Introduction to the Phase-Cavity for the Linac Resonant Cavity Beam Position Monitor System

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In this note we review the details of the phase-cavity employed for the old linac resonant cavity beam-position monitoring system.

This is the fourth note in a series on the E158 BPM system.

1. Introduction

The Two-Mile Accelerator[1] in its original form employed a trio of cavities located every 100 m along the linac, functioning as beam position monitors (BPM). One station is illustrated schematically in Fig. 1. These are asymmetric output, S-Band (2856 MHz), 0.8" diameter aperture cavities, intended to be resonantly excited by an S-Band bunched beam. We will refer to them as the "linac-style" or "in-line" cavity monitors to distinguish them from other resonant BPMs in the Two-Mile complex.

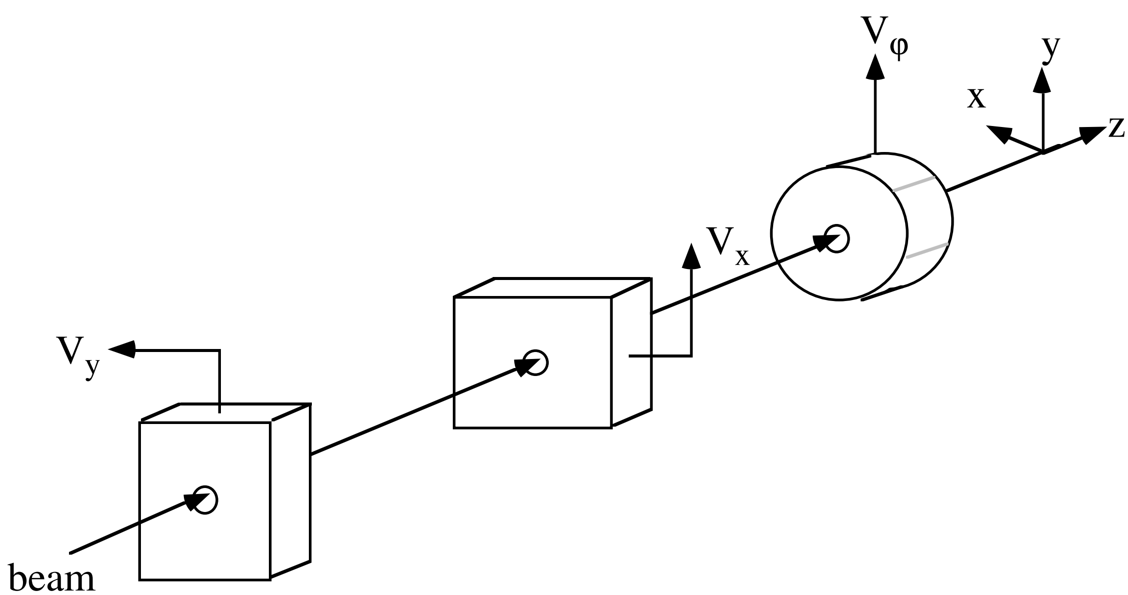

FIGURE 1. One BPM station includes an x-cavity, a y-cavity and a phase or φ-cavity. This sketch can be compared to Fig. 15-4 in the Blue Book (p. 504; in this figure, the girder is viewed from above)

The three cavities in one station are an x-cavity, a y-cavity and a phase or φ-cavity. In this note we review the particulars of the phase cavity, illustrated schematically in Fig. 2.

[1] *The Two-Mile Accelerator*, R. B. Neal, ed., (W. A. Benjamin, New York, 1968) pp. 500-516.

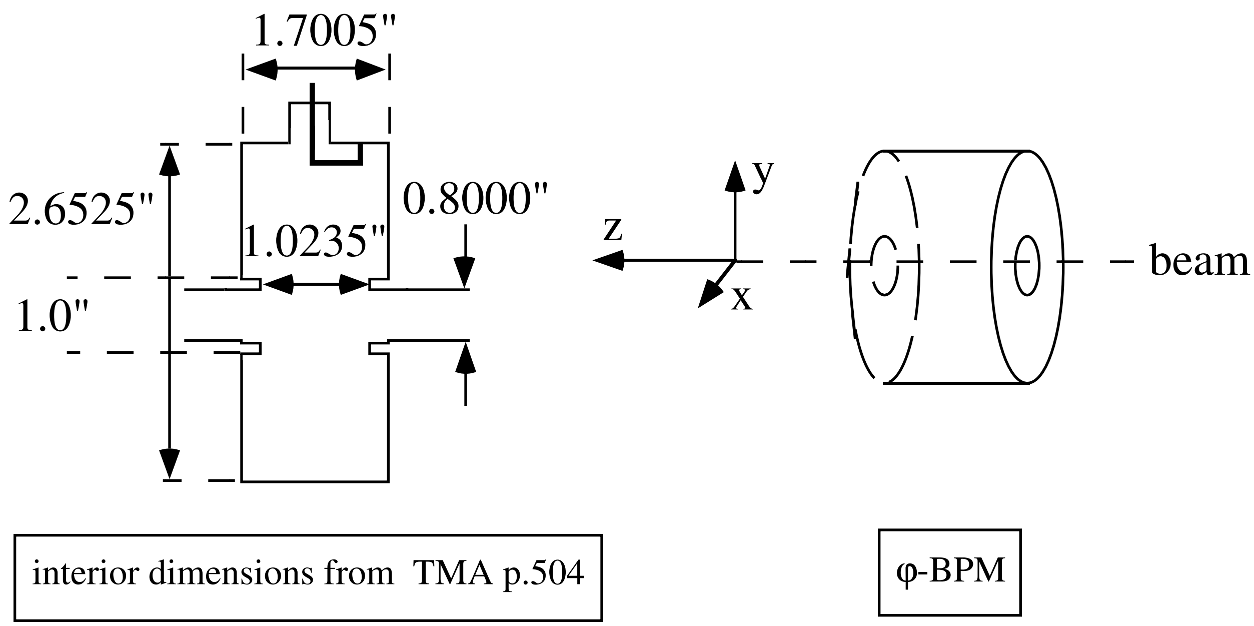

FIGURE 2. Major interior (electrical)dimensions for one linac-style φ-cavity.

2. Cylindrical Pillbox Estimates

To get a rough feeling for the scalings, it is helpful to start with a cylindrical pillbox, the cavity prior to perturbation by the beam-port and nose-cones, and prior to external coupling. The axial electric field varys as $J_0(\beta_0 r)$, with $\beta_0 = j_{01} / R$, $j_{01} \approx 2.40483$, and R the cavity radius. The resonant frequency is $\omega_0 = \beta_0 c$. In terms of the transit angle,

$$\theta = \frac{\omega_0 L}{V} \approx \frac{j_{01} L}{R},$$

and the transit angle factor,

$$T = \frac{\sin\left(\frac{1}{2}\theta\right)}{\left(\frac{1}{2}\theta\right)},$$

the [R/Q] is

$$\left[\frac{R}{Q}\right] \approx 0.982\, Z_0 T^2 \frac{L}{R}.$$

Wall Q is given by

$$Q_w = \frac{j_{on}}{2} \frac{Z_0}{R_{surf}} \frac{L}{L+R} \approx \frac{5.46 \times 10^4}{\sqrt{f(\text{GHz})}} \frac{1}{1 + R/L},$$

where the surface resistance is

$$R_{surf} \approx \frac{1}{\sigma\delta} = 8.3\ \mathrm{m}\Omega\sqrt{f(\mathrm{GHz})} \approx 14\ \mathrm{m}\Omega$$

and $Z_0 = \sqrt{\mu_0 / \varepsilon_0} \approx 376.7\,\Omega$

For operation at $f_0 = c\beta_c / 2\pi \approx 2856\,\mathrm{MHz}$, one estimates

$$2R \approx \frac{c\, j_{01}}{\pi f_0} \approx 8.0\,\mathrm{cm},$$

or 3.2" diameter. There is an optimum in shunt impedance,

$$R_s = Q_w \left[\frac{R}{Q}\right],$$

as a function of transit angle, for $\theta = 158.1°$, and this corresponds to a cavity length,

$$L = \frac{\theta c}{2\pi f_0} \approx 4.6\,\mathrm{cm},$$

or 1.8". Corresponding wall Q is $Q_w \approx 1.73 \times 10^4$. Theoretical shunt impedance in this case is $R_s \approx 3.77\ \mathrm{M}\Omega$, and $[R/Q] \approx 220\ \Omega$.

Actual cavity dimensions were arrived at using tabulated scalings from bench measurements, and with additional "cut & try" on the bench.[2] Final dimensions were chosen to provide a frequency 2 MHz too low prior to brazing, with the idea that frequency could be adjusted by "dimpling". Dimensions account for corrections for the beam-port and nose-cones (to improve [*R/Q*]) and correspond to a gap of 1.02", a cavity width of 1.7" and a cavity diameter of 2.65". Unloaded Q is 9600, and loaded Q is 1200, corresponding to an external Q of 1371. The coupling parameter $\beta = Q_w / Q_e \approx 7.0$. Altenmueller and Brunet quote[3] [*R/Q*] without transit angle correction is $[R/Q]/T^2 \approx 370\,\Omega$, $T \approx 0.819$, and $[R/Q] \approx 248\,\Omega$. Shunt impedance (unloaded) is $R_s = Q_w[R/Q] \approx 2.38\ \mathrm{M}\Omega$. Frequency is 2856 MHz±300 kHz, with temperature detuning of 25 kHz/°F (Blue Book) or, equivalently, 1° of tuning angle for 0.5°F (Altenmueller and Brunet). Field decrement time due to copper losses is

$$T_0 = \frac{2Q_w}{\omega_0} \approx 1.07\ \mu\mathrm{s},$$

and the loaded fill time is

$$T_f = \frac{T_0}{1+\beta} = \frac{2Q_L}{\omega} \approx 133\ \mathrm{ns}.$$

[2] O. Altenmueller and P. Brunet, "Some RF Characteristics of the Beam Phase Reference Cavity" SLAC TN-64-51, Sept. 1964 (available from the SLAC Library).

[3] Taking acount of the fact that Altenmueller and Brunet employ a convention that [*R/Q*] is lower than ours by a factor of 2.

It is helpful, particularly in the case of single-bunch excitation to refer to the *loss-factor*,

$$k_l = \frac{1}{4}\omega_\lambda \left[\frac{R}{Q}\right] \approx 1.11 \text{ V/pC}.$$

Altenmueller and Brunet report experiments in air on the Mark IV accelerator, indicating significant variation in beam-induced voltage depending on transverse position. Initially this leads one to believe that the cavity has some significant deviation from cylindrical symmetry. In fact however, except for the small perturbation due to the coupler, the cavity is approximately symmetric. [It could be that the Mark IV results were due to beam-scraping.] Thus based on previous work,[4] we know that all higher azimuthal harmonics vanish in the synchronous integral,

$$\tilde{w}_\lambda\left(\vec{r}_{\perp b}\right) = \int_{-\infty}^{+\infty} dz\, E_{z\lambda}\left(\vec{r}_\perp = \vec{r}_{\perp b}, z\right) e^{j\omega z/V} = b_0,$$

with the mode longitudinal field. Thus the operating mode is to an excellent approximation, a pure monopole mode,

$$\left[\frac{R}{Q}\right] \approx Z_0 2k_\lambda^{-1} b_0^2.$$

3. Excitation by a Single Bunch

At this point we may proceed to apply the formalism previously developed,[5] to compute the waveforms emanating from the cavity with beam present. Starting from the beam-cavity equation,

$$\left\{\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_L}\frac{d}{dt} + \omega_\lambda^2\right\} V_c = \frac{1}{2}\omega_\lambda \left[\frac{R}{Q}\right] \frac{dI_{b0}}{dt},$$

and defining $\Omega = \sqrt{\omega_0^2 - \nu^2/4}$, $\nu = \omega_0 / Q_L$, we may express the cavity voltage induced by a single Gaussian bunch as

$$V_c(t) = \Re \exp\left[j\Omega(t-T)\right] \tilde{V}_c(t),$$

where the phasor

$$\tilde{V}_c(t) = H(t-T)\exp\left(-\tfrac{1}{2}\nu(t-T)\right)\hat{V}_c,$$

exhibits step-turn on when the beam enters the cavity (we don't resolve temporal features on the transit time scale with a single resonant mode) with complex amplitude

[4] "Mode Classification in Non-Axisymmetric Structures: How Nonlinear Can a Cavity Be?", ARDB TN 152.

[5] David H. Whittum, Beam Excitation of a Non-Axisymmetric Structure: Higher-Moments, Common Mode, Output-Coupling, Resonant and Non-Resonant Operation, ARDB TN 153

$$\hat{V}_c \approx 2k_l\, Q_b \exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z^2\right).$$

The integral

$$\int_{-\infty}^{+\infty} dt\, \tilde{V}_c(t) = R_L\, Q_b \exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z^2\right),$$

gives a measure of the total charge that has passed through the cavity, modulo the form factor. In practical terms, this integral is the area under the waveform obtained by mixing the cavity emitted waveform down to baseband, with a properly phased local oscillator, *i.e.*, it is measurable. In this expression we have abbreviated $R_L = Q_L[R/Q]$.

Notice that the phase of the emitted signal is immune to systematics, even bunch length, since $Q_L \approx 1200 >> 1$. Meanwhile, amplitude is not, and this may be a concern (1) if we are employing the phase-cavity signal for charge normalization of the position cavity signals, and (2) if this factor $\exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z^2\right)$ changes erratically from pulse to pulse. In fact, we favor option (1), and thus must consider feature (2). At the same time, we should realize that when the phase-cavity signal is employed for charge normalization, the form-factor correction, if constant along the macropulse, will divide out, leaving an inferred position independent of bunch length. Thus the actual effect of the form factor on inferred position could be less than one would suspect at first glance.

As a point of reference, voltage is reduced by 1% for $\sigma_z \approx 0.142/\beta_0 \approx 2.4$ mm, and by 0.1% for $\sigma_z \approx 0.045/\beta_0 \approx 0.75$ mm. Let us consider worst-case and typical values one might expect for σ_z. Minimum rms energy spread in the beam emerging from the linac, and neglecting higher-mode effects and injected energy spread, is

$$\delta_E = \frac{\sigma_E}{E} = 2^{1/2} sinh\left(\tfrac{1}{2}\sigma_\psi^2\right) \approx \frac{1}{2^{1/2}}\sigma_\psi^2,$$

where $\sigma_\psi = \beta_0\sigma_z$, and we take account of the fact that our cavities operate at the same frequency as the linac (not a subharmonic). This relation may be expressed as

$$\sigma_\psi(^\circ RF) = 6.8\sqrt{\delta_E(\%)}.$$

If the injector and linac are setup so as to produce a beam with 1% rms energy spread, or smaller, then the phase-width of each bunch will be 6.8° S-Band or smaller. Converting to units of length,

$$1^\circ\ @2856\ \text{MHz} \approx \frac{350.1\ \text{ps}}{360} \approx 0.97\,\text{ps} \approx 0.29\ \text{mm},$$

we find that the bunch length in this case will be 2.0 mm or less. In this case $\exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z\right) \approx 0.993$. In order to effect a 1% change in the amplitude of the φ-cavity signal, the bunch length must in fact change sufficiently to double the energy spread. If the change in bunch length corresponds to 10% correction to energy spread, then the φ-cavity signal suffers a 0.1% correction in amplitude. In other words, an operational scenario where jitter in the form factor correction is likely to produce corrections at the 1% level, would correspond to an almost unusable beam in any case.

Even so, it is quite feasible to monitor bunch-length, over several intervals within the beam macropulse. On the linac at present, in single-bunch mode, this is accomplished by means of toroid gaps as pickups.[6] A controlled study of bunch length-dependence might be easiest in the SLD or FFTB mode of running, where the bunch compressor is available. The cavity BPM LI30 would be natural for such a study. With (1) phase-cavity amplitude and (2) LI25 bunch length monitor (BLM) signal entered in a correlation panel on the SCP, one could take correlation data, with compressor voltage as the step variable in manual acquisition mode. Conditions of one e^- bunch at LI30 would be required to eliminate cross-talk (with the e^+ bunch) as a systematic effect, particularly given that the natural cavity discharge time scale (133 ns) is longer than the nominal bunch separation (59 ns).

Continuing on with our analysis of beam-induced waveforms, our calculation of the voltage-waveform induced by the beam permits us to determine the energy stored in the cavity immediately after the single bunch passes,

$$U = k_l\, Q_b^2 \exp\left(-\beta_0^2\, \sigma_z^2\right).$$

The peak power radiated down the guide is then

$$P_w = \frac{2\beta}{1+\beta}\frac{U_C}{T_f} \approx 1.75\frac{U_C}{T_f}$$

where we make use of $\beta = Q_w / Q_e \approx 7.0$. This result may be expressed generally as

$$P_w = \frac{2\beta}{1+\beta}\frac{1}{T_f} k_l\, Q_b^2 \exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z^2\right) = \frac{2\beta}{T_0} k_l\, Q_b^2 \exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z^2\right) = \frac{1}{4}\frac{\omega_0^2[R/Q]}{Q_e}\, Q_b^2 \exp\left(-\tfrac{1}{2}\beta_0^2\, \sigma_z^2\right),$$

and we provide a few equivalent expressions for clarity.[7] For our parameters we may write

$$U \approx 1.11\, \mu\text{J}\; Q_b^2(\text{nC}), \qquad P_w \approx 1.75\frac{U_C}{T_f} \approx 1.75\frac{1.11\, \mu\text{J}\; Q_b^2(\text{nC})}{133\text{ ns}} \approx 14.6\, W\, Q_b^2(\text{nC}),$$

and it is good to emphasize that this is the *peak* power; the actual power waveform rises as a step

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[6] See R.L. Holtzapple, "Longitudinal Dynamics in the Stanford Linear Collider", Ph.D. Thesis. See also F. Zimmermann, G. Yocky, D. H. Whittum, M. Seidel, D. McCormick, K. Bane, "New bunch-length monitor for the SLC final focus", *Proceedings of the 1997 Particle Accelerator Conference* (IEEE, New York, to be published). See also F. Zimmermann and D. Whittum, "Upgrade to South Final Focus Bunch Length Monitor", ARDB TN 61, "Expected Waveforms, SFF BLM", ARDB TN 62, "External Coupling of a Gap in a Cylindrical Pipe", ARDB TN.

[7] The last form describes the dependence of emitted power on the choice of cavity frequency: [*R/Q*] is a function of aspect ratios and not scale; Q_e could be fixed or scaled with $Q_w \propto \omega_0^{-1/2}$ (where scaling of cavity geometry and surface resistance both enter). Thus peak power developed for a point bunch scales as $\omega_0^2 - \omega_0^{2.5}$ depending on how external Q is scaled. At the same time, our consideration of the form factor, and considerations we have not fully reached yet, of aperture, indicate that perhaps S-Band cavities are small enough. For E158, cost too would be a major issue in changing cavities.

function to this amplitude and decays with an e-folding time of $T_f/2 \approx 66.5$ ns. Some comparisons are helpful. It is conventional to refer to power in units of decibels relative to 1 mW, dBm, so that

$$10\log_{10} P_w(\mathrm{mW}) \approx 41.6 \text{ dBm} + 20\log_{10} Q_b(\mathrm{nC}) \ .$$

For comparison, note that for an SLD high-charge bunch of 5 nC (3.1×10^{10} electrons) this provides 56 dBm. For a single bunch of 17.5 pC (1.1×10^{8} electrons) this provides 6.5 dBm. (As a point of reference, a train of such bunches arriving at 2856 MHz corresponds to 50 mA). These are *large* signals.

4. Excitation by a Bunch Train

Excitation by a bunch train has been considered in ARDB TN153; we cover the highlights here. The signal from a bunch train is computed using superposition. We number bunches n=0,1,..., N-1; let T_{bn} be the arrival time of the n-th bunch; denote $\tau_n = T_{bn} - T_{b0}$. We assume an ideal linac, so that $f_{RF}\tau_n = 1$, with $f_{RF} \approx 2856\,\mathrm{MHz}$ the linac frequency ; however, we permit f_{RF} to differ from $\omega_0/2\pi$, the cavity resonance frequency, as in the case of a failure in temperature regulation.

The voltage waveform induced by the n-th bunch is

$$V_{cn}(t) = \Re\exp\left[j\Omega\left(t-T_{bn}\right)\right]H\left(t-T_{bn}\right)\exp\left(-\frac{1}{2}\nu\left(t-T_{bn}\right)\right)\hat{V}_{cn} = H\left(t-T_{bn}\right)\Re\exp\left[-\Gamma\left(t-T_{bn}\right)\right]\hat{V}_{cn},$$

where $\hat{V}_{cn} \approx 2k_l\, Q_{bn}\exp\left(-\frac{1}{2}\beta_0^2\,\sigma_z^2\right)$ and $\Gamma = \frac{1}{2}\nu - j\Omega$. We may then express the total voltage at any time as

$$V_c(t) = \sum_{n=0}^{N-1} V_{cn}(t) = \Re\sum_{n=0}^{T_n<t}\ \exp\left[-\Gamma\left(t-T_{bn}\right)\right]\hat{V}_{cn} = \Re\exp\left[-\Gamma\left(t-T_{b0}\right)\right]\sum_{n=0}^{T_n<t}\ \exp\left(\Gamma\tau_n\right)\hat{V}_{cn}$$

The sum in the second line is over those bunches that have already passed as of time t. This sum may be employed to study analytically and numerically the effects of systematics due to τ_n, σ_{zn}, and Q_{bn} variations with n.

In the idealized case corresponding to: (1) a "top-hat" current profile, (2) a uniformly bunched beam, and (3) steady or small beam-size along the bunch train, the sum reduces to a geometric series,

$$V_c(t) = \Re\exp\left[\Gamma\left(t-T_{b0}\right)\right]\,\hat{V}_{c0}\,\frac{\exp\left(\Gamma\tau\, n\right)-1}{\exp\left(\Gamma\tau\,\right)-1},$$

where n indexes the last bunch that passed at time t, *i.e.*, $T_n < t < T_{n+1}$. At early times this gives a voltage waveforme rising linearly in time and output power rising quadratically. As time goes on, the system reaches a steady-state between power deposited by the beam, and power absorbed by the walls, and radiated down the waveguide. After the bunch train has gone by, the voltage decays exponentially.

The quantity

$$\Gamma\tau = \frac{1}{2}\nu\tau - j\Omega\tau \approx \frac{\pi}{Q_L} - 2\pi j\left(\frac{\Delta\Omega}{\Omega}\right) - 2\pi j N_h,$$

where the cavity harmonic is $N_h = \Omega / 2\pi f_{RF} = 1$, for the tuned cavity, and we allow for a detuning of the cavity by $\Delta\Omega$ relative to the perfectly bunched beam frequency. Let us abbreviate $\delta_\Omega = \Delta\Omega / \Omega$, noting that for a 1° F cavity temperature error, one may expect a cavity detuning of 25 kHz, corresponding to $\delta_\Omega \approx 9 \times 10^{-6}$. Our voltage phasor is

$$V_c(t) = \Re \exp\left[\Gamma\left(t - T_{b0}\right)\right] \hat{V}_{c0} \frac{\exp\left(n\left[\pi / Q_L - 2\pi j \delta_\Omega\right]\right) - 1}{\exp\left(\pi / Q_L - 2\pi j \delta_\Omega\right) - 1}.$$

For the case of a bunch train long compared to the fill-time, this implies a steady-state amplitude

$$\tilde{V}_{c-s} \approx \frac{Q_L}{\pi} \hat{V}_{c0} \cos\psi e^{j\psi}. \qquad \text{(steady-state)}$$

where the cavity tuning angle is

$$\tan\psi = 2Q_L\delta_\Omega = 2Q_L \frac{\Delta\Omega}{\Omega}.$$

Evidently tuning error introduces both a phase and an amplitude error. The amplitude error is second order in the tuning angle, and the phase-error is just the tuning angle. Tuning angle variation with temperature is 1.2° / °F. Reduction of amplitude by 0.1% corresponds to 2.6° or 2.1°F. So for example, if 1 µm resolution is desired with offsets as large as 100 µm, and if the phase-reference signal is employed for amplitude normalization, then this reference amplitude should be accurate to better than 1%. Good temperature regulation (1-2°F) would be required to accomplish this.

We can illustrate these formulas with a few examples as in Figs. 3 (a)-(d), Fig. 4 (a)-(b), and Fig. 5.

To appreciate the effect of the resonant enhancement in the signal, let us consider the case of a tuned cavity, and a beam of small σ_z. The voltage amplitude in steady-state, $\hat{V}_{cn}$, may be expressed, by comparison with the single-bunch case, in terms of an equivalent charge, Q_{eq}, or the pulse current $I_b = Q_b f_{RF}$, according to

$$\tilde{V}_{c-s} \approx \frac{Q_L}{\pi} 2k_l\, Q_b \equiv 2k_l\, Q_{eq}, \qquad \text{(steady-state)}$$

where the equivalent charge for a single bunch is

$$Q_{eq} = \frac{Q_L}{\pi} Q_b = I_b T_f,$$

just the charge that passes through the cavity in one fill-time ($Q_{eq} \approx 382\, Q_b$ for our parameters). We may also express the steady-state voltage in terms of current as

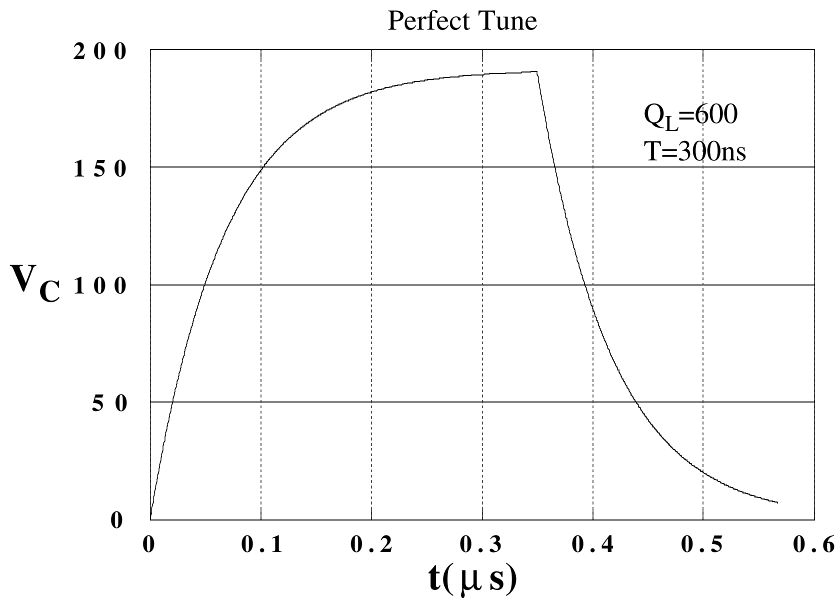

FIGURE 3. (a) Ideal waveform (in units where $2k_lQ_n$=1) for a 300 ns bunch train and a perfectly tuned cavity with Q_L=600. The vertical axis corresponds to $\left|\tilde{V}_c\right|$ in the notation of the text.

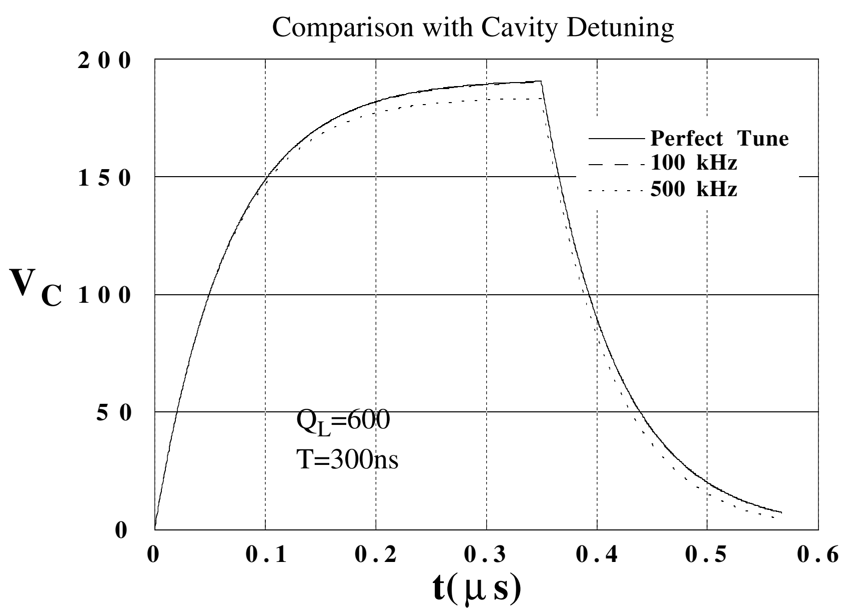

FIGURE 3. (b) Result of cavity detuning; as in (a): in units where $2k_lQ_n$=1, for a 300 ns bunch train and a perfectly tuned cavity with Q_L=600.

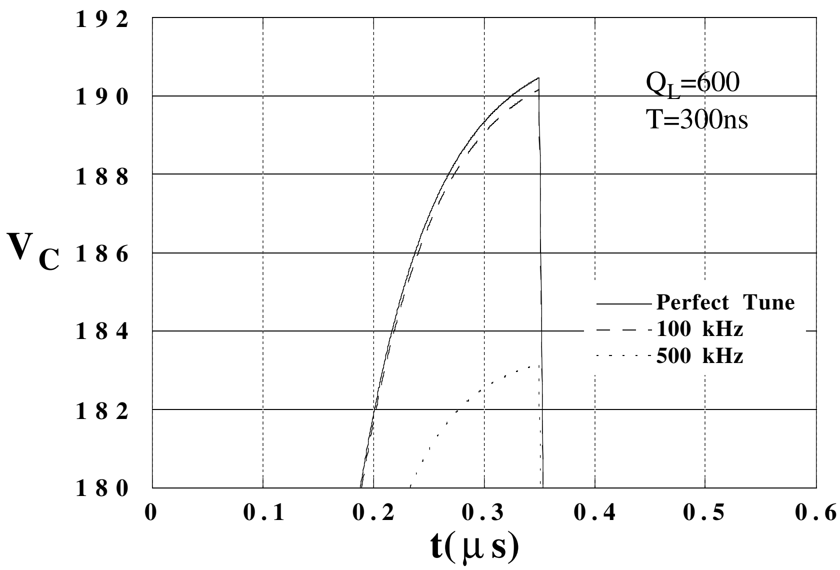

FIGURE 3. (c) A closeup of (b).

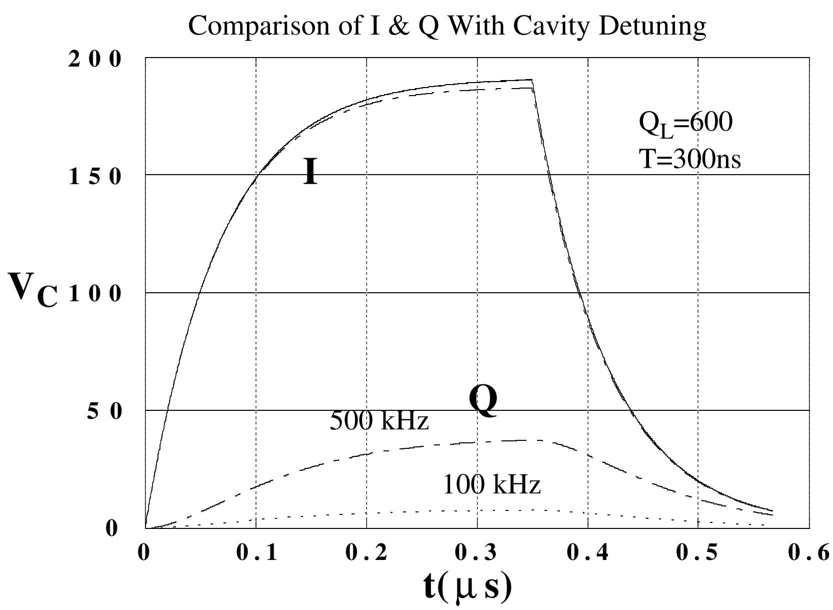

FIGURE 3. (d) A look at the I & Q waveforms corresponding to (b) and (c). The notation is $I = \Re \tilde{V}_c$, $Q = \Im \tilde{V}_c$.

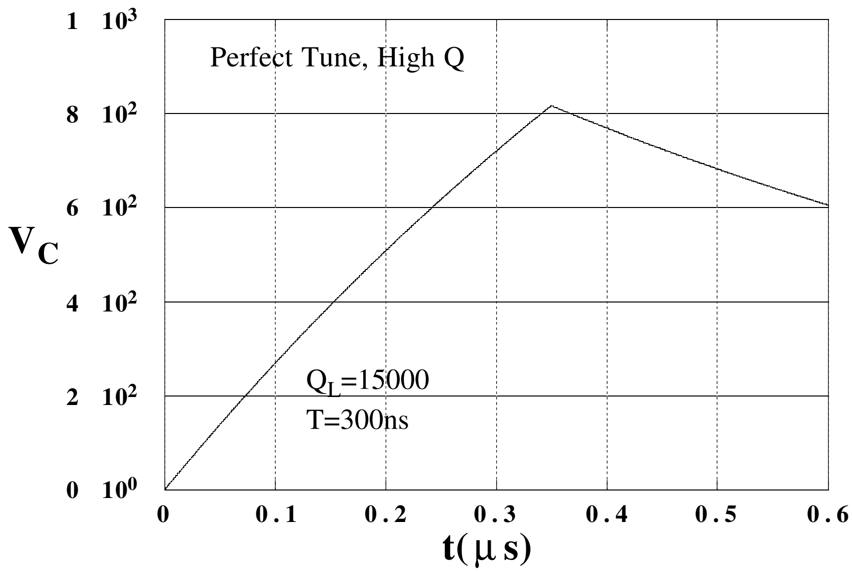

FIGURE 4. (a) Response of a high *Q*, perfectly tuned cavity.

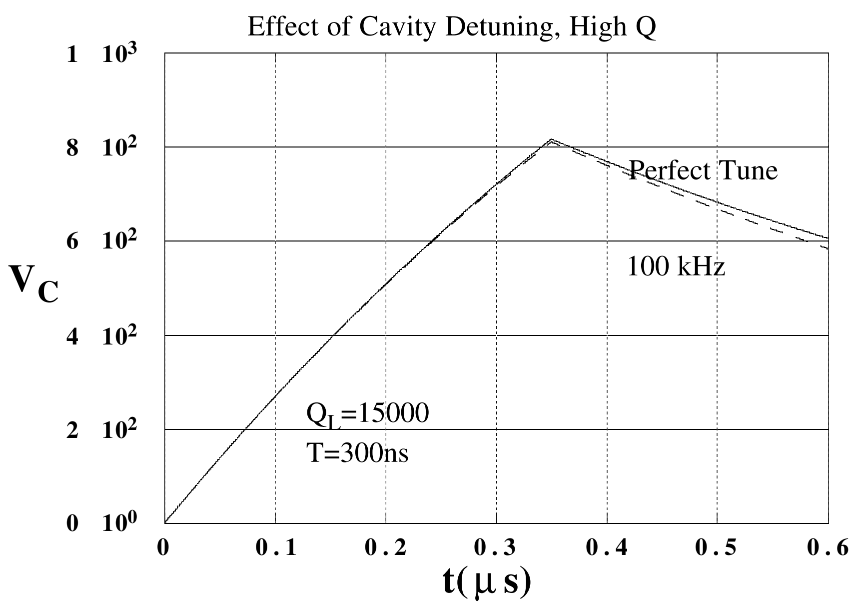

FIGURE 4. (b) Response of a high *Q*, cavity, for perfect tuning, and 100 kHz detuning.

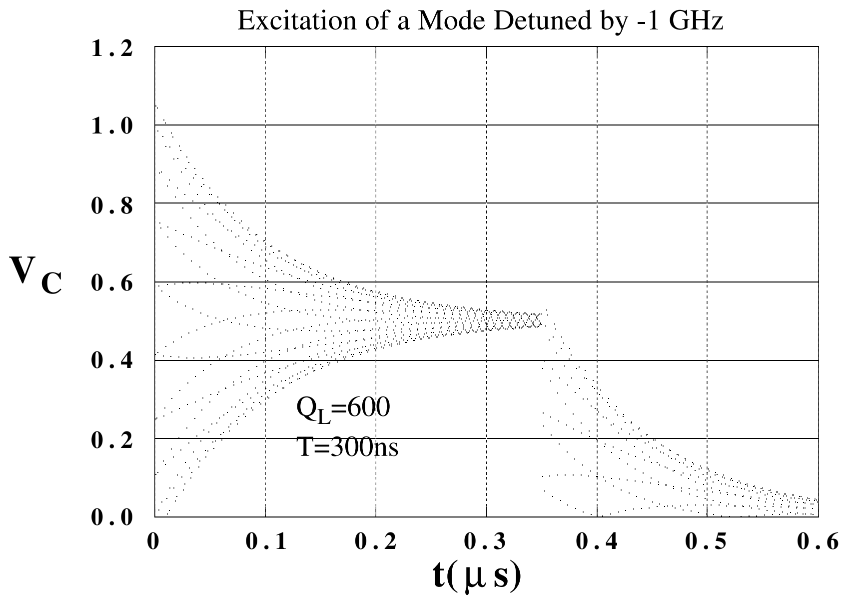

FIGURE 5. Response of a mode detuned by -1 GHz.

$$\tilde{V}_{c-s} = Q_L \left[\frac{R}{Q}\right] I_b = R_L I_b, \qquad \text{(steady-state)}$$

where R_L is the loaded shunt impedance.[8] Steady-state power emanating from the cavity is

$$P_w = \beta \frac{\left|\tilde{V}_{c-s}\right|^2}{R_{shunt}} = \frac{\beta}{(1+\beta)} R_L I_b^2$$

Given the relation to equivalent charge, we may make use of the arithmetic already performed to write

$$10\log_{10} P_w(\text{mW}) \approx 41.6\ \text{dBm} + 20\log_{10} Q_{eq}(\text{nC})\ .$$

It is also helpful to have the expression in terms of current. Since the fill time is 133 ns, 1 nC of equivalent charge corresponds to 7.52 mA of real beam current. Thus

$$10\log_{10} P_w(\text{mW}) \approx 24.1\text{dBm} + 20\log_{10} I_b(\text{mA}).$$

This provides a nominal 257 mW/mA^2 of sensitivity. The Blue Book quotes a value a factor of

[8] It is good to realize, in persuing the literature, that there is a second convention for impedance and current, with current multiplied by two, and impedance divided by two. In this case one is employing $\tilde{I}_b$ as the coefficient in a Fourier series expansion of the current pulse. This coefficient is (for point bunches) just twice the "dc" or "real" or "toroid" current we are denoting by "I_b".

250 mW/mA^2, as does the original technical note on the subject.[9]

In passing we could note that the required sensitivity for the original application of the cavity BPM's was 200 mW/mA^2 . Nominal conditions were intended to be 300 mA corresponding to 17 kW; 50 mA would have correspond to about 500 W. To characterize the signal levels in simpler terms, above 100 mA, care should be employed in disconnecting cables; if one tries to disconnect them with beam on, they will spark, and this can trash other folks electronics. This would also place one at odds with IEEE C95.1-1991.[10]

Before leaving the analysis of the cavity waveform, let us take note of an alternate formulation of the problem. We approximate the beam current using the first harmonic component, $I_{b0} \approx \Re \tilde{I}_{b0} e^{j\omega t}$, and similarly represent the cavity voltage as $V_c \approx \Re \tilde{V}_c e^{j\omega t}$. In the slowly varing envelope (WKB) approximation, the beam-cavity equation reduces to

$$\frac{d\tilde{V}_c}{dt} + \frac{1}{T_f}(1 - j\tan\psi)\tilde{V}_c \approx k_l \tilde{I}_{b0},$$

with tuning angle, filling time, and loss factor given by

$$\tan\psi = Q_L\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) \approx 2Q_L\left(\frac{\omega_0 - \omega}{\omega_0}\right), \qquad T_f = \frac{2Q_L}{\omega_0}, \qquad k_l = \frac{1}{4}\omega_0\left[\frac{R}{Q}\right],$$

For an initially unexcited cavity, and a beam current waveform arriving at t=0,

$$\tilde{V}_c(t) = k_l e^{-\Delta t} \int_0^t dt' \, \tilde{I}_{b0}(t') e^{\Delta t'},$$

abbreviating $\Delta = (1 - j\tan\psi)/T_f$.[11] This form is more amenable to treatment of general current waveforms. For example, modelling intrapulse variation as,

$$\tilde{I}_{b0}(t) = 2I_b \begin{cases} 0\,; & t < 0 \\ e^{-\alpha t}\,; & 0 < t < T_p \\ 0\,; & T_p < t \end{cases}$$

[9] O. Altenmueller and P. Brunet, "Some RF Characteristic of the Beam Phase Reference Cavity" SLAC TN-64-51, Sept. 1964 (available from the SLAC Library).

[10] "IEEE Standard for Safety Levels with Respect to Human Exposure to Radio Frequency Electromagnetic Fields, 3 kHz to 300 GHz", IEEE C95.1-1991 (Revision of ANSI C95.1-1982), IEEE Standards Board (IEEE, New York, 1992).

[11] In making connection with the formulation in terms of a bunch train, the identity

$$\sum_{n=-\infty}^{+\infty} Q_b \delta(t - nT) = \frac{Q_b}{T}\left\{1 + 2\sum_{m=0}^{\infty} \cos\left(2\pi m \frac{t}{T}\right)\right\},$$

is helpful. This implies that a train of bunches each of charge Q_b, spaced at an interval T, and therefore of average current $I_b = Q_b / T$, corresponds to a Fourier component, or rf current, $\tilde{I}_b = 2I_b$.

one finds.

$$\tilde{V}_c(t)=\frac{2k_l I_b}{(\Delta-\alpha)}\begin{cases}\left(e^{-\alpha t}-e^{-\Delta t}\right) & ;\quad 0<t<T_p \\ \left(e^{-\alpha T_p}-e^{-\Delta T_p}\right)e^{-\Delta\left(t-T_p\right)} & ;\quad T_p<t\end{cases}.$$

For the sake of comparison, total pulse charge at time *t* is

$$Q_{tot}(t)=\frac{1}{2}\int_0^t \tilde{I}_{b0}(t)=I_b\begin{cases}\left(\dfrac{1-e^{-\alpha t}}{\alpha}\right) & ;\quad 0<t<T_p \\ \left(\dfrac{1-e^{-\alpha T_p}}{\alpha}\right) & ;\quad T_p<t\end{cases}$$

Both the integral and the discrete-sum expressions for the cavity voltage may be helpful in later analysis of systematics.

6. Numerical Modelling of the Phase-Cavity

To round out our treatment of the cavity, let us model it numerically. This serves as a check of our estimates and those of Altenmueller and Brunet; in addition, it provides a quick means of identifying higher-order modes. For this purpose, we employ the code `GdfidL`,[12] described in a previous note.[13] Dimensions are indicted in Fig. 2. Our input file, and an example output file are provided in the Appendix. The geometry as seen by `GdfidL` is depicted in Fig. 6. A mode listing is given in Table 1.

[12] The acronym stands for the German phrase "Gitter drüber, fertig ist die Laube" or "frame up, finished is the shack", emphasizing the importance of the mesh generator for the finite difference algorithm. See Warner Bruns, "GdfidL: A Finite Difference Program for Arbitrarily Small Perturbations", TET-Note 95/14 (Technische Universität Berlin, Institut für Theoretische Elektrotechnik, Einsteinufer 17, 10587 Berlin. Prof. Heino Henke's group in Berlin has a web site for this and other technical notes.

[13] "Mode Classification in Non-Axisymmetric Structures: How Nonlinear Can a Cavity Be?", ARDB TN 152.

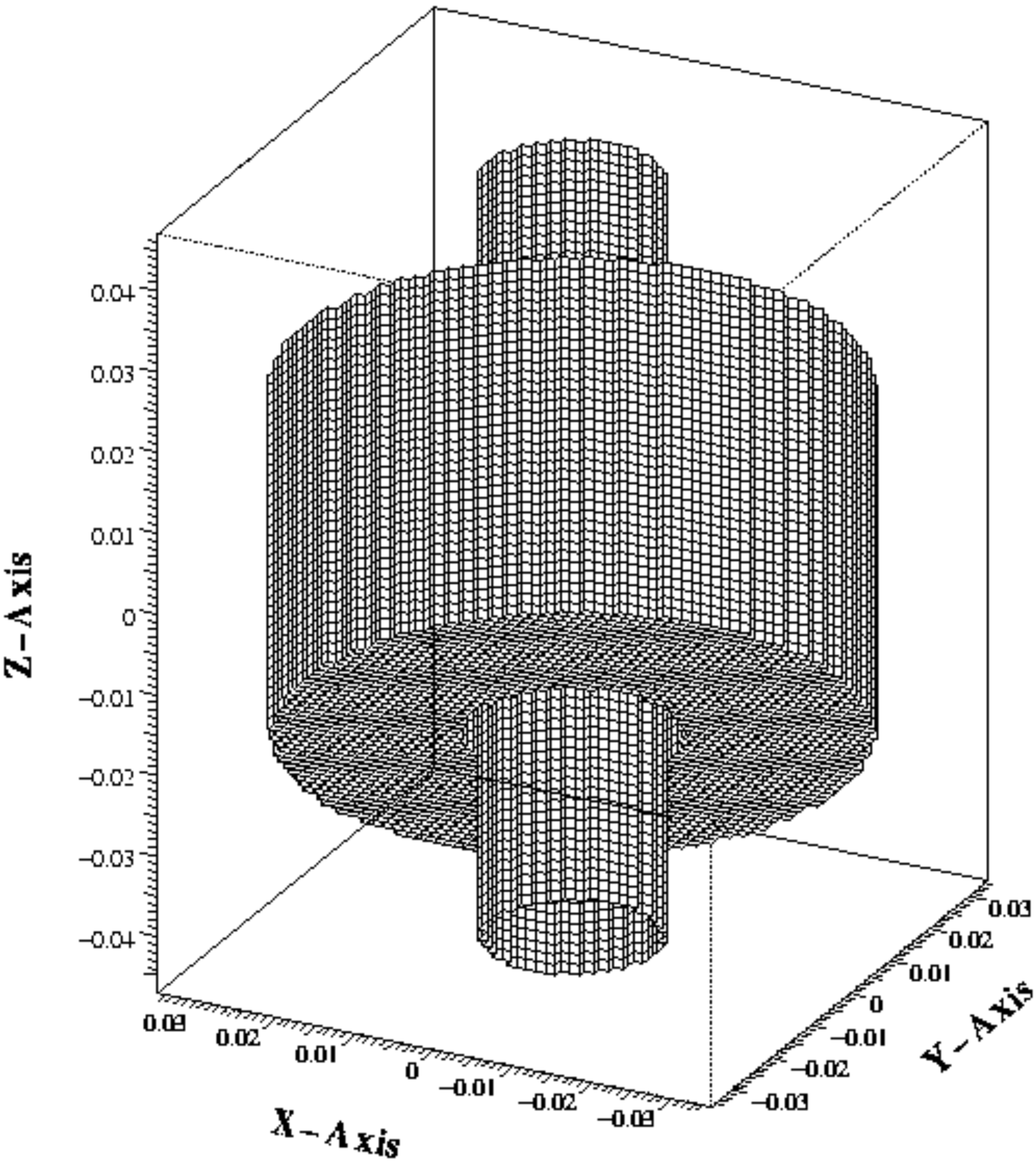

FIGURE 6. Geometry for the Gdfidl calculation for the linac-style phase cavity.

Table 1. Mode listing from `GdfidL` for the phase-cavity. [*R/Q*]-1 is evaluated on the center-axis, [*R/Q*]-2 is evaluated at an xx cm offset.

[in preparation]

For comparison, Altenmueller and Brunet list the following resonances: 2856 MHz, 4210 MHz, 5080 MHz, 6480 MHz. The 4210 MHz mode falls where one would expect a dipole mode. One is pleased, as were Altenmueller and Brunet, to observe that none of these modes falls on a beam harmonic. The frequencies of the first two higher modes could be observed and reconfirmed with a 6 GHz network analyzer with no beam present. During operation, excitation of higher modes can be monitored via WR187 waveguide (high-pass filter, cutting off 2856 MHz), followed by attenuation as needed, and a spectrum analyzer, with a 1 kW limiter on the input port.

7. Conclusions

In this note we have reviewed the basic functioning of the phase-cavity. The larger picture is summarized in ARDB TN 134, together with additional concerns about temperature regulation, network behavior and signal processing.

Appendix: `GdfidL` Input File for the Phase-Cavity

[in preparation]