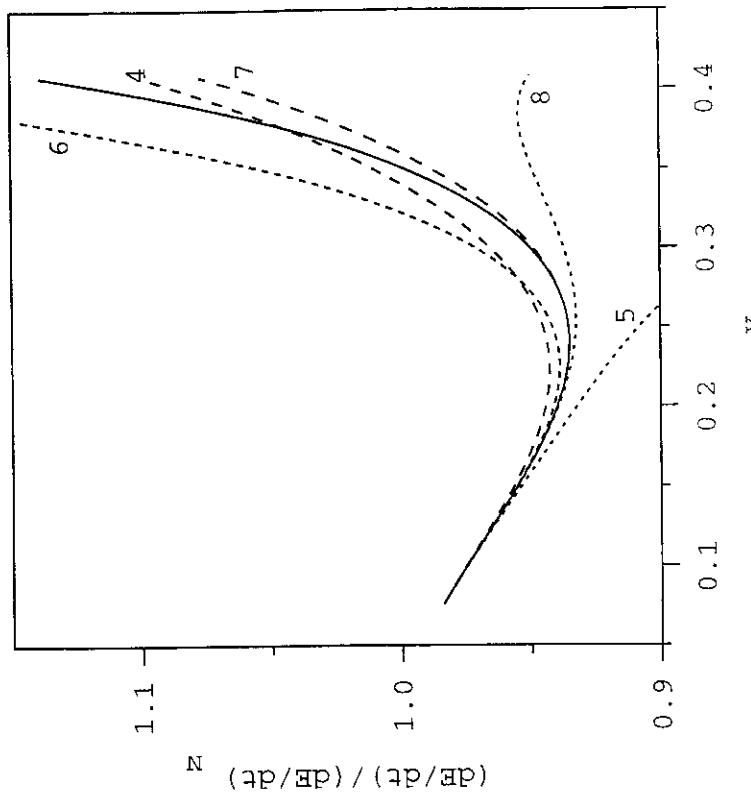
not good enough

- $\Phi(t)$ depends on rad.
reaction

- higher moments

needed to get $\Phi(t)$ right
in important bandwidth

» Poisson: gr-qc/9505030
(1997 version)



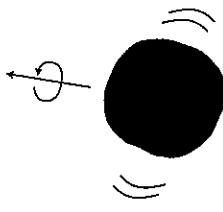
Cosmology & Binary Inspiral

- ◆ Binaries observed at cosmological distances
- ◆ SNR ρ depends on luminosity distance d_L
- ◆ Chirp mass depends on cosmological redshift z
- ◆ \mathcal{M} vs. ρ is redshift *vs.* luminosity distance
 - enough observations determine H_0, Ω_0, Λ
 - direct measurement from grav.-waves: no phenomenology involved

Things that go ‘‘bump’’ in the night

- ◆ Black hole formation
- ◆ Stellar core collapse (supernovae)
- ◆ ?

Black hole formation



- ◆ Formed many ways:

- stellar core collapse,
compact binary
coalescence, *etc.*

- ◆ Waveform:

$$h_{\text{rms}}(t) = 2 \sqrt{\frac{2\varepsilon}{QF}} \frac{M}{r} e^{-\pi ft/Q} \sin 2\pi ft \quad (t > 0)$$

$$f = 12 \text{ KHz} \left(\frac{M_\odot}{M} \right) \left(\frac{F}{37/100} \right)$$

- ◆ Radiation?

- Initial waves depend
on past history
- final waves depend
only on final BH M, J

$$Q \approx 2(1-a)^{-9/20}, \quad F \approx 1 - 63/100(1-a)^{3/10}$$

$$\varepsilon = \frac{E_{\text{Rad}}}{M}, \quad a = J_{\text{BH}} / M^2$$

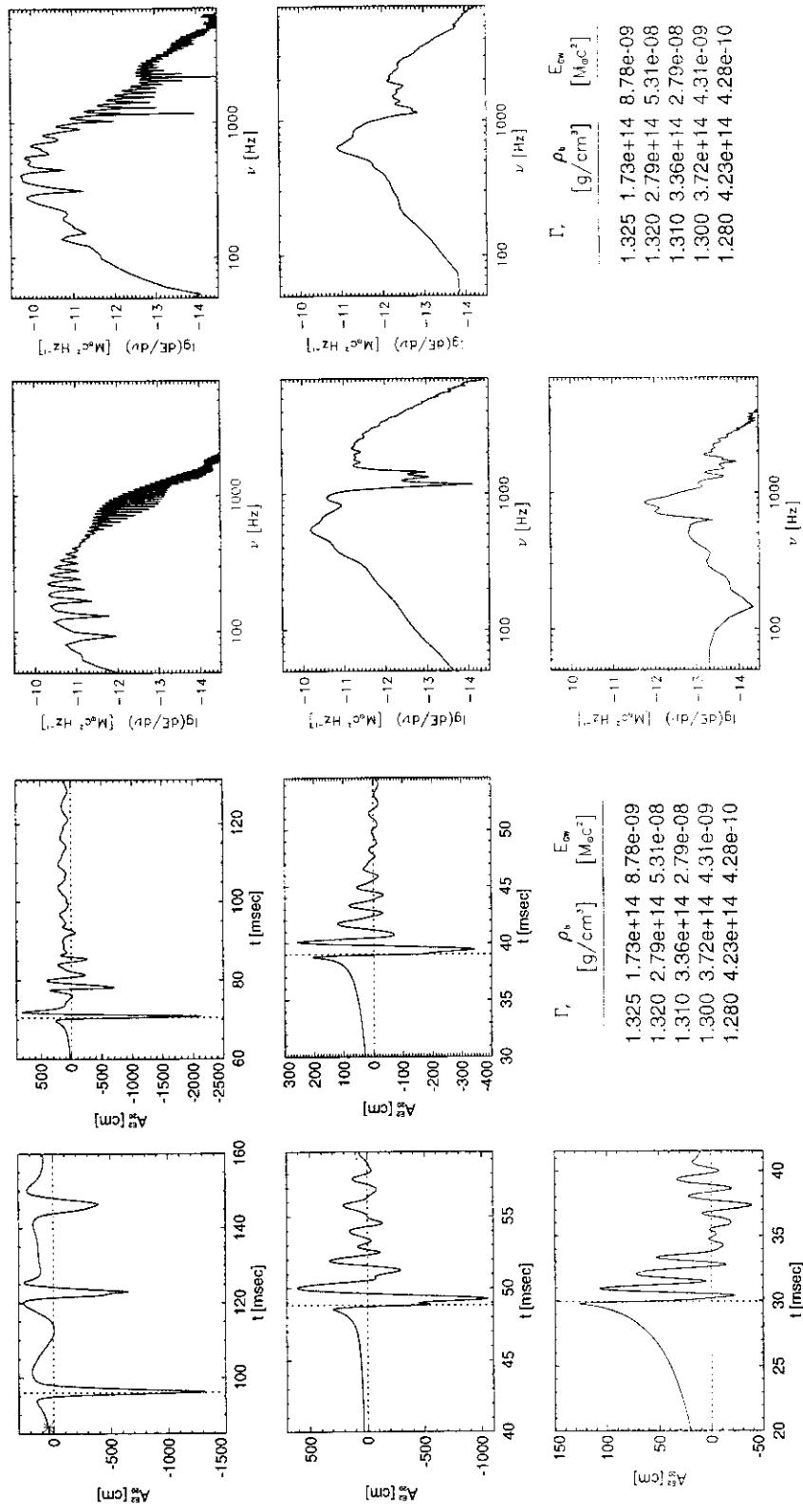
- ◆ Signal-to-noise ratio:

$$\rho^2 \approx 34 \frac{\varepsilon}{10^{-4}} \left(\frac{20 \text{ Mpc}}{r} \right)^2 \left(\frac{M}{13M_\odot} \right)^3 \frac{10^{-46} \text{ Hz}^{-1}}{S(f)}$$

Stellar core collapse

- ◆ Where?
 - Type II Supernovae
 - Accretion-induced collapse
- ◆ Waveform?
 - hydrodynamics
 - nuclear, super-nuclear non-equilibrium EOS
 - γ, v transport, *etc.*

Waveforms & spectral signatures



— from Zwerger & Mueller waveform catalog

» <http://www.mpa-garching.mpg.de/~ewald/GRAV/grav.html>

Detecting the “next” SN1987A

- ◆ Model radiation
 - dL/df constant for $f < 1 \text{ KHz}$
 - $E_+ = E_\times = \epsilon M_\odot/2$
- ◆ Model detector
 - $S(f)$ white from 50 Hz to 1 KHz
- ◆ Signal-to-noise
 - $\rho^2 \approx 100 \frac{\epsilon}{10^{-8}} \left(\frac{50 \text{ Kpc}}{r} \right)^2 \frac{10^{-47} \text{ Hz}^{-1}}{S_n}$

Summary

- ◆ Characterizing Radiation
 - Waveform
 - Total energy radiated
 - Energy spectrum
- ◆ Characterizing Detectors
 - Antenna pattern
 - Noise statistics
 - Noise power spectrum
- ◆ Characterizing detection
 - Bayesian & Frequentist statistics
 - Signal-to-noise ratio